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Natural Projectors in Tensor Spaces*

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Abstract. The aim of this paper is to introduce a method of invariant decompositions of the tensor space $T^r_s\mathbf{R}^n = \mathbf{R}^n \otimes \mathbf{R}^n \otimes \cdots \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*} \otimes \cdots \otimes \mathbf{R}^{n*}$ (r factors $\mathbf{R}^n$, s factors the dual vector space $\mathbf{R}^{n*}$), endowed with the tensor representation of the general linear group $GL_n(\mathbf{R})$. The method is elementary, and is based on the concept of a natural ($GL_n(\mathbf{R})$-equivariant) projector in $T^r_s\mathbf{R}^n$. The case $r = 0$ corresponds with the Young-Kronecker decompositions of $T^0_s\mathbf{R}^n$ into its primitive components. If $r, s \neq 0$, a new, unified invariant decomposition theory is obtained, including as a special case the decomposition theory of tensor spaces by the trace operation.

As an example we find the complete list of natural projectors in $T^1_2\mathbf{R}^n$. We show that there exist families of natural projectors, depending on real parameters, defining new representations of the group $GL_n(\mathbf{R})$ in certain vector subspaces of $T^1_2\mathbf{R}^n$.

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1. Introduction

In this paper we give basic definitions and prove basic results of natural projector theory in tensor spaces over the field or real numbers $\mathbf{R}$. The tensor space of type (r, s) over the

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vector space $\mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \cdots \times \mathbf{R}$ (n factors $\mathbf{R}$) is denoted by $T^r_s\mathbf{R}^n = \mathbf{R}^n \otimes \mathbf{R}^n \otimes \cdots \otimes \mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*} \otimes \cdots \otimes \mathbf{R}^{n*}$ (r factors $\mathbf{R}^n$, s factors the dual vector space $\mathbf{R}^{n*}$). We always suppose $n \geq 2$. $\mathbf{R}^n$ is considered with the canonical left action of the general linear group $GL_n(\mathbf{R})$, and the tensor space $T^r_s\mathbf{R}^n$ is endowed with the induced (tensor) action. Since our discussions are $GL_n(\mathbf{R})$-invariant, the results apply in the well-known sense to any real, n-dimensional vector space E, and to the tensor space $T^r_s E$ of type (r, s) over E.

We wish to describe a method allowing us to find all $GL_n(\mathbf{R})$-invariant vector subspaces of the vector space $T^r_s\mathbf{R}^n$; indeed, this is equivalent to finding all $GL_n(\mathbf{R})$-equivariant projectors $P : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$. In accordance with the terminology of the differential invariant theory, $GL_n(\mathbf{R})$-equivariant projectors are also called *natural*.

This method complements our previous results on decompositions of tensor spaces, which are not based on the group representation theory (see [4, 5]). It can be applied effectively for any concrete r and s. However, a general formula for the decomposition has not been found.

It seems that the idea to apply the theory of projectors to the problem of decomposing a tensor space of type $(r, 0)$, or $(0, s)$ into its primitive components belongs to H. Weyl [7]. However, this idea has never been developed to a complete theory, or used to an analysis of concrete cases. Later, the same author gives preference of the group representation theory over the ideas of the pure projector theory [6]; a standard restrictive assumption in this approach is usually applied from the very beginning, namely the assumption that the representation space is a vector space over an algebraically closed field.

For basic ideas and generalities on natural projectors in tensor spaces we refer to Krupka (see [3], Sections 4.4 and 7.3).

Let us now recall briefly main concepts. A tensor $t \in T^r_s\mathbf{R}^n$ is said to be *invariant*, if $g \cdot t = t$ for all $g \in GL_n(\mathbf{R})$. A theorem of Gurevich says that an invariant tensor of type (r, s), where $r \neq s$, is always the zero tensor, and, if $r = s$, an invariant tensor is always a linear combination $\sum c_\sigma \delta^{j_1}_{i_{\sigma(1)}} \delta^{j_2}_{i_{\sigma(2)}} \cdots \delta^{j_N}_{i_{\sigma(N)}}$ of products of r factors of the Kronecker δ-tensor, where $c_\sigma \in \mathbf{R}$, and σ runs through all permutations of the set $\{1, 2, \ldots, r\}$ (see [1]). Consider a real, N-dimensional vector space E endowed with a left action of $GL_n(\mathbf{R})$. A linear mapping $F : E \to E$ is called $GL_n(\mathbf{R})$-*equivariant*, or *natural*, if $F(g \cdot x) = g \cdot F(x)$ for all $x \in E$ and all $g \in GL_n(\mathbf{R})$. It is a simple observation that F is natural if and only if its components form an invariant tensor [3]. A natural linear mapping $P : E \to E$ which is a projector, i.e., satisfies the projector equation $P^2 = P$, is called a *natural projector*.

In Section 2 we collect standard definitions and facts of the theory of projectors in a vector space (see e.g. [2]). Section 3 is devoted to natural linear operators in a vector space endowed with a left action of $GL_n(\mathbf{R})$. In Section 4 we introduce natural projectors in tensor spaces and related concepts such as natural projector equations, decomposability, reducibility, and primitivity. Section 5 is concerned with the trace decomposition theory; it is shown that the trace decomposition of a tensor is related to a natural projector determined uniquely by certain conditions. Finally, in Section 6 we describe all natural projectors in the tensor space $T^1_2\mathbf{R}^n$.

It should be pointed out that the method of natural projectors allows us to treat in a unique way the case of tensors of type (r, s), where not necessarily $r = 0$, or $s = 0$. In this sense the natural projector theory represents a generalization of the classical Young–Kronecker decomposition theory (see e.g. [6]), as well as of the trace decomposition theory [4, 5].

2. Projectors

This introductory section contains a brief formulation of standard results of the projector theory in a finite-dimensional, real vector space E (see e.g. [2]).

Let E^* be the dual of E, and let $E \times E^* \ni (x, y) \to y(x) = \langle x, y \rangle \in \mathbf{R}$ be the natural pairing. The *dual* $A^* : E^* \to E^*$ of a linear mapping $A : E \to E$ is defined by the condition $\langle Ax, y \rangle = \langle x, A^* y \rangle$ for all $x \in E, y \in E^*$. If $A, B : E \to E$ are two linear mappings, then $(AB)^* = B^* A^*$,

A linear operator $P : E \to E$ is said to be a *projector*, if $P^2 = P$. Clearly, the zero mapping 0, and the identity mapping id_E, are projectors.

Lemma 1. *Let E be a finite-dimensional, real vector space.*

(a) *A projector $P : E \to E$ defines the direct sum decomposition $E = \ker P \oplus \mathrm{im}\, P$.*

(b) *A linear mapping $P : E \to E$ is a projector if and only if $\mathrm{id}_E - P$ is a projector.*

(c) *If $P : E \to E$ is a projector, then $Q = \alpha P$, where $\alpha \in \mathbf{R}$, is a projector if and only if $\alpha = 0, 1$.*

(d) *Let $P, Q : E \to E$ be two projectors such that $\mathrm{im}\, P = \mathrm{im}\, Q = F$. Then there exists a unique linear isomorphism $U : F \to F$ such that $P = U \circ Q$.*

Let $u^* : E^* \to E^*$ denote the dual of a linear mapping $u : E \to E$. We say that two projectors $P, Q : E \to E$ are *orthogonal*, if $\langle Px, Q^* y \rangle = 0$ and $\langle Qx, P^* y \rangle = 0$ for all $x \in E, y \in E^*$. Obviously, P and Q are orthogonal if and only if $QP = 0$ and $PQ = 0$. For every projector P, the projectors P and $\mathrm{id}_E - P$ are orthogonal.

Lemma 2. *Let $P, Q : E \to E$ be projectors.*

(a) *$P + Q$ is a projector if and only if P and Q are orthogonal.*

(b) *$P - Q$ is a projector if and only if $PQ = QP = Q$.*

(c) *If P and Q commute, $PQ - QP = 0$, then $R = PQ = QP$ is a projector, and $\mathrm{im}\, R = \mathrm{im}\, P \cap \mathrm{im}\, Q$.*

(d) $\ker P = \mathrm{im}\,(\mathrm{id} - P)$.

Remark 1. If $P + Q$ is a projector, then condition (a) implies $PQ = QP = 0$ hence by (c), $\mathrm{im}\, P \cap \mathrm{im}\, Q = \{0\}$. Thus $\mathrm{im}\,(P + Q) = \mathrm{im}\, P + \mathrm{im}\, Q$ is the direct sum of its subspaces $\mathrm{im}\, P$ and $\mathrm{im}\, Q$.

Remark 2. If $P - Q$ is a projector, condition (b) together with (c) imply that $\mathrm{im}\, Q \subset \mathrm{im}\, P$.

3. Natural linear operators in tensor spaces

Let E be a finite-dimensional, real vector space, endowed with a left action of the general linear group $GL_n(\mathbf{R})$, denoted multiplicatively. A linear operator $F : E \to E$ is said to be $GL_n(\mathbf{R})$-*equivariant*, or *natural*, if $F(A \cdot x) = A \cdot F(x)$ for every $x \in E$ and every $A \in GL_n(\mathbf{R})$. The vector space of natural linear operators on E is denoted $\mathcal{N}E$.

The kernel and the image of a natural linear operator $F : E \to E$ are $GL_n(\mathbf{R})$-invariant vector subspaces of E.

Our aim in this section is to study natural linear operators in the tensor space $T^r_s\mathbf{R}^n$. If the canonical basis of $\mathbf{R}^n$ is denoted by e_i, and e^i is the dual basis of $\mathbf{R}^{n*}$, then any tensor $t \in T^r_s\mathbf{R}^n$ is uniquely expressible in the form

$$t = t^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_s}, \tag{3.1}$$

where the real numbers $t = t^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ are the components of t. We usually write $t = t^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$.

Let $(A, x) \to \bar{x} = A \cdot x$ be the canonical left action of $GL_n(\mathbf{R})$ on $\mathbf{R}^n$; in the canonical basis of $\mathbf{R}^n$, $\bar{x}^i = A^i_j x^j$, where $A = A^i_j$. If $B = A^{-1}$, $B = B^i_j$, the tensor action of $GL_n(\mathbf{R})$ on $T^r_s\mathbf{R}^n$ is given by

$$\bar{t} = A \cdot t = \bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_s}, \tag{3.2}$$

where

$$\bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = A^{i_1}_{k_1} A^{i_2}_{k_2} \cdots A^{i_r}_{k_r} B^{l_1}_{j_1} B^{l_2}_{j_2} \cdots B^{l_s}_{j_s} t^{k_1 k_2 \cdots k_r}_{l_1 l_2 \cdots l_s}. \tag{3.3}$$

A tensor $t \in T^r_s\mathbf{R}^n$ is said to be *invariant*, if $A \cdot t = t$ for all $A \in GL_n(\mathbf{R})$. The following theorem describes all invariant tensors (see [1], and [3]).

Let S_r denote the group of permutations σ of the set $\{1, 2, \ldots, r\}$.

Lemma 3. *Let $t \in T^r_s\mathbf{R}^n$.*

(a) *Assume that $r \neq s$. Then t is invariant if and only if $t = 0$.*

(b) *Assume that $r = s$. Then t is invariant if and only if*

$$t^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_r} = \sum_{\sigma \in S_r} a^\sigma \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_r}_{j_{\sigma(r)}} \tag{3.4}$$

for some $a^\sigma \in \mathbf{R}$.

Invariant tensors in $T^r_r\mathbf{R}^n$ form a real vector space. This vector space is spanned by the invariant tensors

$$\begin{aligned} E_\sigma &= \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_r}_{j_{\sigma(r)}} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r} \\ &= e_{j_{\sigma(1)}} \otimes e_{j_{\sigma(2)}} \otimes \cdots \otimes e_{j_{\sigma(r)}} \otimes e^{j_1} \otimes e^{j_2} \otimes \cdots \otimes e^{j_r}. \end{aligned} \tag{3.5}$$

Note that any invariant tensor can be expressed, instead of (3.4), by

$$t = \sum_{\sigma \in S_r} a^\sigma E_\sigma. \tag{3.6}$$

Now we apply Lemma 3 to natural linear mappings $F : T^r_s\mathbf{R}^n \to T^p_q\mathbf{R}^n$. We have the following simple observation ([3], Section 4.4).

Lemma 4. *Let $F : T^r_s\mathbf{R}^n \to T^p_q\mathbf{R}^n$ be a linear mapping,*

$$\bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = F^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}{}^{k_1 k_2 \cdots k_q}_{l_1 l_2 \cdots l_p} t^{l_1 l_2 \cdots l_p}_{k_1 k_2 \cdots k_q} \tag{3.7}$$

its expression relative to the canonical basis of $\mathbf{R}^n$. F is natural if and only if its components $F^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}{}^{k_1 k_2 \cdots k_q}_{l_1 l_2 \cdots l_p}$ are components of an invariant tensor.

If F is identified with a tensor, F becomes an element of the tensor space $T^{r+q}_{s+p}\mathbf{R}^n$. Thus by Lemma 3, a nontrivial natural linear mapping $F : T^r_s\mathbf{R}^n \to T^p_q\mathbf{R}^n$ exists if and only if $r + q = s + p$.

Let us discuss the case $p = r$, $q = s$. Then by Lemma 3 (b), F has an expression

$$F^{i_1 i_2 \cdots i_r \; i_{r+1} i_{r+2} \cdots i_{r+s}}_{j_1 j_2 \cdots j_s \; j_{s+1} j_{s+2} \cdots j_{s+r}} = \sum_{\sigma \in S_{r+s}} a_\sigma \delta^{i_1}_{j_{\sigma(1)}} \delta^{i_2}_{j_{\sigma(2)}} \cdots \delta^{i_r}_{j_{\sigma(r)}} \delta^{i_{r+1}}_{j_{\sigma(r+1)}} \delta^{i_{r+2}}_{j_{\sigma(r+2)}} \cdots \delta^{i_{r+s}}_{j_{\sigma(r+s)}}, \tag{3.8}$$

where $a_\sigma \in \mathbf{R}$. Clearly, the same is expressed by the equation

$$\bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = \sum_{\mu \in S_r,\, \nu \in S_s} a_\sigma t^{l_{\mu(1)} l_{\mu(2)} \cdots l_{\mu(r)}}_{k_{\nu(1)} k_{\nu(2)} \cdots k_{\nu(s)}} + \tau^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}, \tag{3.9}$$

where the summation takes place through $\sigma \in S_{r+s}$ of the form of the product of two permutations $\sigma = \mu\nu$, $\nu \in S_r$, $\mu \in S_s$ and $\tau^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ contains all the remaining terms. Note that each term in $\tau^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ contains at least as one factor the Kronecker δ-tensor multiplied by an expression obtained from $t^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s}$ by the trace operation in one superscript and one subscript.

Since $F^{i_1 i_2 \cdots i_r \; k_1 k_2 \cdots k_q}_{j_1 j_2 \cdots j_s \; l_1 l_2 \cdots l_p}$ are components of an invariant tensor, F can also be expressed by means of (3.6) as

$$F = \sum_{\sigma \in S_{r+s}} a^\sigma E_\sigma. \tag{3.10}$$

If $F, G : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$ are two natural linear operators, given in components by

$$\bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = F^{i_1 i_2 \cdots i_r \; k_1 k_2 \cdots k_s}_{j_1 j_2 \cdots j_s \; l_1 l_2 \cdots l_r} t^{l_1 l_2 \cdots l_r}_{k_1 k_2 \cdots k_s}, \quad \bar{t}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = G^{i_1 i_2 \cdots i_r \; k_1 k_2 \cdots k_s}_{j_1 j_2 \cdots j_s \; l_1 l_2 \cdots l_r} t^{l_1 l_2 \cdots l_r}_{k_1 k_2 \cdots k_s} \tag{3.11}$$

then the composed natural linear operator is given by

$$\bar{\bar{t}}^{i_1 i_2 \cdots i_r}_{j_1 j_2 \cdots j_s} = G^{i_1 i_2 \cdots i_r \; k_1 k_2 \cdots k_s}_{j_1 j_2 \cdots j_s \; l_1 l_2 \cdots l_r} F^{l_1 l_2 \cdots l_r \; p_1 p_2 \cdots p_s}_{k_1 k_2 \cdots k_s \; q_1 q_2 \cdots q_r} t^{q_1 q_2 \cdots q_r}_{p_1 p_2 \cdots p_s}. \tag{3.12}$$

To obtain an explicit formula, one should substitute from (3.8) into (3.12); indeed, this cannot be done effectively in general, but in every concrete case.

4. Natural projectors in tensor spaces

Let E be a finite-dimensional, real vector space, endowed with a left action of the general linear group $GL_n(\mathbf{R})$. By a *natural projector* on E we mean a natural linear operator $F : E \to E$ which is a projector. A natural linear operator F is a natural projector if and only if it satisfies the *projector equation* $F^2 = F$. The projector equation represents a system of quadratic equations for the components of F.

If $P : E \to E$ is a natural projector, then both vector subspaces $\operatorname{im} P$, $\ker P$ of E are $GL_n(\mathbf{R})$-invariant ([2], § 43).

A natural projector $P : E \to E$ is said to be *decomposable*, if there exist a natural projector $Q \neq 0, P$ and a natural projector R, such that $P = Q + R$. In this case Q and R are orthogonal (Lemma 2 (a)). A natural projector which is not decomposable is called *indecomposable.*

P is said to be *reducible*, if there exists a natural projector $Q \neq 0$ such that $\operatorname{im} Q \subset \operatorname{im} P$ and $\operatorname{im} Q \neq \operatorname{im} P$. If P is not reducible, it is called *irreducible*, or *primitive.*

Remark 3. Examples show that there exist reducible natural projectors which are not decomposable. Consider the family P_λ of natural linear operators in $T^1_2\mathbf{R}^n$ defined by the equations

$$\bar{t}^i_{jk} = \delta^i_k t^p_{pj} + \lambda\delta^i_k(-nt^p_{pj} + t^p_{jp}). \tag{4.1}$$

One can easily verify that (4.1) consists of natural projectors. Indeed, contracting (4.1) we obtain $\bar{t}^p_{pj} = t^p_{pj} + \lambda(-nt^p_{pj} + t^p_{jp})$, $\bar{t}^p_{jp} = nt^p_{pj} + \lambda n(-nt^p_{pj} + t^p_{jp})$, and then

$$\begin{aligned}\bar{\bar{t}}^i_{jk} &= \delta^i_k \bar{t}^p_{pj} + \lambda\delta^i_k(-n\bar{t}^p_{pj} + \bar{t}^p_{jp})\\ &= \delta^i_k(t^p_{pj} + \lambda(-nt^p_{pj} + t^p_{jp})) - \lambda n\delta^i_k(t^p_{pj} + \lambda(-nt^p_{pj} + t^p_{jp}))\\ &\quad +\lambda\delta^i_k(nt^p_{pj} + \lambda n(-nt^p_{pj} + t^p_{jp}))\\ &= \delta^i_k t^p_{pj} - \delta^i_k\lambda nt^p_{pj} + \delta^i_k\lambda t^p_{jp} = \delta^i_k t^p_{pj} + \lambda\delta^i_k(-nt^p_{pj} + t^p_{jp}) = \bar{t}^i_{jk}\end{aligned}$$

verifying the projector equations $P^2_\lambda = P_\lambda$. Note that the family (4.1) includes the natural projector $\bar{t}^i_{jk} = \delta^i_k t^p_{pj}$, and the natural projector $\bar{t}^i_{jk} = (1/n)\delta^i_k t^p_{jp}$ defined by taking $\lambda = 1/n$. The family $\lambda\delta^i_k(-nt^p_{pj} + t^p_{jp}$ in (4.1) does not consist of projectors, because λ serves as a multiplicative parameter, and two non-zero projectors cannot differ by a factor different from 1. Indeed, writing $\bar{t}^p_{qr} = \lambda\delta^p_r(-nt^s_{sq} + t^s_{qs})$, we get $\bar{t}^p_{pj} = \lambda(-nt^s_{sj} + t^s_{js}), \bar{t}^p_{jp} = \lambda n(-nt^s_{sj} + t^s_{js})$ hence $\bar{\bar{t}}^i_{jk} = \lambda\delta^i_k(-n\bar{t}^p_{pj} + \bar{t}^p_{jp}) = -ln\delta^i_k\bar{t}^p_{jp} + \lambda\delta^i_k\bar{t}^p_{jp} = -\lambda^2 n\delta^i_k(-nt^s_{sj} + t^s_{js}) + \lambda^2 n\delta^i_k(-nt^s_{sj} + t^s_{js}) = 0 \neq \bar{t}^i_{jk}$.

From now on we consider natural projectors on a tensor space $T^r_s\mathbf{R}^n$.

Theorem 1. *Let $P : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$ be a natural projector.*

(a) *P is decomposable if and only if there exists a natural projector $Q \neq 0, P$ such that*

$$PQ = Q, \quad QP = Q. \tag{4.2}$$

(b) *P is reducible if and only if there exists a natural projector $Q \neq 0, P$ such that*

$$PQ = Q, \quad \operatorname{im} Q \neq \operatorname{im} P. \tag{4.3}$$

Proof. (a) If P is decomposable, we have two natural projectors Q and R such that $R = P - Q$ and $QR = 0$, $RQ = 0$ (Lemma 2 (a)). Thus, $Q(P-Q) = (P-Q)Q = 0$, i.e., $QP = PQ = Q$. Conversely, assume that we have a natural projector Q satisfying (4.2). Define $R = P - Q$; R is a natural linear operator (Lemma 3, Lemma 4), and $R^2 = P - PQ - QP + Q = P - Q - Q + Q = P - Q = R$ as required.

(b) Let P be reducible. Then there exists a natural projector $Q \neq 0$ such that $\operatorname{im} Q \subset \operatorname{im} P$ and $\operatorname{im} Q \neq \operatorname{im} P$. Thus, to any $t \in T^r_s\mathbf{R}^n$ there exists $t' \in T^r_s\mathbf{R}^n$ such that $Qt = Pt' = P(Pt') = PQt$ hence $PQ = Q$. Conversely, assume that we have a natural projector $Q \neq 0$ satisfying (4.3). Then $\operatorname{im} Q = Q(T^r_s\mathbf{R}^n) = P(Q(T^r_s\mathbf{R}^n)) \subset P(T^r_s\mathbf{R}^n) = \operatorname{im} P$ as required.

Equations from Theorem 1 (a) for a projector Q

$$PQ = Q, \quad QP = Q, \quad Q^2 = Q \tag{4.4}$$

are equivalent with the equations

$$PQP = Q, \quad Q^2 = Q. \tag{4.5}$$

Indeed, (4.4) implies (4.5), and vice versa: $QP = PQPP = PQP = Q$, $PQ = PPQP = PQP = Q$. Each of the systems (4.4) and (4.5) is called the *decomposability equation* of P. Equation $PQ = Q$ from Theorem 1 (b) is called the *reducibility equation.*

Now we study indecomposability, and primitivity.

Theorem 2. *Let $P : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$ be a natural projector.*

(a) *P is indecomposable if and only if the decomposability equation of P has exactly one nontrivial solution, $Q = P$.*

(b) *P is primitive if and only if the reducibility equation of P has no nontrivial solution.*

Proof. Both assertions are immediate consequences of Theorem 1.

(a) If P is indecomposable, there is no $Q \neq 0, P$ such that $PQ = Q$, $QP = Q$, which means that the decomposability equations have only one nontrivial solution, $Q = P$. The converse is obvious.

(b) If P is primitive, then by definition, (4.3) has only the trivial solution, and vice versa.

Now we consider properties of primitive natural projectors.

Theorem 3.

(a) *Any two different primitive natural projectors in $T^r_s\mathbf{R}^n$ are orthogonal.*

(b) *The number of different nontrivial natural projectors in $T^r_s\mathbf{R}^n$ is finite.*

(c) *The sum of any two primitive natural projectors is a natural projector.*

(d) *Let M be the number of different nontrivial primitive natural projectors in $T^r_s\mathbf{R}^n$. If a natural projector in $T^r_s\mathbf{R}^n$ admits a decomposition $P = p_1 + p_2 + \cdots + p_K$, where $p_1, p_2, \ldots, p_K$ are primitive natural projectors, then $K \leq M$, the primitive natural projectors $p_1, p_2, \ldots, p_K$ are mutually different, and this decomposition is unique.*

(e) *The identity natural projector* $\mathrm{id} : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$ *admits the decomposition*

$$\mathrm{id} = p_1 + p_2 + \cdots + p_M \tag{4.6}$$

where $\{p_1, p_2, \ldots, p_M\}$ is the set of nonzero primitive natural projectors.

Proof. (a) If P_1, P_2 are two different primitive natural projectors, then $\mathrm{im}\, P_1P_2 = \mathrm{im}\, P_2P_1 = 0$ hence $P_1P_2 = P_2P_1 = 0$.

(b) Since $\dim T^r_s\mathbf{R}^n$ is finite, this assertion follows from (a).

(c) By (a), any two different primitive natural projectors p_1, p_2 are orthogonal. Thus, by Lemma 2 (a), p_1+p_2 is always a projector; p_1+p_2 is obviously a natural projector (Lemma 4).

(d) Assume that $P = p_1 + p_2 + \cdots + p_K = q_1 + q_2 + \cdots + q_L$. Then by orthogonality, $p_l^2 = p_l = p_l(q_1 + q_2 + \cdots + q_L)$, where at most one term on the right is nonzero. But $p_l \neq 0$ hence exactly one term on the right, say p_lq_k, is nonzero, and is equal to p_l, i.e., $p_l = p_lq_k = q_kp_l$. Since different primitive projectors are orthogonal (see (a)), we have $q_k = p_l$. In particular, the two sums $p_1 + p_2 + \cdots + p_K$, $q_1 + q_2 + \cdots + q_L$ may differ only by the order of the summation.

(e) If $P = p_1 + p_2 + \cdots + p_M \neq \mathrm{id}$, we have a nonzero natural projector $Q = \mathrm{id} - P$, which is a contradiction with maximality of the set $\{p_1, p_2, \ldots, p_M\}$.

5. The trace decomposition

For basic notions of the trace decomposition theory as used in this section, we refer to [4], [5]. The following assertion can be used when calculating the trace decomposition of concrete tensor spaces.

Theorem 4. *Let $r, s \geq 1$. There exists a unique natural linear operator $Q : T^r_s\mathbf{R}^n \to T^r_s\mathbf{R}^n$ satisfying the following two conditions:*

1. *Qt is traceless for every $t \in T^r_s\mathbf{R}^n$.*
2. *$(\mathrm{id} - Q)t = t - Qt$ is δ-generated for every $t \in T^r_s\mathbf{R}^n$.*

Q is a natural projector.

Proof. Existence and uniqueness of Q follows from the decomposition $t = Qt + (\mathrm{id} - Q)t$, and from the trace decomposition theorem. We prove that Q is a projector. By hypothesis, Qt is traceless for every $t \in T^r_s\mathbf{R}^n$, hence $Q^2t = Q(Qt)$ is also traceless for every t. Similarly, since $t - Qt$ is δ-generated for every $t \in T^r_s\mathbf{R}^n$, the formula

$$(\mathrm{id} - Q^2)t = (\mathrm{id} - Q + Q - Q^2)t = (\mathrm{id} - Q)t + (\mathrm{id} - Q)Qt \tag{5.1}$$

shows that $(\mathrm{id}-Q^2)t$ must also be δ-generated. Since $t = Q^2t+(\mathrm{id}-Q^2)t$, then by uniqueness, $Q^2 = Q$.

In a concrete case, the natural projector Q can be determined from the conditions (1) and (2) of Theorem 4. Clearly, given Q, the *trace decomposition* of a tensor $t \in T^r_s\mathbf{R}^n$ is obtained by the formula

$$t = Qt + (\mathrm{id} - Q)t. \tag{5.2}$$

6. Natural projectors in $\mathbf{R}^n \otimes \mathbf{R}^{n*} \otimes \mathbf{R}^{n*}$

As an application of the natural projector theory, we find the complete list of natural projectors in the space of tensors of type (1,2) $T^1_2\mathbf{R}^n$. Since our discussions are $GL_n(\mathbf{R})$-invariant, the results apply in the well-known sense to any real, finite-dimensional vector space E, and to the tensor space of type (1,2) over E.

First let us describe natural linear operators in $T^1_2\mathbf{R}^n$. Using the canonical basis e_i of $\mathbf{R}^n$ and the dual basis e^j of $\mathbf{R}^{n*}$, we usually express a tensor $t \in T^1_2\mathbf{R}^n$ in terms of its components as $t = t^i_{jk}e_i \otimes e^j \otimes e^k$, and we write $t = t^i_{jk}$. If $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$ is a linear operator, we write $P = P^{i}_{jk}{}^{qr}_{p}$, where $P^{i}_{jk}{}^{qr}_{p}$ are the components of P, and the indices i, j, k, p, q, r run through the set $\{1, 2, \ldots, n\}$. The equations of P are usually written in the form $\bar{t}^i_{jk} = P^{i}_{jk}{}^{qr}_{p} t^p_{qr}$. P is natural if and only if

$$P^{i}_{jk}{}^{qr}_{p} = a\delta^i_j\delta^q_k\delta^r_p + b\delta^i_j\delta^q_p\delta^r_k + c\delta^i_k\delta^q_p\delta^r_j + d\delta^i_k\delta^q_j\delta^r_p + e\delta^i_p\delta^q_j\delta^r_k + f\delta^i_p\delta^q_k\delta^r_j, \tag{6.1}$$

where a, b, c, d, e, f are some real numbers. In view of (6.1), we also write

$$P = (a, b, c, d, e, f). \tag{6.2}$$

We denote by $\mathcal{N}(T^1_2\mathbf{R}^n)$ the real vector space of natural linear operators $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$; by (6.1), $\dim \mathcal{N}(T^1_2\mathbf{R}^n) = 6$.

We find the composition law for the natural linear operators. Consider a natural linear operator (6.1), and another natural linear operator $Q = Q^{a}_{bc}{}^{qr}_{p}$, where

$$Q^{a}_{bc}{}^{qr}_{p} = a'\delta^a_b\delta^q_c\delta^r_p + b'\delta^a_b\delta^q_p\delta^r_c + c'\delta^a_c\delta^q_p\delta^r_b + d'\delta^a_c\delta^q_b\delta^r_p + e'\delta^a_p\delta^q_b\delta^r_c + f'\delta^a_p\delta^q_c\delta^r_b. \tag{6.3}$$

Lemma 6. *The composed natural linear operator* $R = PQ = R^{i}_{jk}{}^{qr}_{p}$ *is expressed by*

$$R^{i}_{jk}{}^{qr}_{p} = a''\delta^i_j\delta^q_k\delta^r_p + b''\delta^i_j\delta^q_p\delta^r_k + c''\delta^i_k\delta^q_p\delta^r_j + d''\delta^i_k\delta^q_j\delta^r_p + e''\delta^i_p\delta^q_j\delta^r_k + f''\delta^i_p\delta^q_k\delta^r_j, \tag{6.4}$$

where

$$\begin{aligned}
a'' &= a'a + nd'a + e'a + na'b + f'b + d'b + a'e + d'f,\\
b'' &= b'a + nc'a + f'a + nb'b + c'b + e'b + b'e + c'f,\\
c'' &= nb'c + c'c + e'c + b'd + nc'd + f'd + c'e + b'f,\\
d'' &= na'c + d'c + f'c + a'd + nd'd + e'd + d'e + a'f,\\
e'' &= e'e + f'f,\\
f'' &= f'e + e'f.
\end{aligned} \tag{6.5}$$

Proof. Since for any $t \in T^1_2\mathbf{R}^n$, $t = t^p_{qr}$, $Rt = \bar{\bar{t}}^i_{jk} = P^{i}_{jk}{}^{bc}_{a}\bar{t}^a_{bc} = P^{i}_{jk}{}^{bc}_{a}Q^{a}_{bc}{}^{qr}_{p}t^p_{qr} = R^{i}_{jk}{}^{qr}_{p}t^p_{qr}$, the coefficients $R^{i}_{jk}{}^{qr}_{p}$ are obtained from the formula

$$R^{i}_{jk}{}^{qr}_{p} = P^{i}_{jk}{}^{bc}_{a}Q^{a}_{bc}{}^{qr}_{p}. \tag{6.6}$$

Now we derive the equations for natural projectors in $T^1_2\mathbf{R}^n$.

Lemma 7. *A natural linear operator* $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$ *expressed by* (6.1), *is a natural projector if and only if*

$$\begin{aligned}
&a^2 + (nb + nd + 2e - 1)\,a + bd + (b + d)f = 0,\\
&nb^2 + (a + c + 2e - 1)\,b + nca + (a + c)f = 0,\\
&c^2 + (nb + nd + 2e - 1)\,c + bd + (b + d)f = 0,\\
&nd^2 + (a + c + 2e - 1)\,d + nac + (a + c)f = 0,\\
&e = e^2 + f^2,\\
&f = 2ef.
\end{aligned} \tag{6.7}$$

Proof. The components of P satisfy the projector equation $P^{i}_{jk}{}^{vw}_{u}P^{u}_{vw}{}^{qr}_{p} = P^{i}_{jk}{}^{qr}_{p}$, which can be obtained by substituting $Q = P$ and $R = P$ in (6.5).

Equations (6.7) are referred to as the *natural projector equations.* These equations represent a system of six quadratic equations for six unknowns (a, b, c, d, e, f).

Remark 4. If P is a natural projector, then the complementary projector $\mathrm{id} - P$ is also natural. Thus, if P (6.1) satisfies (6.7), then $\mathrm{id} - P$ also satisfies (6.7). Indeed,

$$\mathrm{id} - P = a'\delta^i_j\delta^q_k\delta^r_p + b'\delta^i_j\delta^q_p\delta^r_k + c'\delta^i_k\delta^q_p\delta^r_j + d'\delta^i_k\delta^q_j\delta^r_p + e'\delta^i_p\delta^q_j\delta^r_k + f'\delta^i_p\delta^q_k\delta^r_j, \tag{6.8}$$

where

$$a' = -a,\ b' = -b,\ c' = -c,\ d' = -d,\ e' = 1 - e,\ f' = -f. \tag{6.9}$$

The transformation (6.9) leaves invariant the system (6.7).

It is easily seen that the formulas

$$\begin{aligned} &a' = c,\ b' = b,\ c' = a,\ d' = d,\ e' = e,\ f' = f, \\ &a' = a,\ b' = d,\ c' = c,\ d' = b,\ e' = e,\ f' = f \end{aligned} \tag{6.10}$$

also define invariant transformations of (6.7). Consequently, if (a, b, c, d, e, f) is a natural projector, then also $(-a, -b, -c, -d, 1 - e, -f)$, (c, b, a, d, e, f), (a, d, c, b, e, f), are natural projectors.

We are now in a position to find all solutions of the natural projector equations (6.7). We write these solutions in the form of their equations $\bar{t}^i_{jk} = P^{i\ \ qr}_{jk\ p} t^p_{qr}$, i.e., as

$$\bar{t}^i_{jk} = a\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + et^i_{jk} + ft^i_{kj}. \tag{6.11}$$

Here (a, b, c, d, e, f) are the components (6.2) of a natural projector, expressed by (6.1). Note that the list (A1), (A2), ..., (D4) below includes one-, and two-parameter families of natural projectors.

We define

$$\begin{aligned} &A_1 = nd + d^2 - n^2d^2, && A_2 = -nd + d^2 - n^2d^2, \\ &A_3 = n^2d^2 - d^2 - d, && A_4 = n^2d^2 - d^2 + d, \\ &B_1 = n^2c^2 - c^2 + c, && B_2 = n^2c^2 - c^2 - c, \\ &C_1 = 4d + 4d^2 - 4n^2d^2 + 1, && C_2 = -4d + 4d^2 - 4n^2d^2 + 1. \end{aligned} \tag{6.12}$$

Theorem 5. *The following list contains all natural projectors* $P : T^1_2\mathbf{R}^n \to T^1_2\mathbf{R}^n$*:*

$$\begin{aligned} &\bar{t}^i_{jk} = 0, \\ &\bar{t}^i_{jk} = \tfrac{1}{2(n-1)}(-\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} - \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}), \\ &\bar{t}^i_{jk} = \tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}), \\ &\bar{t}^i_{jk} = -\tfrac{1}{n^2-1}\delta^i_j t^s_{ks} + \tfrac{n}{n^2-1}\delta^i_j t^s_{sk} - \tfrac{1}{n^2-1}\delta^i_k t^s_{sj} + \tfrac{n}{n^2-1}\delta^i_k t^s_{js}, \end{aligned} \tag{A1}$$

$$\begin{aligned} &\bar{t}^i_{jk} = \tfrac{-d+\sqrt{A_1}}{n}\delta^i_j t^s_{ks} + \tfrac{n+2d-n^2d-2\sqrt{A_1}}{n^2}\delta^i_j t^s_{sk} + \tfrac{-d+\sqrt{A_1}}{n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js}, \\ &\bar{t}^i_{jk} = -\tfrac{d+\sqrt{A_1}}{n}\delta^i_j t^s_{ks} + \tfrac{n+2d-n^2d+2\sqrt{A_1}}{n^2}\delta^i_j t^s_{sk} - \tfrac{d+\sqrt{A_1}}{n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js}, \\ &d \in\ [0, n/(n^2 - 1)], \end{aligned} \tag{A2}$$

$$\begin{aligned} \bar{t}^i_{jk} &= \left(1 - c - 2n\left(-nc + \sqrt{B_1}\right)\right)\delta^i_j t^s_{ks} + \left(-nc + \sqrt{B_1}\right)\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} \\ &\quad + \left(-nc + \sqrt{B_1}\right)\delta^i_k t^s_{js}, \\ \bar{t}^i_{jk} &= \left(1 - c + 2n\left(nc + \sqrt{B_1}\right)\right)\delta^i_j t^s_{ks} - \left(nc + \sqrt{B_1}\right)\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} \\ &\quad - \left(nc + \sqrt{B_1}\right)\delta^i_k t^s_{js}, \\ c &\in (-\infty, -1/(n^2 - 1)] \cup [0, \infty), \end{aligned} \tag{A3}$$

$$\begin{aligned} &\bar{t}^i_{jk} = (1 - nb)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk}, \\ &\bar{t}^i_{jk} = \tfrac{d-nd^2-cd}{d+nc}\delta^i_j t^s_{ks} + \tfrac{c-c^2-ncd}{d+nc}\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js}, \quad d + nc \neq 0, \\ &\bar{t}^i_{jk} = -nb\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} - \tfrac{1}{n^2-1}\delta^i_k t^s_{sj} + \tfrac{n}{n^2-1}\delta^i_k t^s_{js}, \end{aligned} \tag{A4}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = t^i_{jk},\\
&\bar{t}^i_{jk} = \tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} - \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} - \delta^i_k t^s_{js}) + t^i_{jk},\\
&\bar{t}^i_{jk} = -\tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}) + t^i_{jk},\\
&\bar{t}^i_{jk} = \tfrac{1}{n^2-1}(\delta^i_j t^s_{ks} - n\delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} - n\delta^i_k t^s_{js}) + t^i_{jk},
\end{aligned} \tag{B1}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = \tfrac{-d+\sqrt{A_2}}{n}\delta^i_j t^s_{ks} + \tfrac{-n+2d-n^2d-2\sqrt{A_2}}{n^2}\delta^i_j t^s_{sk} + \tfrac{-d+\sqrt{A_2}}{n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + t^i_{jk},\\
&\bar{t}^i_{jk} = -\tfrac{d+\sqrt{A_2}}{n}\delta^i_j t^s_{ks} + \tfrac{-n+2d-n^2d+2\sqrt{A_2}}{n^2}\delta^i_j t^s_{sk} - \tfrac{d+\sqrt{A_2}}{n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + t^i_{jk},\\
&d \in\ [-n/(n^2-1), 0],
\end{aligned} \tag{B2}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = \left(1 - c - 2n\left(-nc + \sqrt{B_2}\right)\right)\delta^i_j t^s_{ks} + \left(-nc + \sqrt{B_2}\right)\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj}\\
&\qquad + \left(-nc + \sqrt{B_2}\right)\delta^i_k t^s_{js} + t^i_{jk},\\
&\bar{t}^i_{jk} = \left(1 - c - 2n\left(-nc - \sqrt{B_2}\right)\right)\delta^i_j t^s_{ks} - \left(nc + \sqrt{B_2}\right)\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj}\\
&\qquad - \left(nc + \sqrt{B_2}\right)\delta^i_k t^s_{js} + t^i_{jk},\\
&c \in (-\infty, 0] \cup [1/(n^2-1), \infty),
\end{aligned} \tag{B3}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = (1-nb)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} + t^i_{jk},\\
&\bar{t}^i_{jk} = -\tfrac{d+nd^2+cd}{d+nc}\delta^i_j t^s_{ks} - \tfrac{c+c^2+ncd}{d+nc}\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + t^i_{jk}, \quad d + nc \neq 0,\\
&\bar{t}^i_{jk} = -nb\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} + \tfrac{1}{n^2-1}\delta^i_k t^s_{sj} - \tfrac{n}{n^2-1}\delta^i_k t^s_{js} + t^i_{jk},
\end{aligned} \tag{B4}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = \tfrac{1}{2(n-1)}(-\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} - \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}) + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = -\tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}) + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = -\tfrac{n}{n^2-1}\delta^i_j t^s_{ks} + \tfrac{1}{n^2-1}\delta^i_j t^s_{sk} - \tfrac{n}{n^2-1}\delta^i_k t^s_{sj} + \tfrac{1}{n^2-1}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},
\end{aligned} \tag{C1}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = -\tfrac{1+2d+\sqrt{C_1}}{2n}\delta^i_j t^s_{ks} + \tfrac{2d-n^2d+1+\sqrt{C_1}}{n^2}\delta^i_j t^s_{sk} - \tfrac{1+2d+\sqrt{C_1}}{2n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + et^i_{jk} + ft^i_{kj},\\
&\bar{t}^i_{jk} = \tfrac{-1-2d+\sqrt{C_1}}{2n}\delta^i_j t^s_{ks} + \tfrac{2d-n^2d+1-\sqrt{C_1}}{n^2}\delta^i_j t^s_{sk} + \tfrac{-1-2d+\sqrt{C_1}}{2n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&d \in\ [-\tfrac{1}{2}(n+1), \tfrac{1}{2}(n+1)],
\end{aligned} \tag{C2}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = -\left(nd + \sqrt{A_3}\right)\delta^i_j t^s_{ks} + d\delta^i_j t^s_{sk} + (-nd + \sqrt{A_3})\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = (-nd + \sqrt{A_3})\delta^i_j t^s_{ks} + d\delta^i_j t^s_{sk} - \left(nd + \sqrt{A_3}\right)\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&d \in (-\infty, 0] \cup [1/(n^2-1), \infty),
\end{aligned} \tag{C3}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = -\left(\tfrac{1}{2} + nb\right)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} - \tfrac{1}{2(n-1)}\delta^i_k t^s_{sj} + \tfrac{1}{2(n-1)}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = -\tfrac{c+2nd^2+2cd}{2d+2nc+1}\delta^i_j t^s_{ks} - \tfrac{d+2ncd+2c^2}{2d+2nc+1}\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},\\
&\qquad 2d + 2nc + 1 \neq 0,\\
&\bar{t}^i_{jk} = \left(\tfrac{1}{2} - nb\right)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} - \tfrac{1}{2(n-1)}\delta^i_k t^s_{sj} - \tfrac{1}{2(n-1)}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} + \tfrac{1}{2}t^i_{kj},
\end{aligned} \tag{C4}$$

$$\begin{aligned}
\bar{t}^i_{jk} &= \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
\bar{t}^i_{jk} &= \tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} - \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} - \delta^i_k t^s_{js}) + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
\bar{t}^i_{jk} &= \tfrac{1}{2(n-1)}(\delta^i_j t^s_{ks} + \delta^i_j t^s_{sk} + \delta^i_k t^s_{sj} + \delta^i_k t^s_{js}) + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
\bar{t}^i_{jk} &= \tfrac{n}{n^2-1}\delta^i_j t^s_{ks} - \tfrac{1}{n^2-1}\delta^i_j t^s_{sk} + \tfrac{n}{n^2-1}\delta^i_k t^s_{sj} - \tfrac{1}{n^2-1}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},
\end{aligned} \tag{D1}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = \tfrac{1-2d-\sqrt{C_2}}{2n}\delta^i_j t^s_{ks} + \tfrac{2d-n^2d-1+\sqrt{C_2}}{n^2}\delta^i_j t^s_{sk} + \tfrac{1-2d-\sqrt{C_2}}{2n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = \tfrac{1-2d+\sqrt{C_2}}{2n}\delta^i_j t^s_{ks} + \tfrac{2d-n^2d-1-\sqrt{C_2}}{n^2}\delta^i_j t^s_{sk} + \tfrac{1-2d+\sqrt{C_2}}{2n}\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
&d \in [-1/(n^2-1), 1/(n^2-1)],
\end{aligned} \tag{D2}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = -\left(nd + \sqrt{A_4}\right)\delta^i_j t^s_{ks} + d\delta^i_j t^s_{sk} + \left(-nd + \sqrt{A_4}\right)\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = \left(-nd + \sqrt{A_4}\right)\delta^i_j t^s_{ks} + d\delta^i_j t^s_{sk} - \left(nd + \sqrt{A_4}\right)\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
&d \in (-\infty, -1/(n^2-1)] \cup [0,\infty),
\end{aligned} \tag{D3}$$

$$\begin{aligned}
&\bar{t}^i_{jk} = -\left(\tfrac{1}{2} + nb\right)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} + \tfrac{1}{2(n-1)}\delta^i_k t^s_{sj} + \tfrac{1}{2(n-1)}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}t^i_{kj},\\
&\bar{t}^i_{jk} = \tfrac{c-2nd^2-2cd}{2d+2nc-1}\delta^i_j t^s_{ks} + \tfrac{d-2ncd-2c^2}{2d+2nc-1}\delta^i_j t^s_{sk} + c\delta^i_k t^s_{sj} + d\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}ft^i_{kj},\\
&\qquad 2d + 2nc - 1 \neq 0,\\
&\bar{t}^i_{jk} = \left(\tfrac{1}{2} - nb\right)\delta^i_j t^s_{ks} + b\delta^i_j t^s_{sk} + \tfrac{1}{2(n-1)}\delta^i_k t^s_{sj} - \tfrac{1}{2(n-1)}\delta^i_k t^s_{js} + \tfrac{1}{2}t^i_{jk} - \tfrac{1}{2}ft^i_{kj}.
\end{aligned} \tag{D4}$$

Proof. (6.7) splits into the following 16 cases to be considered separately:

$$(e,f) = (0,0),\ a = c,\ b = d, \tag{A1}$$

$$(e,f) = (0,0),\ a = c,\ b = -d - \frac{1}{n}(a + c + 2e - 1), \tag{A2}$$

$$(e,f) = (0,0),\ a = -c - nb - nd - 2e + 1,\ b = d, \tag{A3}$$

$$(e,f) = (0,0),\ a = -c - nb - nd - 2e + 1, b = -d - \frac{1}{n}(a + c + 2e - 1), \tag{A4}$$

$$(e,f) = (1,0),\ a = c,\ b = d, \tag{B1}$$

$$(e,f) = (1,0),\ a = c,\ b = -d - \frac{1}{n}(a + c + 2e - 1), \tag{B2}$$

$$(e,f) = (1,0),\ a = -c - nb - nd - 2e + 1,\ b = d, \tag{B3}$$

$$(e,f) = (1,0),\ a = -c - nb - nd - 2e + 1,\ b = -d - \frac{1}{n}(a + c + 2e - 1), \tag{B4}$$

$$(e,f) = \left(\frac{1}{2}, \frac{1}{2}\right),\ a = c,\ b = d, \tag{C1}$$

$$(e,f) = \left(\frac{1}{2}, \frac{1}{2}\right),\ a = c,\ b = -d - \frac{1}{n}(a + c + 2e - 1), \tag{C2}$$

$$(e,f) = \left(\frac{1}{2}, \frac{1}{2}\right),\ a = -c - nb - nd - 2e + 1,\ b = d, \tag{C3}$$

$$(e,f)=\left(\frac{1}{2},\frac{1}{2}\right),\ a=-c-nb-nd-2e+1,\ b=\ -d-\frac{1}{n}(a+c+2e-1) \tag{C4}$$

$$(e,f)=\left(\frac{1}{2},-\frac{1}{2}\right),\ a=c,\ b=d, \tag{D1}$$

$$(e,f)=\left(\frac{1}{2},-\frac{1}{2}\right),\ a=c,\ b=-d-\frac{1}{n}(a+c+2e-1), \tag{D2}$$

$$(e,f)=\left(\frac{1}{2},-\frac{1}{2}\right),\ a=-c-nb-nd-2e+1,\ b=d, \tag{D3}$$

$$(e,f)=\left(\frac{1}{2},-\frac{1}{2}\right),\ a=-c-nb-nd-2e+1,\ b=\ -d-\frac{1}{n}(a+c+2e-1). \tag{D4}$$

Each of these cases is subject to the conditions

$$\begin{aligned} &a^2+c^2+(nb+nd+2e-1)(a+c)+2bd+2(b+d)f=0,\\ &nb^2+nd^2+(a+c+2e-1)(b+d)+2nac+2(a+c)f=0. \end{aligned} \tag{6.13}$$

To complete the proof, we solve the system (6.13) of two quadratic equations for every of the possibilities (A1), (A2), …, (D4). We get, using *MAPLE*,

$$\begin{aligned} &(0,0,0,0,0,0),\\ &\left(-\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},0,0\right),\\ &\left(\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},0,0\right),\\ &\left(-\tfrac{1}{n^2-1},\tfrac{n}{n^2-1},-\tfrac{1}{n^2-1},\tfrac{n}{n^2-1},0,0\right), \end{aligned} \tag{A1}$$

$$\begin{aligned} &\left(\tfrac{-d+\sqrt{A_1}}{n},\tfrac{n+2d-n^2d-2\sqrt{A_1}}{n^2},\tfrac{-d+\sqrt{A_1}}{n},d,0,0\right),\\ &\left(-\tfrac{d+\sqrt{A_1}}{n},\tfrac{n+2d-n^2d+2\sqrt{A_1}}{n^2},-\tfrac{d+\sqrt{A_1}}{n},d,0,0\right),\\ &d\in\ [0,n/(n^2-1)], \end{aligned} \tag{A2}$$

$$\begin{aligned} &\left(1-c-2n\left(-nc+\sqrt{B_1}\right),-nc+\sqrt{B_1},c,-nc+\sqrt{B_1},0,0\right),\\ &\left(1-c+2n\left(nc+\sqrt{B_1}\right),-\left(nc+\sqrt{B_1}\right),c,-\left(nc+\sqrt{B_1}\right),0,0\right),\\ &c\in\ (-\infty,-1/(n^2-1)]\cup[0,\infty), \end{aligned} \tag{A3}$$

$$\begin{aligned} &(1-nb,b,0,0,0,0),\\ &\left(\tfrac{d-nd^2-cd}{d+nc},\tfrac{c-c^2-ncd}{d+nc},c,d,0,0\right),\quad d+nc\neq 0,\\ &\left(-nb,b,-\tfrac{1}{n^2-1},\tfrac{n}{n^2-1},0,0\right), \end{aligned} \tag{A4}$$

$$\begin{aligned} &(0,0,0,0,1,0),\\ &\left(\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},1,0\right),\\ &\left(-\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},-\tfrac{1}{2(n-1)},1,0\right),\\ &\left(\tfrac{1}{n^2-1},-\tfrac{n}{n^2-1},\tfrac{1}{n^2-1},-\tfrac{n}{n^2-1},1,0\right), \end{aligned} \tag{B1}$$

$$\begin{aligned}&\left(\tfrac{-d+\sqrt{A_2}}{n}, \tfrac{-n+2d-n^2d-2\sqrt{A_2}}{n^2}, \tfrac{-d+\sqrt{A_2}}{n}, d, 0, 0\right),\\ &\left(-\tfrac{d+\sqrt{A_2}}{n}, \tfrac{-n+2d-n^2d+2\sqrt{A_2}}{n^2}, -\tfrac{d+\sqrt{A_2}}{n}, d, 0, 0\right),\\ &d \in \ [-n/(n^2-1), 0],\end{aligned} \tag{B2}$$

$$\begin{aligned}&\left(1-c-2n\left(-nc+\sqrt{B_2}\right), -nc+\sqrt{B_2}, c, -nc+\sqrt{B_2}, 1, 0\right),\\ &\left(1-c-2n\left(-nc-\sqrt{B_2}\right), -nc-\sqrt{B_2}, c, -nc-\sqrt{B_2}, 1, 0\right),\\ &c \in \ (-\infty, 0] \cup [1/(n^2-1), \infty),\end{aligned} \tag{B3}$$

$$\begin{aligned}&(1-nb, b, 0, 0, 1, 0),\\ &\left(-\tfrac{d+nd^2+cd}{d+nc}, -\tfrac{c+c^2+ncd}{d+nc}, c, d, 1, 0\right), \quad d+nc \neq 0,\\ &\left(-nb, b, \tfrac{1}{n^2-1}, -\tfrac{n}{n^2-1}, 1, 0\right),\end{aligned} \tag{B4}$$

$$\begin{aligned}&\left(0, 0, 0, 0, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(-\tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(-\tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(-\tfrac{n}{n^2-1}, \tfrac{1}{n^2-1}, -\tfrac{n}{n^2-1}, \tfrac{1}{n^2-1}, \tfrac{1}{2}, \tfrac{1}{2}\right),\end{aligned} \tag{C1}$$

$$\begin{aligned}&\left(-\tfrac{1+2d+\sqrt{C_1}}{2n}, \tfrac{2d-n^2d+1+\sqrt{C_1}}{n^2}, -\tfrac{1+2d+\sqrt{C_1}}{2n}, d, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(\tfrac{-1-2d+\sqrt{C_1}}{2n}, \tfrac{2d-n^2d+1-\sqrt{C_1}}{n^2}, \tfrac{-1-2d+\sqrt{C_1}}{2n}, d, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &d \in \ [-\tfrac{1}{2}(n+1), \tfrac{1}{2}(n+1)],\end{aligned} \tag{C2}$$

$$\begin{aligned}&\left(-nd-\sqrt{A_3}, d, -nd+\sqrt{A_3}, d, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(-nd+\sqrt{A_3}, d, -nd-\sqrt{A_3}, d, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &d \in (-\infty, 0] \cup [1/(n^2-1), \infty),\end{aligned} \tag{C3}$$

$$\begin{aligned}&\left(-\tfrac{1}{2}-nb, b, -\tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2}, \tfrac{1}{2}\right),\\ &\left(-\tfrac{c+2nd^2+2cd}{2d+2nc+1}, -\tfrac{d+2ncd+2c^2}{2d+2nc+1}, c, d, \tfrac{1}{2}, \tfrac{1}{2}\right), \quad 2d+2nc+1 \neq 0,\\ &\left(\tfrac{1}{2}-nb, b, -\tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2}, \tfrac{1}{2}\right),\end{aligned} \tag{C4}$$

$$\begin{aligned}&\left(0, 0, 0, 0, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\ &\left(\tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\ &\left(\tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\ &\left(\tfrac{n}{n^2-1}, -\tfrac{1}{n^2-1}, \tfrac{n}{n^2-1}, -\tfrac{1}{n^2-1}, \tfrac{1}{2}, -\tfrac{1}{2}\right),\end{aligned} \tag{D1}$$

$$\begin{aligned}&\left(\tfrac{1-2d-\sqrt{C_2}}{2n}, \tfrac{2d-n^2d-1+\sqrt{C_2}}{n^2}, \tfrac{1-2d-\sqrt{C_2}}{2n}, d, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\ &\left(\tfrac{1-2d+\sqrt{C_2}}{2n}, \tfrac{2d-n^2d-1-\sqrt{C_2}}{n^2}, \tfrac{1-2d+\sqrt{C_2}}{2n}, d, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\ &d \in \ [-1/(n^2-1), 1/(n^2-1)],\end{aligned} \tag{D2}$$

$$\begin{aligned}&\left(-nd-\sqrt{A_4}, d, -nd+\sqrt{A_4}, d, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\&\left(-nd+\sqrt{A_4}, d, -nd-\sqrt{A_4}, d, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\&d\in(-\infty,-1/(n^2-1)]\cup[0,\infty),\end{aligned} \tag{D3}$$

$$\begin{aligned}&\left(-\tfrac{1}{2}-nb, b, \tfrac{1}{2(n-1)}, \tfrac{1}{2(n-1)}, \tfrac{1}{2}, -\tfrac{1}{2}\right),\\&\left(\tfrac{c-2nd^2-2cd}{2d+2nc-1}, \tfrac{d-2ncd-2c^2}{2d+2nc-1}, c, d, \tfrac{1}{2}, -\tfrac{1}{2}\right),\ 2d+2nc-1\neq 0,\\&\left(\tfrac{1}{2}-nb, b, \tfrac{1}{2(n-1)}, -\tfrac{1}{2(n-1)}, \tfrac{1}{2}, -\tfrac{1}{2}\right).\end{aligned} \tag{D4}$$

Now our assertion follows from (6.11).

Remark 5. Note that Theorem 5 gives us a complete answer to the problem of finding all natural projectors in $T^1_2\mathbf{R}^n$. Properties of these natural projectors can be obtained from this list (A1), (A2), ..., (D4) by a direct analysis.

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