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A Simpler Construction of Volume Polynomials for a Polyhedron

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Abstract. This paper is related to the third author's previous result on the existence of volume polynomials for a given polyhedron having only triangular faces. We simplify his original proof in the case when the polyhedron is homeomorphic to the 2-sphere. Our approach exploits the fact that any such polyhedron contains a so-called clean vertex – that is, a vertex not incident with any nonfacial cycle composed of 3 edges. This fact appears as one of the main results of the article. Also, we characterize triangulations reducible to a tetrahedron by repeatedly removing 3-valent vertices, and estimate the degree of volume polynomials. We address the torus case too.

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1. Introduction

Let K^2 be a geometric simplicial 2-complex in Euclidean 3-space E^3 and let $|K^2|$ denote the carrier (or underlying space) of K^2. Throughout this paper, we will assume that $|K^2|$ is homeomorphic to a closed, compact, orientable 2-manifold of a given genus $g \geq 0$.

By a *polyhedron* (or *polyhedral surface*) P we will mean a simplexwise linear continuous mapping $P : |K^2| \to E^3$. Especially, a *spherical polyhedron* is one of genus $g = 0$, a *torus polyhedron* has genus $g = 1$. The image $P(|K^2|)$ in E^3 is often also called a polyhedron, with combinatorial structure K^2. The images of the 0-, 1- and 2-simplices of K^2 are called the *vertices*, *edges* and *faces* of P, respectively. A cycle formed by some three edges of a polyhedron P is called a 3-cycle. A 3-cycle of a polyhedron P is called *empty* (or *nonfacial*) if it does not bound a face of the polyhedron.

In this paper we prove that the combinatorial structure of any polyhedron P homeomorphic to the 2-sphere belongs to the class K_0 introduced in [7, 8]. That class consists of the triangulations containing at least one vertex which is not incident with any empty 3-cycle. In reality our Theorem 2 affirms even more: P in fact contains at least two such vertices. As matter of language, a vertex v not incident with an empty 3-cycle will be called a *clean* vertex; otherwise v will be *unclean*.

A (*simplicial*) *triangulation* T of a given 2-manifold M^2 is defined to be the space of M^2 with the structure of a simplicial 2-complex, induced by a homeomorphism η from the carrier of a simplicial 2-complex K^2 to M^2. The images of the 0-, 1-, and 2-simplices of K^2 are called the vertices, edges, and faces of T, respectively. The *graph* $G(T)$ *of* T is the graph defined by the vertices and edges of T. Since T has the structure of a simplicial 2-complex, loops and multiple edges are disallowed in $G(T)$.

If v is a vertex of T, then the *star* of v, $\mathrm{st}(v)$, is the family of all simplices in T of which v is a vertex. If σ is a simplex, the *open simplex* $\mathrm{o}(\sigma)$ associated with σ consists of those points of σ all of whose barycentric coordinates are positive (in some geometric representation of σ). The *open star* of v, $\mathrm{ost}(v)$, is the union of all the open simplices $\mathrm{o}(\sigma)$ for which v is a vertex of σ. Note that $v \notin \mathrm{ost}(v)$.

With each polyhedron P having combinatorial structure K^2 and homeomorphic to a given 2-manifold M^2, we associate the triangulation induced by a homeomorphism $\eta : |K^2| \to M^2$. Two polyhedra have the same combinatorial structure if and only if the corresponding triangulations are *isomorphic* – that is, there exists a bijection between their vertex sets which preserves edges and faces. The concepts of an empty 3-cycle and a clean vertex are defined for a triangulation in the same way as for a polyhedron. Strictly speaking, being clean (or unclean) is a combinatorial property of a vertex in the triangulation T (or its combinatorial structure K^2) rather than a geometric property in a particular polyhedron realizing T.

The *degree* of a vertex v in a triangulation T is the number of edges of T incident with v. Especially, 3-*valent vertices* (that is, vertices of degree 3) are needed when applying our construction. Note that 3-valent vertices are clean.

The article is organized as follows. In Section 2 we simplify the original construction [7, 8] of volume polynomials for a given spherical polyhedron. Our construction requires the existence of at least one clean vertex. In Section 3 we justify that construction, establishing the existence of a clean vertex in any triangulation of the 2-sphere. In that section we also obtain some sufficient conditions for the existence of 3-valent vertices. In general there exist many volume polynomials and the question about the minimum possible degree of such polynomials is not a straightforward problem. In Section 4 we outline some approach to this problem – more precisely, we characterize the triangulations of the 2-sphere reducible to a tetrahedron by repeatedly removing 3-valent vertices (together with their open stars) and estimate below the maximum value among the minimum possible degrees of volume polynomials for spherical polyhedra with a given number of vertices as parameter.

Sections 2–4 have been written by the first and third authors. Section 5, written by the first and second authors, deals with triangulations of the torus along these lines.

2. Construction of a volume polynomial

The third author [7, 8] previously generalized Heron's formula from the area of a triangle to the volume of an orientable polyhedron (Theorem 1 below). Since a given polyhedron P need not be embedded, or even immersed, in E^3, we will have to use the concept of generalized volume instead of the usual volume of P. The *generalized volume* of an oriented polyhedron P in 3-space E^3 is defined to be the sum of the oriented volumes of the tetrahedra spanned by a fixed point O and the coherently oriented faces of P (thus the volume may be negative or equal 0). Note that the so defined volume does not depend on the choice of O and that, when P is embedded in E^3, it coincides with the usual oriented volume of P.

For a given polyhedron P with q edges, denote by $L = \{l_k\} = \{l_1, l_2, \ldots, l_q\}$ the sequence of the lengths of its edges, in an arbitrarily fixed order. Then, denote by $\mathcal{P}(P, K^2, L)$ the family of polyhedra in E^3 with a given combinatorial structure K^2, whose edges have the same lengths as the corresponding edges of P.

Theorem 1. (Sabitov [7, 8]) *Let P be an oriented polyhedron in E^3 with a given combinatorial structure K^2 and given lengths $\{l_k\}$ of the edges. Then there exists a polynomial,*

$$Q(V) = V^{2N} + a_1(L)V^{2N-2} + \cdots + a_N(L),$$

so that the generalized volume of any polyhedron in $\mathcal{P}(P, K^2, L)$ is a root of this polynomial. Furthermore, the coefficients $a_i(L)$ are themselves polynomials in $\{l_1^2, l_2^2, \ldots, l_q^2\}$ with rational coefficients depending on K^2. □

The proof [7, 8] of this theorem is constructive. That construction heavily depends on the existence, or nonexistence, of a clean vertex. In the next section we shall establish a crucial result, Theorem 2, stating that at least two clean vertices always can be found, once we restrict our attention to spherical polyhedra. This result allows a significant simplification of the proof [7, 8] of Theorem 1. More precisely, in the spherical case, Lemma 1 ("Main Lemma") of [7, 8] is always applicable. That lemma states that if Theorem 1 holds for each n-vertex polyhedron P_n then it also holds for any $(n+1)$-vertex polyhedron P_{n+1} of the same genus as P_n provided P_{n+1} has a clean vertex. Fortunately, for genus $g = 0$, thanks to Theorem 2, P_{n+1} always has even two clean vertices! Furthermore, the determination of polynomial $Q(V)$ is now much simpler: there is no need to cut the polyhedra around some empty 3-cycle as required in the above cited works.

We now describe our construction of a volume polynomial for a given spherical n-vertex polyhedron $P = P_n$. Choose a clean vertex of minimum degree, remove it together with its open star, and close the hole with a collection of triangles determined by the diagonals and the sides of the hole. Observe that no multiple edges can be produced while closing the hole because the chosen vertex was clean. Denote by P_{n-1} the resulting polyhedron; it has $n-1$ vertices. Repeat this process to obtain an $(n-2)$-vertex polyhedron P_{n-2}, and so forth. Finally, we will come to a tetrahedron P_4. Now, starting from that P_4, we can successively write volume polynomials $Q_4(V)$, $Q_5(V), \ldots,$ $Q_{n-1}(V)$ and $Q_n(V)$ satisfying Theorem 1, for the volumes of P_4, P_5, $\ldots$, P_{n-1} and $P = P_n$, respectively. The method for writing those polynomials is elaborated in [7, 8] and we omit the details here.

The method is especially efficient in the particular case in which the removed vertices are 3-valent. In that case, we can write the volume polynomials $Q_i(V)$ (for $i = 5, \ldots, n$) by repeatedly adding the volume of the tetrahedron removed to the volume of P_{i-1} with plus or minus sign. Then the degree of the resulting polynomial $Q_n(V)$ is minimum possible.

In the general case, the proposed construction does not necessarily lead to a minimum-degree volume polynomial. But in the spherical case, Astrelin and the third author have proved [1] that a minimum-degree volume polynomial Q_0 is unique and is a divisor of any volume polynomial $Q_n(V)$. Therefore, Q_0 is a common divisor of all such polynomials.

3. Existence of clean vertices

The primary purpose of this section is to justify our construction, namely: establish the existence of a clean vertex in any spherical polyhedron.

Theorem 2. *Any triangulation T of the 2-sphere has at least two clean vertices. Moreover, if the number of vertices of T is greater than 4, then among the clean vertices there are at least two which are not adjacent.*

Proof. Note that a tetrahedron has all the four vertices clean and pairwise adjacent. For the general case we proceed by induction on the number of vertices of T. We begin with the case of 5-vertex triangulations. In fact we have only one such triangulation, up to isomorphism – namely, an embedding of the complete graph K_5 with one edge deleted in the 2-sphere. For this triangulation the statement is obvious. Assume that the statement is true for all triangulations having less than n vertices ($n \geq 5$), and let T be a triangulation of the 2-sphere with n vertices. If T does not have an empty 3-cycle, the statement obviously holds. If T has such a cycle, denote it by C, cut the 2-sphere open around it and close the two resulting triangular holes with faces. As a result of such a surgery, we obtain a pair of triangulations, T_1 and T_2, of the 2-sphere, both having less than n vertices.

Firstly, assume T_1 and T_2 both have more than four vertices. Then apply the induction assumption to find, in both triangulations, a pair of nonadjacent clean vertices. Obviously these vertices cannot be both on C. Thus we can find a pair of vertices, one in T_1 and the other in T_2, so that both are clean in T and not incident with C. Clearly, then, such a pair of vertices (in T) are as desired.

Secondly, assume T_1 has four vertices, that is, is a tetrahedron. Denote by u_1 the vertex of T_1 not incident with C. If T_2 has at least five vertices, we can find in it, by the induction assumption, a clean vertex u_2 not incident with C and not adjacent with u_1. If T_2 has four vertices, T has five vertices and the statement is true by the base of induction. □

As we have seen in Section 2, it is advantageous to have 3-valent vertices for our construction.

Theorem 3. *If a triangulation T of the 2-sphere with at least five vertices has exactly two clean vertices, then both are 3-valent.*

Proof. Denote by u and v the two clean vertices of T. Observe firstly that any empty 3-cycle of T separates u and v, for otherwise we could not find two nonadjacent clean vertices in the part missing u and v (with the hole closed with a triangular face), which contradicts Theorem 2. Note that that part would have to have at least five vertices, since otherwise a third clean vertex would be found in T. Denote by $C(u)$ the empty 3-cycle in T so that the part of T separated by $C(u)$ and containing u does not have an empty 3-cycle inside itself. The existence of $C(u)$ follows from the following construction: Pick any empty 3-cycle of T as $C(u)$, then repeatedly proceed to a smaller 3-cycle around u, in the sense that the part separated by a previous 3-cycle contains the next one. This process is obviously finite, and we denote by $C(u)$ the smallest 3-cycle in the sense that $C(u)$ separates a part containing u but no more empty 3-cycle. Then $C(u)$ has the property that it separates a part of T having u as the sole interior vertex, for otherwise a second interior vertex would be obviously unclean and we could find another empty 3-cycle in that part. It follows that u is 3-valent in T. Similarly, v is 3-valent. □

One might expect that if a triangulation of the 2-sphere has all its clean vertices pairwise nonadjacent, then each clean vertex is 3-valent. However, this is not generally true.

Example 1. Consider a cube with a pyramid attached to its upper and lower faces (the bases of those pyramids are supposed to be removed so that we have a spherical polyhedron again), dissect the four quadrilateral faces by diagonals, and attach a triangular pyramid to each of the resulting triangles. Then, in the resulting triangulation, the eight original vertices of the cube are unclean, while the ten newly added vertices are clean. This triangulation has all clean vertices pairwise nonadjacent, but two of them have degree four.

Sphere Conjecture. If the stars of the clean vertices of a triangulation of the 2-sphere are pairwise vertex-disjoint, then each clean vertex is 3-valent.

4. Decomposition of a polyhedron into tetrahedra

In the light of our construction (Section 2), it is an important combinatorial problem to characterize triangulations T of the 2-sphere reducible to a triangulation with four vertices ("tetrahedron") by repeatedly removing a 3-valent vertex, always together with its open star, followed by closing the hole with a single triangular face. Observe that such an operation does not alter the genus. We will say, then, that triangulation T is *stellarly reducible.* This term is motivated as follows. Recall that a *stellar subdivision* applied to a triangulation consists of adding a vertex in some face f and joining it to the three vertices in the boundary of f. Therefore the replacement of the three faces incident with a 3-valent vertex by a single triangular face can be regarded as an operation inverse to the stellar subdivision: it decreases the number of vertices by one.

Let B be a convex 3-polytope in E^3 which contains a 3-valent vertex, v. Cut B by the plane passing through the three boundary edges of $\mathrm{st}(v)$ and discard the part that contains v. The result, B', is again a convex 3-polytope, with one less vertex. If B' also contains a 3-valent vertex, we can repeat the operation and cut B'. If, proceeding in this fashion, we can reduce B to a single solid tetrahedron, we will say that B is *decomposable into tetrahedra* or, more briefly, 3-*decomposable*.

This geometric construction can be interpreted in combinatorial terms. Let T be a triangulation of the 2-sphere. By well-known Steinitz's Theorem, T can be realized in E^3 as the boundary 2-complex of a convex 3-polytope B. Suppose T is stellarly reducible. Then the removal of a 3-valent vertex of T corresponds to some cutting of B as described in the preceding paragraph; the converse is also true. Therefore T is stellarly reducible if and only if B is 3-decomposable.

The process of decomposition of B into solid tetrahedra can be also regarded as a decomposition of some abstract simplicial 3-complex K^3 into 3-simplices where a 3-simplex is removable if, and only if, it shares precisely one of its 2-faces with another 3-simplex of K^3.

Example 2. It is easy to verify that any solid pyramid with its base arbitrarily triangulated is decomposable into tetrahedra.

The following is a more sophisticated example.

Example 3. If we cut the triangulation T 1 (Figure 1 on page 270) around the 3-cycle $(3, 1, 2, 3)$ and close the two holes with triangular faces, we obtain the triangulation T 1′ of the 2-sphere depicted in Figure 2(a). The vertices of the face corresponding to the upper side of the rectangle (Figure 1) are denoted by $1'$, $2'$, $3'$, and the corresponding vertices of the lower face by $1''$, $2''$, $3''$, respectively. We assume that the 3-cycle $(3'', 1'', 2'', 3'')$ bounds a face ("outer face"). Then, T 1′ is stellarly reducible by removing the 3-valent vertices $1'$, $2'$, $3'$, 4, 5, and 6, one by one in this order. This process is illustrated in Figure 2.

It is an open combinatorial problem to characterize triangulations of a given genus g reducible by successively removing 3-valent vertices to a vertex-minimum triangulation admissible by the (orientable or nonorientable) 2-manifold of genus g. In this section we settle this problem for $g = 0$. Our first criterion (Theorem 4) is rather theoretical, but the second (Theorem 5) is readily checkable and therefore is more useful on the practical side.

For a given simplicial 3-complex K^3, we define the *dual graph*, $G^*(K^3)$, to be the graph whose vertices correspond to the 3-simplices of K^3 and two vertices are adjacent if and only if the corresponding 3-simplices share a 2-face.

Theorem 4. *A convex* 3*-polytope B is* 3*-decomposable if and only if B is the carrier of some* 3*-complex K^3 having a tree as its dual graph $G^*(K^3)$.*

Proof. Sufficiency: Since $G^*(K^3)$ is a tree, it has a vertex of degree one. The 3-simplex of K^3 corresponding to that vertex is removable. Furthermore, after the removal of that 3-simplex, the dual of the resulting 3-complex is also a tree and we can repeat this process as many times as needed to obtain a single tetrahedron.

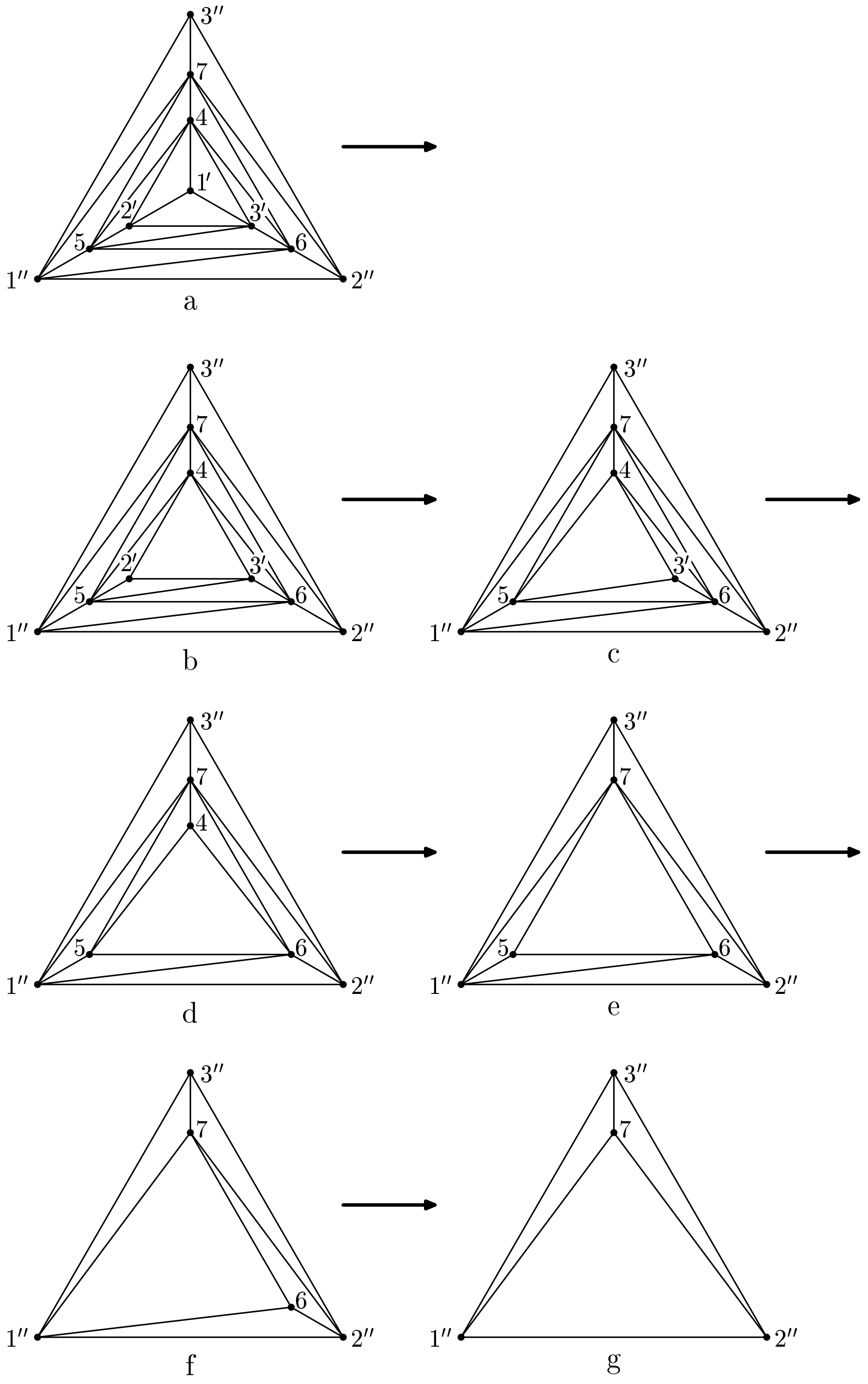

Fig. 2. 3-decomposition of T 1 cut around $(3, 1, 2, 3)$.

Necessity: Assume to the contrary that $G^*(K^3)$ is not a tree. Hence, $G^*(K^3)$ has a cycle C. Consider the simplicial 3-complex determined by the 3-simplices of K^3 corresponding to the vertices of C. This 3-complex bears resemblance to a "cycle" made up of tetrahedra. Then, however, none of its 3-simplices would be ever removable. This contradicts the hypothesis that B can be 3-decomposed. □

Remark 1. Theorem 4 obviously extends to a 3-polytope B which is not necessarily convex if by a 3-polytope we mean a simplexwise linear continuous mapping to E^3 from the carrier of some 3-complex K^3, homeomorphic to the unit 3-ball, under the condition that all vertices of K^3 are mapped onto the vertices of B.

Theorem 5. *An n-vertex triangulation T of the 2-sphere is stellarly reducible if and only if it has at least $n-4$ empty 3-cycles.*

To prove this theorem, we need to develop two lemmas.

Lemma 1. *Any n-vertex triangulation T of the 2-sphere can have at most $n-4$ empty 3-cycles.*

Proof. This proceeds by induction on the number of vertices of T. For $n=5$, the only 5-vertex triangulation has exactly one empty 3-cycle and the statement is true. Assume that the statement is true for all triangulations having less than n vertices ($n \geq 5$), and let T be an n-vertex triangulation of the 2-sphere. Suppose for a contradiction that some $n-3$ empty 3-cycles are found in T. Cut T around one of those 3-cycles into two pieces. Denote by T_1 and T_2 the two triangulations obtained by closing the holes of the pieces by triangular faces. By Jordan curve theorem, any other empty 3-cycle lies entirely in T_1 or T_2. Denote by k and m the numbers of vertices and empty 3-cycles in T_1, respectively. Then T_2 has $n-k+3$ vertices and at least $n-m-4$ empty 3-cycles. The case $m > k-4$ is impossible by the induction assumption for T_1, so that $m \leq k-4$, so that $n-m-4 \geq n-k > n-k-1$, which contradicts the induction assumption applied to T_2. □

Lemma 2. *If an n-vertex triangulation T of the 2-sphere has $n-4$ empty 3-cycles, then T has at least two nonadjacent 3-valent vertices.*

Proof. This proceeds by induction on the number of vertices of T. For the 5-vertex triangulation the statement is true. Assume that the statement is true for all triangulations having less than n vertices, and let T be an n-vertex triangulation of the 2-sphere having $n-4$ empty 3-cycles. We apply the same surgery as in the proof of Lemma 1, using the same notation. The k-vertex triangulation T_1 has exactly $k-4$ empty 3-cycles. For, it cannot have more, by Lemma 1, but if it had less, we would have a contradiction with Lemma 1 for the triangulation T_2.

Suppose first that $k > 4$ and $n-k > 1$. Denote by C_1 and C_2 the two banks of the cut of T. Then, after closing the holes, T_1 has face C_1, and T_2 has face C_2. By the induction assumption, T_1 has two nonadjacent 3-valent vertices. At least one of them, designate it by v_1, is not in C_1. Similarly, T_2 has two nonadjacent 3-valent vertices, and at least one of them, v_2, is not in C_2. The vertices v_1 and v_2 are nonadjacent 3-valent vertices in T, as desired.

Suppose now that $k = 4$ or $n - k = 1$. Both possibilities are treated similarly and we assume the former. The sole vertex of T_1 not in C_1, designate it by v_1, has degree 3 in T. Now, if $n - k = 1$, the sole vertex v_2 of T_2 not in C_2 is 3-valent in T. If $n - k > 1$, take as v_2 any 3-valent vertex of T_2 not in C_2. Such a vertex exists by the induction assumption. In both cases the pair $\{v_1, v_2\}$ is as desired. The lemma has been proved. □

Proof of Theorem 5. Sufficiency: By induction on the number of vertices of T. Assume that an n-vertex triangulation T has at least $n - 4$ empty 3-cycles. Then, by Lemma 1, T has exactly $n-4$ empty 3-cycles. By Lemma 2, T has two nonadjacent 3-valent vertices. Remove one of those vertices together with its open star and close the hole with a single triangular face. The resulting $(n - 1)$-vertex triangulation has $n - 5$ empty 3-cycles. Applying the induction assumption finishes the proof of the "if" part.

Necessity: Observe first that no edge is added under the removal of a 3-valent vertex. Assume T is stellarly reducible. The removal of each 3-valent vertex v destroys one empty 3-cycle – namely, the one on the boundary of $\mathrm{st}(v)$. Therefore, to arrive at a tetrahedron, we have to destroy some $n - 4$ empty 3-cycles. □

Recall that a minimum-degree volume polynomial for a given spherical polyhedron is unique.

Theorem 6. *There exists an n-vertex spherical polyhedron so that any of its volume polynomials has degree at least 2^{n-3} and this bound is attained by some volume polynomial.*

Proof. Let P be an n-vertex stellarly reducible spherical polyhedron. By Theorem 5 and Lemma 1, it has exactly $n-4$ empty 3-cycles and, by Lemma 2, there are exactly 2^{n-4} different sequences of tetrahedron deletions transforming P into a single tetrahedron. Recall that when constructing a volume polynomial for P as described in Section 2, we successively remove tetrahedra from P, and for each tetrahedron removed we add its volume to, or subtracted from, the volume of the preceding polyhedron. The statement is now obvious. □

It follows from this proof that the statement of Theorem 6 holds for any pyramid with n vertices; see Example 2.

Corollary 1. *The maximum degree of the minimum-degree volume polynomials over n-vertex spherical polyhedra is at least 2^{n-3}.* □

5. The torus case

In [7, 8], a detailed description is given as how to write a volume polynomial by way of surgery – that is, cutting the polyhedron. It is important to keep in mind the allowance of self-intersections in a polyhedron; those can be of two kinds: self-crossings and self-touchings. That means that by cutting a torus polyhedron around some empty 3-cycle and closing the holes with triangles f_1 and f_2, one obtains a spherical polyhedron with an obvious self-touching – namely, $f_1 = f_2$ as point sets in E^3. Thus, by way of surgery, a torus polyhedron can be reduced to a spherical one. We are interested in whether this process can produce clean (especially 3-valent) vertices, since such vertices are playing an important role in the spherical case. The purpose of this section is to study torus polyhedra along these lines.

By straightforward algebraic manipulations involving Euler's equation and the fact that the sum of vertex degrees in any graph is twice the number of edges, it can be easily seen that for each triangulation T of the torus the mean degree of a vertex is equal to 6. It follows that any triangulation of the torus has a vertex of degree 5, 4, or 3 unless it is 6-*regular* – that is, each vertex has degree 6. Therefore, 7 is the minimum number of vertices that a triangulation of the torus can have. Such a vertex-minimum torus triangulation indeed exists and is known [5, 6] to be unique up to isomorphism. This triangulation is shown in Figure 1 and will be denoted by T 1; identify the opposite sides of the rectangle to obtain a torus.

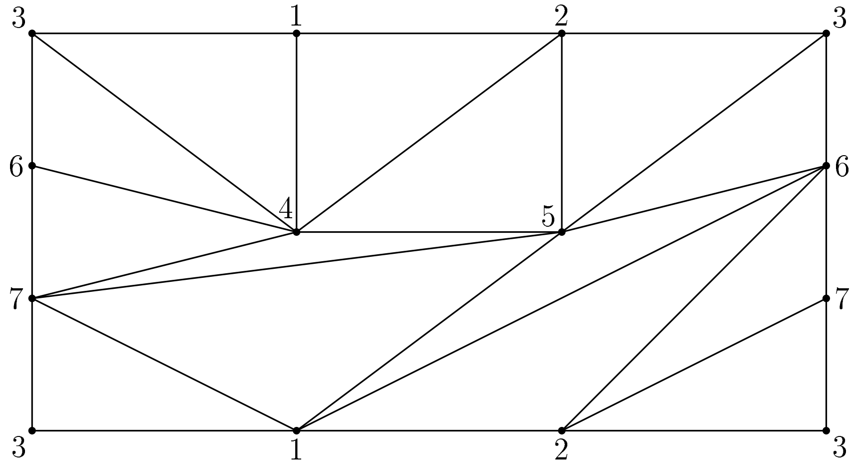

Fig. 1. 7-vertex triangulation T 1 of a torus.

Czászár has established [4] the existence of an embedding of T 1 in E^3. In [3], the reader may find instructions on how to make a cardboard model of this remarkable minimum torus polyhedron without diagonals (with a photograph) realizing T 1. Bokowski and Eggert have suggested a method for constructing all embeddings of T 1 in E^3.

Observe that T 1 (as well as any polyhedron realizing this triangulation) has the property that all its vertices are unclean. Furthermore, any vertex-minimum triangulation of each nonspherical 2-manifold has this property, for otherwise we could decrease the number of vertices by removing a clean vertex together with its open star and closing the hole with a collection of triangular faces.

All 6-regular triangulations of the torus have been classified [6] by the second author. We reproduce that classification shortly, but first we address a byproduct of it: there exist infinitely many 6-regular torus triangulations without clean vertices. Whence, there does not exist a constant N so that any torus polyhedron with at least N vertices has a clean vertex. Therefore, for a torus polyhedron and, in general, for a polyhedron of genus $g > 0$ it is hopeless to avoid surgery in constructing a volume polynomial.

We now proceed to classify 6-regular triangulations of the torus. Let T be a 6-regular torus triangulation. For a given vertex v of T, denote by v_1, v_2, v_3, v_4, v_5, and v_6 the adjacent vertices in cyclic order as they occur around v in T. We say that the three pairs of edges $\{vv_1, vv_4\}$, $\{vv_2, vv_5\}$, and $\{vv_3, vv_6\}$ are the "diagonals" of $\mathrm{st}(v)$. A cycle C in T is *straight* at vertex v when the two edges of C incident with v form a diagonal of $\mathrm{st}(v)$. A cycle is *geodesic* if it is straight at each vertex.

According to the classification [6], every 6-regular torus triangulation can be written in a *standard form* as $T(p,q,r)$. Such a form is not well-defined. The set of admissible triples of integers (p,q,r) breaks into orbits by the action of some translation group, where the triples in one orbit represent isomorphic triangulations.

Let (p,q,r) be a given triple of integers satisfying the inequalities,

$$p > 0, \quad r > 0, \quad \text{and} \quad 0 \le q < p.$$

We now describe the construction for $T(p,q,r)$. Consider the 6-regular (infinite) triangulation of the xy-plane $E^2(x,\, y)$, defined by the three families of lines: $x = k$, $y = l$, and $y = x + m$, where k, l, and m are arbitrary integers. We suppose that a vertex is placed in each point at which the lines cross. In the so-triangulated plane consider the rectangle,

$$\{(x,y) : \; 0 \le x \le r, \; 0 \le y \le p\}.$$

Identify the opposite sides of this rectangle so that vertex $(0,\, k)$ merges with $(r,\, k-q \,(\mathrm{mod}\, p))$ for each $k = 0,\dots,p$, and $(l,\, 0)$ merges with $(l,\, p)$ for $l = 0,\dots,r$. It is a simple matter to verify that we thus obtain a triangular embedding of a *pseudograph* (that is, a graph with loops and multiple edges possible) in the torus. That embedding is in a standard form, denoted by $T(p,q,r)$. The second author has characterized [6] all triples (p,q,r) for which $T(p,q,r)$ is a triangulation (in the sense of the definition given in the Introduction). Its graph is free from loops and multiple edges.

Example 4. The vertex-minimum triangulation T 1 of Figure 1 can be written as $T(7,2,1)$ or $T(7,4,1)$.

Lemma 3. *A 6-regular torus triangulation T satisfies the following two conditions,*
(a) *every vertex belongs to some empty 3-cycle, and*
(b) *every empty 3-cycle is geodesic,*
if and only if T is either $T(3,0,3)$ or $T(3,s,t)$ for some $s \in \{0,1,2\}$ and $t \ge 4$.

Proof. Necessity: Conditions (a) and (b) ensure the existence of a geodesic 3-cycle and this implies that T can be written in a standard form as $T(3,q,r)$. Since pr is equal to the number of vertices in $T(p,q,r)$ and the smallest triangulation of the torus has 7 vertices, it follows that $r \ge 3$. For $r = 3$ we obtain a triangulation $T(3,q,3)$ with 9 vertices. It is a result of [6] that this triangulation is one of the following two types:

Type 1: $T(9,2,1) = T(9,3,1) = T(9,5,1) = T(9,6,1) = T(3,1,3) = T(3,2,3)$;
Type 2: $T(3,0,3)$.

The first type admits six standard forms while the second admits only one such form. They both satisfy condition (a) since they have a geodesic 3-cycle. It is straightforward to inspect which of them satisfy condition (b): $T(3,0,3)$ does, but $T(3,1,3)$ does not because it contains an empty 3-cycle not geodesic.

Sufficiency: We already know that $T(3,0,3)$ satisfies conditions (a) and (b). If $r > 3$, there is no empty 3-cycle in $T(3,q,r)$ other than the geodesic cycles of length p. Therefore each $T(3,q,r)$ with $r > 3$ satisfies both conditions (a) and (b). The proof is complete. □

Theorem 7. *Let T be a triangulation of the torus without a clean vertex. Then T can be cut open around some empty 3-cycle C of T so that the resulting triangulation T', with the holes closed with triangular faces, is a triangulation of the 2-sphere with at least two clean vertices. Furthermore, if T is not one of $T(3,0,3)$, or $T(3,s,t)$ where $s \in \{0,1,2\}$ and $t \geq 4$, C can be chosen so that at least one of the clean vertices of T' is a 3-valent vertex. If T is one of the above exceptions, it is impossible to cut T around an empty 3-cycle with producing a 3-valent vertex.*

Torus Conjecture. The exceptional triangulations can be cut producing at least two clean vertices of degree 4.

Proof of Theorem 7. By the hypothesis, each vertex of T is incident with an empty 3-cycle. Take as C any empty 3-cycle of T, cut T open around C and close the holes with triangular faces. The resulting space is still connected. To see this, assume the contrary. Then one of the components, P', would be necessarily homeomorphic to the 2-sphere. If P' had 4 vertices, T would have a 3-valent, and thereby clean, vertex. This case is excluded by the hypothesis. If P' had at least 5 vertices, it would contain at least two nonadjacent clean vertices, by Theorem 2. One of those vertices, v, would not lie on the bank of C in P' and thereby would be a clean vertex of T. This, however, contradicts our hypothesis. Therefore, by way of cutting T around C followed by closing the holes with triangular faces, we obtain a triangulation T' as desired, by Theorem 2.

Let us now prove the second statement. If T is not 6-regular, we can choose as C an empty 3-cycle incident with a vertex of degree 5 or 4. Furthermore, if T is 6-regular, but has an empty 3-cycle not geodesic, we can again cut T into an annulus with a vertex of degree 3 on the boundary. Then we will obviously have a desired 3-valent vertex in T'. Therefore it only remains to consider the case in which every empty 3-cycle of T is geodesic. In that case, Lemma 3 applies to T, since all vertices of T are unclean, by the hypothesis, and T is one of the exceptions. The last statement also follows immediately from Lemma 3. The proof is complete. $\square$

Presently, little is known about nonorientable polyhedra of low genus in regard to the existence of clean vertices.

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