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Homogeneous Lorentz Manifolds with Simple Isometry Group

Dave Witte

Department of Mathematics, Oklahoma State University
Stillwater, OK 74078
e-mail: dwitte@math.okstate.edu

Abstract. Let H be a closed, noncompact subgroup of a simple Lie group G, such that G/H admits an invariant Lorentz metric. We show that if $G = \mathrm{SO}(2,n)$, with $n \geq 3$, then the identity component H° of H is conjugate to $\mathrm{SO}(1,n)^\circ$. Also, if $G = \mathrm{SO}(1,n)$, with $n \geq 3$, then H° is conjugate to $\mathrm{SO}(1,n-1)^\circ$.

1. Introduction

Definition 1.1.

- *A Minkowski form on a real vector space V is a nondegenerate quadratic form that is isometric to the form $-x_1^2 + x_2^2 + \cdots + x_{n+1}^2$ on $\mathbb{R}^{n+1}$, where $\dim V = n+1 \geq 2$.*
- *A Lorentz metric on a smooth manifold M is a choice of Minkowski metric on the tangent space T_pM, for each $p \in M$, such that the form varies smoothly as p varies.*

A. Zeghib [14] classified the compact homogeneous spaces that admit an invariant Lorentz metric. In this note, we remove the assumption of compactness, but add the restriction that the transitive group G is almost simple. Our starting point is a special case of a theorem of N. Kowalsky.

Theorem 1.2. (N. Kowalsky, cf. [11, Thm. 5.1]) *Let G/H be a nontrivial homogeneous space of a connected, almost simple Lie group G with finite center. If there is a G-invariant Lorentz metric on G/H, then either*

1) *there is also a G-invariant Riemannian metric on G/H; or*

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2) *G is locally isomorphic to either* $\mathrm{SO}(1,n)$ *or* $\mathrm{SO}(2,n)$*, for some* n.

As explained in the following elementary proposition, it is easy to characterize the homogeneous spaces that arise in Conclusion (1) of Theorem 1.2, although it is probably not reasonable to expect a complete classification.

Notation 1.3. We use $\mathfrak{g}$ to denote the Lie algebra of a Lie group G, and $\mathfrak{h} \subset \mathfrak{g}$ to denote the Lie algebra of a Lie subgroup H of G.

Proposition 1.4. (cf. [11, Thm. 1.1]) *Let* G/H *be a homogeneous space of a Lie group* G*, such that* $\mathfrak{g}$ *is simple and* $\dim G/H \geq 2$*. The following are equivalent.*

1) *The homogeneous space* G/H *admits both a* G*-invariant Riemannian metric and a* G*-invariant Lorentz metric.*
2) *The closure of* $\mathrm{Ad}_G H$ *is compact, and leaves invariant a one-dimensional subspace of* $\mathfrak{g}$ *that is not contained in* $\mathfrak{h}$.

The two main results of this note examine the cases that arise in Conclusion (2) of Theorem 1.2. It is well known [10, Egs. 2 and 3] that $\mathrm{SO}(1,n)^\circ/\mathrm{SO}(1,n-1)^\circ$ and $\mathrm{SO}(2,n)^\circ/\mathrm{SO}(1,n)^\circ$ have invariant Lorentz metrics. Also, for any discrete subgroup Γ of $\mathrm{SO}(1,2)$, the Killing form provides an invariant Lorentz metric on $\mathrm{SO}(1,2)^\circ/\Gamma$. We show that these are essentially the only examples.

Note that $\mathrm{SO}(1,1)$ and $\mathrm{SO}(2,2)$ fail to be almost simple. Thus, in 1.2(2), we may assume

- G is locally isomorphic to $\mathrm{SO}(1,n)$, and $n \geq 2$; or
- G is locally isomorphic to $\mathrm{SO}(2,n)$, and $n \geq 3$.

Proposition 2.4′. *Let* G *be a Lie group that is locally isomorphic to* $\mathrm{SO}(1,n)$*, with* $n \geq 2$*. If* H *is a closed subgroup of* G*, such that*

- *the closure of* $\mathrm{Ad}_G H$ *is not compact, and*
- *there is a* G*-invariant Lorentz metric on* G/H*,*

then either

1) *after any identification of* $\mathfrak{g}$ *with* $\mathfrak{so}(1,n)$*, the subalgebra* $\mathfrak{h}$ *is conjugate to a standard copy of* $\mathfrak{so}(1,n-1)$ *in* $\mathfrak{so}(1,n)$*, or*
2) $n=2$ *and* H *is discrete.*

Theorem 3.5′. *Let* G *be a Lie group that is locally isomorphic to* $\mathrm{SO}(2,n)$*, with* $n \geq 3$*. If* H *is a closed subgroup of* G*, such that*

- *the closure of* $\mathrm{Ad}_G H$ *is not compact, and*
- *there is a* G*-invariant Lorentz metric on* G/H*,*

then, after any identification of $\mathfrak{g}$ *with* $\mathfrak{so}(2,n)$*, the subalgebra* $\mathfrak{h}$ *is conjugate to a standard copy of* $\mathfrak{so}(1,n)$ *in* $\mathfrak{so}(2,n)$.

N. Kowalsky announced a much more general result than Theorem 3.5′ in [10, Thm. 4], but it seems that she did not publish a proof before her premature death. She announced a version of Proposition 2.4′ (with much more general hypotheses and a somewhat weaker conclusion) in [10, Thm. 3], and a proof appears in her Ph.D. thesis [9, Cor. 6.2].

Remark 1.5. It is easy to see that there is a G-invariant Lorentz metric on G/H if and only if there is an $(\mathrm{Ad}_G H)$-invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$. Thus, although Proposition 2.4′ and Theorem 3.5′ are geometric in nature, they can be restated in more algebraic terms. It is in such a form that they are proved in §2 and §3.

Proposition 2.4′ and Theorem 3.5′ are used in work of S. Adams [3] on nontame actions on Lorentz manifolds. See [16, 11, 4, 15, 1, 2] for some other research concerning actions of Lie groups on Lorentz manifolds.

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2. Homogeneous spaces of $\mathrm{SO}(1, n)$

The following lemma is elementary.

Lemma 2.1. *Let π be the standard representation of $\mathfrak{g} = \mathfrak{so}(1,k)$ on $\mathbb{R}^{k+1}$, and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ be an Iwasawa decomposition of $\mathfrak{g}$.*

1) *The representation π has only one positive weight (with respect to $\mathfrak{a}$), and the corresponding weight space is 1-dimensional.*
2) *There are subspaces V and W of $\mathbb{R}^{k+1}$, such that*
   (a) $\dim(\mathbb{R}^{k+1}/V) = 1$;
   (b) $\dim W = 1$;
   (c) $\pi(\mathfrak{n})V \subset W$;
   (d) *for all nonzero $u \in \mathfrak{n}$, we have $\pi(u)^2\mathbb{R}^{k+1} = W$; and*
   (e) *for all nonzero $u \in \mathfrak{n}$ and $v \in \mathbb{R}^{k+1}$, we have $\pi(u)^2 v = 0$ if and only if $v \in V$.*

Corollary 2.2. *Let $\mathfrak{h}$ be a subalgebra of a real Lie algebra $\mathfrak{g}$, let Q be a Minkowski form on $\mathfrak{g}/\mathfrak{h}$, and define $\pi\colon N_G(\mathfrak{h}) \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h})$ by $\pi(g)(v + \mathfrak{h}) = (\mathrm{Ad}_G\, g)v + \mathfrak{h}$.*

1) *Suppose T is a connected Lie subgroup of G that normalizes H, such that $\pi(T) \subset \mathrm{SO}(Q)$ and $\mathrm{Ad}_G T$ is diagonalizable over $\mathbb{R}$. Then, for any ordering of the T-weights on $\mathfrak{g}$, the subalgebra $\mathfrak{h}$ contains codimension-one subspaces of both $\mathfrak{g}^+$ and $\mathfrak{g}^-$, where $\mathfrak{g}^+$ is the sum of all the positive weight spaces of T, and $\mathfrak{g}^-$ is the sum of all the negative weight spaces of T.*
2) *If U is a connected Lie subgroup of G that normalizes H, such that $\pi(U) \subset \mathrm{SO}(Q)$ and $\mathrm{Ad}_G U$ is unipotent, then there are subspaces $V/\mathfrak{h}$ and $W/\mathfrak{h}$ of $\mathfrak{g}/\mathfrak{h}$, such that*
   (a) $\dim(\mathfrak{g}/V) = 1$;
   (b) $\dim(W/\mathfrak{h}) = 1$;
   (c) $[V, \mathfrak{u}] \subset W$;

(d) *for each* $u \in \mathfrak{u}$, *either* $W = \mathfrak{h} + (\mathrm{ad}_{\mathfrak{g}} u)^2 \mathfrak{g}$, *or* $[\mathfrak{g}, u] \subset \mathfrak{h}$; *and*

(e) *for all* $u \in \mathfrak{u}$, *we have* $(\mathrm{ad}_{\mathfrak{g}} u)^2 V \subset \mathfrak{h}$.

For ease of reference, let us record the following well known fact from the theory of real algebraic groups.

Lemma 2.3. *Let* $\overline{H}$ *be a Zariski closed, noncompact subgroup of* $\mathrm{GL}(m, \mathbb{R})$, *for some* m. *If* $\overline{H}$ *does not contain any nontrivial hyperbolic elements, then there exist a compact subgroup* M *and a nontrivial unipotent subgroup* U, *such that* $\overline{H} = M \ltimes U$.

Proof. The algebraic Levi decomposition [13, Thm. 6.4, p. 286], [7, Prop. 8.4.2, p. 117] provides Zariski closed subgroups M and U of $\overline{H}$, such that

- $\overline{H} = M \ltimes U$;
- M is reductive; and
- U is unipotent.

Because M is reductive and, being a subgroup of $\overline{H}$, does not contain hyperbolic elements, we know that M is compact [5, Cor. 9.4, p. 127]. However, $M \ltimes U = \overline{H}$ is not compact, so this implies that U cannot be compact; hence, U is nontrivial. □

Proposition 2.4. *Let* H *be a Lie subgroup of* $G = \mathrm{SO}(1, n)$, *with* $n \geq 2$, *such that*

- *the closure of* H *is not compact; and*
- *there is an* $(\mathrm{Ad}_G H)$*-invariant Minkowski form on* $\mathfrak{g}/\mathfrak{h}$.

Then either

1) H° *is conjugate to a standard copy of* $\mathrm{SO}(1, n-1)^\circ$ *in* $\mathrm{SO}(1, n)$, *or*
2) $n = 2$ *and* H° *is trivial.*

Proof. Let $\overline{H}$ be the Zariski closure of H, and note that the Minkowski form is also invariant under $\mathrm{Ad}_G \overline{H}$. Replacing H by a finite-index subgroup, we may assume $\overline{H}$ is Zariski connected.

Let $G = KAN$ be an Iwasawa decomposition of G.

Case 1. Assume $n \geq 3$ and $A \subset \overline{H}$. From Corollary 2.2(1) , we see that $\mathfrak{h}$ contains codimension-one subspaces of both $\mathfrak{n}$ and $\mathfrak{n}^-$. (Note that this implies H° is nontrivial.) This implies that $\overline{H}$ is reductive. (Because $(H \cap N)^\circ \,\mathrm{unip}\, \overline{H}$ is a unipotent subgroup that intersects N nontrivially (and $\mathbb{R}\text{-rank}\, G = 1$), it must be contained in N, so $\mathrm{unip}\, \overline{H} \subset N$. Similarly, $\mathrm{unip}\, \overline{H} \subset N^-$. Therefore $\mathrm{unip}\, \overline{H} \subset N \cap N^- = e$.) Then, since $\overline{H}$ contains a codimension-one subgroup of N, and since $A \subset \overline{H}$, it follows that $\overline{H}$ is conjugate to either $\mathrm{SO}(1, n-1)$ or $\mathrm{SO}(1, n)$. Because H° is a nontrivial, connected, normal subgroup of $\overline{H}$, we conclude that H° is conjugate to either $\mathrm{SO}(1, n-1)^\circ$ or $\mathrm{SO}(1, n)^\circ$. Because $\mathfrak{g}/\mathfrak{h} \neq 0$ (else $\dim \mathfrak{g}/\mathfrak{h} = 0 < 2$, which contradicts the fact that there is a Minkowski form on $\mathfrak{g}/\mathfrak{h}$), we see that H° is conjugate to $\mathrm{SO}(1, n-1)^\circ$.

Case 2. Assume $n \geq 3$ and $\overline{H}$ does not contain any nontrivial hyperbolic elements. From Lemma 2.3, we know there exist a compact subgroup M and a nontrivial unipotent subgroup U, such that $\overline{H} = M \ltimes U$. Replacing H by a conjugate, we may assume, without loss of generality, that $U \subset N$.

Let us show, for every nonzero $u \in \mathfrak{u}$, that $[\mathfrak{g}, u] \not\subset \mathfrak{h}$. From the Morosov Lemma [8, Thm. 17(1), p. 100], we know there exists $v \in \mathfrak{g}$, such that $[v, u]$ is hyperbolic (and nonzero). If $[v, u] \in \mathfrak{h}$, this contradicts the fact that $\overline{H}$ does not contain nontrivial hyperbolic elements.

Let $V/\mathfrak{h}$ and $W/\mathfrak{h}$ be subspaces of $\mathfrak{g}/\mathfrak{h}$ as in Corollary 2.2(2). Because $(\operatorname{ad}_{\mathfrak{g}} u)^2 \mathfrak{g} = \mathfrak{n}$ for every nonzero $u \in \mathfrak{n}$, we have $W = \mathfrak{n} + \mathfrak{h}$ (see 2.2(2d)), so $\dim \mathfrak{n}/(\mathfrak{h} \cap \mathfrak{n}) = 1$ (see 2.2(2b)) and

$$[\mathfrak{u}, V] \subset W = \mathfrak{n} + \mathfrak{h} \subset \mathfrak{n} + \overline{\mathfrak{h}} = \mathfrak{n} + \mathfrak{m} \tag{2.5}$$

(see 2.2(2c)).

Assume, for the moment, that $n \geq 4$. Then

$$\begin{aligned} \dim \mathfrak{u} + \dim(V \cap \mathfrak{n}^-) &\geq \dim(\mathfrak{h} \cap \mathfrak{n}) + \dim(V \cap \mathfrak{n}^-) \\ &\geq (\dim \mathfrak{n} - 1) + (\dim \mathfrak{n}^- - 1) \\ &= (n-2) + (n-2) \\ &\geq n \\ &> \dim \mathfrak{n}. \end{aligned}$$

This implies that there exist $u \in \mathfrak{u}$ and $v \in V \cap \mathfrak{n}^-$, such that $\langle u, v \rangle \cong \mathfrak{sl}(2, \mathbb{R})$, with $[u, v]$ hyperbolic (and nonzero). This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ has no nontrivial hyperbolic elements.

We may now assume that $n = 3$. For any nonzero $u \in \mathfrak{n}$, we have

$$\dim[u, V] \geq \dim[u, \mathfrak{g}] - 1 = \dim \mathfrak{n} + 1 > \dim \mathfrak{n},$$

so $[\mathfrak{u}, V] \not\subset \mathfrak{n}$. Then, from (2.5), we conclude that $\mathfrak{m} \neq 0$, so $\mathfrak{m}$ acts irreducibly on $\mathfrak{n}$. This contradicts the fact that $\mathfrak{h} \cap \mathfrak{n}$ is a codimension-one subspace of $\mathfrak{n}$ that is normalized by $\mathfrak{m}$.

Case 3. Assume $n = 2$. We may assume H° is nontrivial (otherwise Conclusion (2) holds). We must have $\dim \mathfrak{g}/\mathfrak{h} \geq 2$, so we conclude that $\dim H^\circ = 1$ and $\dim \mathfrak{g}/\mathfrak{h} = 2$. Because $\mathrm{SO}(1,1)$ consists of hyperbolic elements, this implies that $\operatorname{Ad}_G h$ acts diagonalizably on $\mathfrak{g}/\mathfrak{h}$, for every $h \in H$. Therefore H° is conjugate to A, and, hence, to $\mathrm{SO}(1,1)^\circ$. □

3. Homogeneous spaces of $\mathrm{SO}(2, n)$

Theorem 3.1. (Borel-Tits [6, Prop. 3.1]) *Let H be an F-subgroup of a reductive algebraic group G over a field F of characteristic zero. Then there is a parabolic F-subgroup P of G, such that* $\operatorname{unip} H \subset \operatorname{unip} P$ *and* $H \subset N_G(\operatorname{unip} H) \subset P$.

Notation 3.2. Let $k = \lfloor n/2 \rfloor$. Identifying $\mathbb{C}^{k+1}$ with $\mathbb{R}^{2k+2}$ yields an embedding of $\mathrm{SU}(1,k)$ in $\mathrm{SO}(2, 2k)$. Then the inclusion $\mathbb{R}^{2k+2} \hookrightarrow \mathbb{R}^{2k+3}$ yields an embedding of $\mathrm{SU}(1,k)$ in $\mathrm{SO}(2, 2k+1)$. Thus, we may identify $\mathrm{SU}\big(1, \lfloor n/2 \rfloor\big)$ with a subgroup of $\mathrm{SO}(2, n)$.

We use the following well-known result to shorten one case of the proof of Theorem 3.5.

Lemma 3.3. ([12, Lem. 6.8]) *If L is a connected, almost-simple subgroup of* $\mathrm{SO}(2, n)$*, such that* $\mathbb{R}$*-rank* $L = 1$ *and* $\dim L > 3$*, then L is conjugate under* $\mathrm{O}(2, n)$ *to a subgroup of either* $\mathrm{SO}(1, n)$ *or* $\mathrm{SU}\big(1, \lfloor n/2 \rfloor\big)$.

Corollary 3.4. *Let L be a connected, reductive subgroup of $G = \mathrm{SO}(2,n)$, such that $\mathbb{R}$-rank $L = 1$. Then $\dim U \le n-1$, for every connected, unipotent subgroup U of L.*

Furthermore, if $\dim U = n-1$, then either

1) *L is conjugate to $\mathrm{SO}(1,n)^\circ$; or*
2) *n is even, and L is conjugate under $\mathrm{O}(2,n)$ to $\mathrm{SU}(1,n/2)$.*

Theorem 3.5. *Let H be a Lie subgroup of $G = \mathrm{SO}(2,n)$, with $n \ge 3$, such that*

- *the closure of H is not compact, and*
- *there is an $(\mathrm{Ad}_G H)$-invariant Minkowski form on $\mathfrak{g}/\mathfrak{h}$.*

Then H° is conjugate to a standard copy of $\mathrm{SO}(1,n)^\circ$ in $\mathrm{SO}(2,n)$.

Proof. Let $\overline{H}$ be the Zariski closure of H, and note that the Minkowski form is also invariant under $\mathrm{Ad}_G \overline{H}$. Replacing H by a finite-index subgroup, we may assume $\overline{H}$ is Zariski connected.

Let $G = KAN$ be an Iwasawa decomposition of G. For each real root ϕ of $\mathfrak{g}$ (with respect to the Cartan subalgebra $\mathfrak{a}$), let $\mathfrak{g}_\phi$ be the corresponding root space, and let $\mathrm{proj}_\phi \colon \mathfrak{g} \to \mathfrak{g}_\phi$ and $\mathrm{proj}_{\phi\oplus-\phi} \colon \mathfrak{g} \to \mathfrak{g}_\phi + \mathfrak{g}_{-\phi}$ be the natural projections. Fix a choice of simple real roots α and β of $\mathfrak{g}$, such that $\dim \mathfrak{g}_\alpha = 1$ and $\dim \mathfrak{g}_\beta = n-2$ (so the positive real roots are α, β, $\alpha+\beta$, and $\alpha+2\beta$). Replacing N by a conjugate under the Weyl group, we may assume $\mathfrak{n} = \mathfrak{g}_\alpha + \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. From the classification of parabolic subgroups [5, Prop. 5.14, p. 99], we know that the only proper parabolic subalgebras of $\mathfrak{g}$ that contain $\mathfrak{n}_{\mathfrak{g}}(\mathfrak{n})$ are

$$\mathfrak{n}_{\mathfrak{g}}(\mathfrak{n}), \;\; \mathfrak{p}_\alpha = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n}) + \mathfrak{g}_{-\alpha}, \;\text{ and }\; \mathfrak{p}_\beta = \mathfrak{n}_{\mathfrak{g}}(\mathfrak{n}) + \mathfrak{g}_{-\beta}. \tag{3.6}$$

Case 1. Assume $\overline{\mathfrak{h}}$ contains nontrivial hyperbolic elements. Let $\mathfrak{t} = \overline{\mathfrak{h}} \cap \mathfrak{a}$. Replacing H by a conjugate, we may assume $\mathfrak{t} \neq 0$.

Subcase 1.1. Assume $\mathfrak{t} \in \{\ker(\alpha+\beta), \ker\beta\}$.

Subsubcase 1.1.1. Assume $\overline{H}$ is reductive. We may assume $\mathfrak{t} = \ker(\alpha+\beta)$ (if necessary, replace H with its conjugate under the Weyl reflection corresponding to the root α). Then, from Corollary 2.2(1), we see that $\mathfrak{h}$ contains a codimension-one subspace of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$. (Note that this implies H° is nontrivial.)

Let $\mathfrak{n}' = \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$, so $\mathfrak{n}'$ is the Lie algebra of a maximal unipotent subgroup of G. (In fact, $\mathfrak{n}'$ is the image of $\mathfrak{n}$ under the Weyl reflection corresponding to the root α.) From the preceding paragraph, we know that

$$\dim(\overline{\mathfrak{h}} \cap \mathfrak{n}') \ge \dim(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}) - 1 = n-1.$$

Therefore, Corollary 3.4 implies that $\overline{H}$ is conjugate (under $\mathrm{O}(2,n)$) to either $\mathrm{SO}(1,n)$ or $\mathrm{SU}(1,n/2)$. It is easy to see that $\overline{H}$ is not conjugate to $\mathrm{SU}(1,n/2)$. (See [12, proof of Thm. 1.5] for an explicit description of $\mathfrak{su}(1,n/2) \cap \mathfrak{n}$. If n is even, then $n > 3$, so $\mathfrak{su}(1,n/2)$ does not contain a codimension-one subspace of any $(n-2)$-dimensional root space, but $\overline{\mathfrak{h}}$ does contain a codimension-one subspace of $\mathfrak{g}_\beta$.) Therefore, we conclude that $\overline{H}$ is conjugate to $\mathrm{SO}(1,n)$. Then, because H° is a nontrivial, connected, normal subgroup of $\overline{H}$, we conclude that $H^\circ = (\overline{H})^\circ$ is conjugate to $\mathrm{SO}(1,n)^\circ$.

Subsubcase 1.1.2. Assume $\overline{H}$ is not reductive. Let P be a maximal parabolic subgroup of G that contains $\overline{H}$ (see Theorem 3.1). By replacing P and H with conjugate subgroups, we may assume that P contains the minimal parabolic subgroup $N_G(N)$. Therefore, the classification of parabolic subalgebras (3.6) implies that P is either P_α or P_β.

Subsubsubcase 1.1.2.1. Assume $\mathfrak{t} = \ker(\alpha+\beta)$. From Corollary 2.2(1), we see that $\mathfrak{h}$ (and hence also $\mathfrak{p}$) contains codimension-one subspaces of $\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta + \mathfrak{g}_{-\alpha}$ and $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha$. Because $\mathfrak{p}_\alpha$ does not contain such a subspace of $\mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha$, we conclude that $P = P_\beta$. Furthermore, because the intersection of $\mathfrak{p}_\beta$ with each of these subspaces does have codimension one, we conclude that $\mathfrak{h}$ has precisely the same intersection; therefore $(\mathfrak{g}_{\alpha+2\beta} + \mathfrak{g}_\beta) + (\mathfrak{g}_{-\beta} + \mathfrak{g}_\alpha) \subset \mathfrak{h}$. Hence $\mathfrak{h} \supset [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. We now have

$$(\operatorname{ad}_\mathfrak{g} \mathfrak{g}_{\alpha+\beta})^2 \mathfrak{g} = \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta} \equiv 0 \pmod{\mathfrak{h}},$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+\beta}] \supset [\mathfrak{g}_{-\alpha-\beta}, \mathfrak{g}_{\alpha+\beta}] \supset \ker\beta.$$

This contradicts the fact that $\overline{\mathfrak{h}} \cap \mathfrak{a} = \mathfrak{t} = \ker(\alpha+\beta)$.

Subsubsubcase 1.1.2.2. Assume $\mathfrak{t} = \ker\beta$. From Corollary 2.2(1), we see that $\mathfrak{h}$ (and hence also $\mathfrak{p}$) contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. Because neither $\mathfrak{p}_\alpha$ nor $\mathfrak{p}_\beta$ contains such a subspace, this is a contradiction.

Subcase 1.2. Assume $\mathfrak{t} \in \{\ker\alpha, \ker(\alpha+2\beta)\}$. We may assume $\mathfrak{t} = \ker\alpha$ (if necessary, replace H with its conjugate under the Weyl reflection corresponding to the root β). From Corollary 2.2(1), we see that $\mathfrak{h}$ contains a codimension-one subspace of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$. Because any codimension-one subalgebra of a nilpotent Lie algebra must contain the commutator subalgebra, we conclude that $\mathfrak{h}$ contains $\mathfrak{g}_{\alpha+2\beta}$. Then we have

$$(\operatorname{ad}_\mathfrak{g} \mathfrak{g}_{\alpha+2\beta})^2 \mathfrak{g} = \mathfrak{g}_{\alpha+2\beta} \equiv 0 \pmod{\mathfrak{h}},$$

so Corollary 2.2(2d) implies

$$\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}] \supset \mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}.$$

Similarly, we also have $\mathfrak{h} \supset \mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$. It is now easy to show that $\mathfrak{h} \supset \mathfrak{g}_\phi$ for every real root ϕ, so $\mathfrak{h} = \mathfrak{g}$. This contradicts the fact that $\mathfrak{g}/\mathfrak{h} \neq 0$.

Subcase 1.3. Assume $\mathfrak{t}$ contains a regular element of $\mathfrak{a}$. Replacing H by a conjugate under the Weyl group, we may assume that $\mathfrak{n}$ is the sum of the positive root spaces, with respect to $\mathfrak{t}$. Then, from Corollary 2.2(1), we see that $\mathfrak{h}$ contains codimension-one subspaces of both $\mathfrak{n}$ and $\mathfrak{n}^-$. Therefore, $\mathfrak{h}$ contains codimension-one subspaces of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$ and $\mathfrak{g}_{-\beta} + \mathfrak{g}_{-\alpha-\beta} + \mathfrak{g}_{-\alpha-2\beta}$, so the argument of Subcase 1.2 applies.

Case 2. Assume that $\overline{\mathfrak{h}}$ does not contain nontrivial hyperbolic elements. From Lemma 2.3, we know there exist a compact subgroup M and a nontrivial unipotent subgroup U, such that $\overline{H} = M \ltimes U$. Choose subspaces $V/\mathfrak{h}$ and $W/\mathfrak{h}$ of $\mathfrak{g}/\mathfrak{h}$ as in Corollary 2.2(2).

Let P be a proper parabolic subgroup of G, such that $U \subset \operatorname{unip} P$ and $H \subset P$ (see Theorem 3.1). Replacing H and P by conjugates, we may assume, without loss of generality, that P contains the minimal parabolic subgroup $N_G(N)$ (so $\operatorname{unip} P \subset N$). From the classification of parabolic subalgebras (3.6), we know that there are only three possibilities for P. We consider each of these possibilities separately.

First, though, let us show that

$$\text{for every nonzero } u \in \mathfrak{u}, \text{ we have } [\mathfrak{g}, u] \not\subset \mathfrak{h}. \tag{3.7}$$

From the Morosov Lemma [8, Thm. 17(1), p. 100], we know there exists $v \in \mathfrak{g}$, such that $[v, u]$ is hyperbolic (and nonzero). If $[v, u] \in \mathfrak{h}$, this contradicts the fact that $\overline{\mathfrak{h}}$ does not contain nontrivial hyperbolic elements.

Subcase 2.1. Assume $P = N_G(N)$ is a minimal parabolic subgroup of G.

Subsubcase 2.1.1. Assume $\operatorname{proj}_\beta \mathfrak{u} \neq 0$. Choose $u \in \mathfrak{u}$, such that $\operatorname{proj}_\beta u \neq 0$, and let $Z = (\operatorname{ad}_\mathfrak{g} u)^2 \mathfrak{g}_{-\alpha-2\beta}$. (So $\dim Z = 1$, $\operatorname{proj}_{-\alpha} Z \neq 0$, and $\operatorname{proj}_{-\alpha-\beta} Z = 0$.) From Corollary 2.2(2d), we know that $Z \subset W$. Then, because $\operatorname{proj}_{-\alpha} \mathfrak{h} \subset \operatorname{proj}_{-\alpha} \mathfrak{p} = 0$, we conclude, from Corollary 2.2(2b), that $W = \mathfrak{h} + Z$.

Because $W = \mathfrak{h} + Z \subset \mathfrak{p} + Z$, we have $\operatorname{proj}_{-\alpha-\beta} W = 0$. Therefore, because $\operatorname{proj}_\beta u \neq 0$, we conclude, from Corollary 2.2(2c), that $\operatorname{proj}_{-\alpha-2\beta} V = 0$, so Corollary 2.2(2a) implies that $V = \ker(\operatorname{proj}_{-\alpha-2\beta})$. In particular, we have $\mathfrak{g}_{-\beta} \subset V$, so Corollary 2.2(2c) implies $[\mathfrak{g}_{-\beta}, u] \subset W$. Therefore, we have

$$\begin{aligned}
[\mathfrak{g}_{-\beta}, \operatorname{proj}_\beta u] &\subset [\mathfrak{g}_{-\beta}, u + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta})] \\
&= [\mathfrak{g}_{-\beta}, u] + [\mathfrak{g}_{-\beta}, \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}] \\
&\subset W + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta}) \\
&= \mathfrak{h} + Z + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta}) \\
&\subset \mathfrak{m} + \mathfrak{n} + Z.
\end{aligned}$$

Because $\operatorname{proj}_{-\alpha}[\mathfrak{g}_{-\beta}, \operatorname{proj}_\beta u] = 0$, we conclude that $[\mathfrak{g}_{-\beta}, \operatorname{proj}_\beta u] \subset \mathfrak{m} + \mathfrak{n}$. This contradicts the fact that $\mathfrak{m} + \mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subsubcase 2.1.2. Assume $\operatorname{proj}_\beta \mathfrak{u} = 0$. Replacing H by a conjugate under N, we may assume $\mathfrak{m} \subset \mathfrak{g}_0$, so $\operatorname{proj}_\beta \overline{\mathfrak{h}} = 0$.

We have $\mathfrak{u} \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$, so $(\operatorname{ad}_\mathfrak{g} u)^2 \mathfrak{g} \subset \mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}$ for every $u \in \mathfrak{u}$. Thus, Corollary 2.2(2d) implies $W \subset (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) + \mathfrak{h}$.

We have

$$\operatorname{proj}_{\beta \oplus -\beta} W \subset \operatorname{proj}_{\beta \oplus -\beta}(\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) + \operatorname{proj}_{\beta \oplus -\beta} \mathfrak{h} = 0,$$

so Corollary 2.2(2c) implies that $\operatorname{proj}_{\beta \oplus -\beta}\big((\operatorname{ad}_\mathfrak{g} \mathfrak{u})V\big) = 0$.

Subsubsubcase 2.1.2.1. Assume $\operatorname{proj}_\alpha u \neq 0$, for some $u \in \mathfrak{u}$. From the conclusion of the preceding paragraph, we know that $\operatorname{proj}_{-\beta}\big((\operatorname{ad}_\mathfrak{g} u)V\big) = 0$. Because $\operatorname{proj}_\beta u = 0$ and $\operatorname{proj}_\alpha \neq 0$, this implies $\operatorname{proj}_{-\alpha-\beta} V = 0$, so $V = \ker(\operatorname{proj}_{-\alpha-\beta})$ (see 2.2(2a)). In particular, $\mathfrak{g}_{-\alpha} \subset V$, so Corollary 2.2(2c) implies

$$\begin{aligned}
[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] &\subset [\mathfrak{u} + (\mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}), \mathfrak{g}_{-\alpha}] \subset [\mathfrak{u}, V] + [\mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha}] \\
&\subset W + \mathfrak{g}_\beta \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}.
\end{aligned}$$

This contradicts the fact that $\mathfrak{m}+\mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subsubsubcase 2.1.2.2. Assume $\operatorname{proj}_{\alpha+\beta} u \neq 0$, for some $u \in \mathfrak{u}$. From Subsubsubcase 2.1.2.1, we may assume $\operatorname{proj}_\alpha u = 0$. Because $0 = \operatorname{proj}_{\beta\oplus-\beta}\big((\operatorname{ad}_\mathfrak{g} u)V\big)$ has codimension ≤ 1 in $\operatorname{proj}_{\beta\oplus-\beta}\big((\operatorname{ad}_\mathfrak{g} u)\mathfrak{g}\big)$ (see 2.2(2a)), which contains the 2-dimensional subspace $\operatorname{proj}_{\beta\oplus-\beta}\big([u, \mathfrak{g}_{-\alpha-2\beta} + \mathfrak{g}_{-\alpha}]\big)$, we have a contradiction.

Subsubsubcase 2.1.2.3. Assume $\mathfrak{u} = \mathfrak{g}_{\alpha+2\beta}$. (This argument is similar to Subsubsubcase 2.1.2.1.) Because $\operatorname{proj}_\beta\big((\operatorname{ad}_\mathfrak{g} \mathfrak{u})V\big) = 0$, we know that $\operatorname{proj}_{-\alpha-\beta} V = 0$, so $V = \ker(\operatorname{proj}_{-\alpha-\beta})$ (see 2.2(2a)). In particular, $\mathfrak{g}_{-\alpha-2\beta} \subset V$, so Corollary 2.2(2c) implies

$$[\mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta}] \subset [\mathfrak{u}, V] \subset W \subset \mathfrak{h} + \mathfrak{n} \subset \mathfrak{m} + \mathfrak{n}.$$

This contradicts the fact that $\mathfrak{m}+\mathfrak{n}$ does not contain nontrivial hyperbolic elements.

Subcase 2.2. Assume $P = P_\alpha$. We may assume there exists $x \in \mathfrak{h}$, such that $\operatorname{proj}_{-\alpha} x \neq 0$ (otherwise, $H \subset N_G(N)$, so Subcase 2.1 applies). Note that, because $U \subset \operatorname{unip} P$, we have $\operatorname{proj}_\alpha \mathfrak{u} = 0$.

Subsubcase 2.2.1. Assume $\operatorname{proj}_{\alpha+\beta} \mathfrak{u} \neq 0$. Choose $u \in \mathfrak{u}$, such that $\operatorname{proj}_{\alpha+\beta} u \neq 0$. Then $[x,u] \in [\mathfrak{h},\mathfrak{u}] \subset \mathfrak{u}$, and $\big[[x,u],u\big]$ is a nonzero element of $\mathfrak{g}_{\alpha+2\beta}$, so we see that $\mathfrak{g}_{\alpha+2\beta} \subset [\mathfrak{u},\mathfrak{u}]$. Because every unipotent subgroup of $\mathrm{SO}(1,k)$ is abelian, we conclude that $\operatorname{ad}_\mathfrak{g} \mathfrak{g}_{\alpha+2\beta}$ acts trivially on $\mathfrak{g}/\mathfrak{h}$, which means $\mathfrak{h} \supset [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}]$. This contradicts (3.7).

Subsubcase 2.2.2. Assume $\operatorname{proj}_{\alpha+\beta} \mathfrak{u} = 0$. We may assume, furthermore, that $\operatorname{proj}_\alpha \mathfrak{h} \neq 0$ (otherwise, by replacing H with its conjugate under the Weyl reflection corresponding to the root α, we could revert to Subcase 2.1). Then, because $[\mathfrak{h},\mathfrak{u}] \subset \mathfrak{u}$, we must have $\operatorname{proj}_\beta \mathfrak{u} = 0$. Thus, $\mathfrak{u} = \mathfrak{g}_{\alpha+2\beta}$. From Corollary 2.2(2d), we have

$$W = [\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{\alpha+2\beta}] + \mathfrak{h} = \mathfrak{g}_{\alpha+2\beta} + \mathfrak{h} \subset \mathfrak{u} + \overline{\mathfrak{h}} = \overline{\mathfrak{h}},$$

so

$$\begin{aligned} W \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) &\subset \overline{\mathfrak{h}} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = (\overline{\mathfrak{h}} \cap \mathfrak{n}) \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) \\ &= \mathfrak{u} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = \mathfrak{g}_{\alpha+2\beta} \cap (\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}) = 0. \end{aligned}$$

On the other hand, from Corollary 2.2(2c), we know that W contains a codimension-one subspace of $[\mathfrak{g}, \mathfrak{g}_{\alpha+2\beta}]$, so W contains a codimension-one subspace of $\mathfrak{g}_\beta + \mathfrak{g}_{\alpha+\beta}$. This is a contradiction.

Subcase 2.3. Assume $P = P_\beta$. Note that, because $U \subset \operatorname{unip} P$, we have $\operatorname{proj}_\beta \mathfrak{u} = 0$.

From Corollary 2.2(2d), we have

$$\begin{aligned} W &= \mathfrak{h} + (\operatorname{ad}_\mathfrak{g} u)^2 \mathfrak{g} \subset \mathfrak{h} + (\mathfrak{g}_\alpha + \mathfrak{g}_{\alpha+\beta} + \mathfrak{g}_{\alpha+2\beta}) \\ &= \mathfrak{h} + \operatorname{unip} \mathfrak{p}_\beta \subset (\mathfrak{m} + \mathfrak{u}) + \operatorname{unip} \mathfrak{p}_\beta = \mathfrak{m} + \operatorname{unip} \mathfrak{p}_\beta. \end{aligned}$$

Subsubcase 2.3.1. Assume there is some nonzero $u \in \mathfrak{u}$, such that $\operatorname{proj}_\alpha u = 0$. Replacing H by a conjugate (under $G_{-\beta}$), we may assume $\operatorname{proj}_{\alpha+\beta} u \neq 0$.

Let $V' = V \cap (\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta})$. Because V' contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-\beta}$ (see Corollary 2.2(2a)), one of the following two subsubsubcases must apply.

Subsubsubcase 2.3.1.1. Assume there exists $v \in V'$, such that $\operatorname{proj}_{-\alpha-\beta} v = 0$. From Corollary 2.2(2c), we have $[u, v] \in W$. Then, because $[u, v]$ is a nonzero element of $\mathfrak{g}_\beta$, we conclude that

$$0 \neq W \cap \mathfrak{g}_\beta \subset (\mathfrak{m} + \operatorname{unip} \mathfrak{p}_\beta) \cap \mathfrak{g}_\beta = 0.$$

This contradicts the fact that M, being compact, has no nontrivial unipotent elements.

Subsubsubcase 2.3.1.2. Assume $\operatorname{proj}_{-\alpha-\beta} V' = \mathfrak{g}_{-\alpha-\beta}$. For $v \in V'$, we have $\operatorname{proj}_0[u, v] = [\operatorname{proj}_{\alpha+\beta} u, \operatorname{proj}_{-\alpha-\beta} v]$. Thus, there is some $v \in V'$, such that $\operatorname{proj}_0[u, v]$ is hyperbolic (and nonzero). On the other hand, from Corollary 2.2(2c), we have $[u, v] \in W = \mathfrak{m} + \operatorname{unip} \mathfrak{p}_\beta$. This contradicts the fact that $\mathfrak{m} \subset \overline{\mathfrak{h}}$ does not contain nonzero hyperbolic elements.

Subsubcase 2.3.2. Assume $\operatorname{proj}_\alpha u \neq 0$, for every nonzero $u \in \mathfrak{u}$. Fix some nonzero $u \in \mathfrak{u}$. Because $\dim \mathfrak{u}_\alpha = 1$, we must have $\dim \mathfrak{u} = 1$ (so $\mathfrak{u} = \mathbb{R}u$). Replacing H by a conjugate (under G_β), we may assume $\operatorname{proj}_{\alpha+\beta} u = 0$. Also, we may assume $\operatorname{proj}_{\alpha+2\beta} u \neq 0$ (otherwise, we could revert to Subsubcase 2.3.1 by replacing H with its conjugate under the Weyl reflection corresponding to the root β).

Let $\mathfrak{t} = [u, \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-2\beta}]$. Because $\langle \mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha} \rangle$ and $\langle \mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta} \rangle$ centralize each other, we see that $\mathfrak{t} = [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] + [\mathfrak{g}_{\alpha+2\beta}, \mathfrak{g}_{-\alpha-2\beta}]$ is a two-dimensional subspace of $\mathfrak{g}$ consisting entirely of hyperbolic elements. Because V contains a codimension-one subspace of $\mathfrak{g}_{-\alpha} + \mathfrak{g}_{-\alpha-2\beta}$ (see Corollary 2.2(2a)), and $[u, V] \subset W$ (see Corollary 2.2(2c)), we see that W contains a codimension-one subspace of $\mathfrak{t}$, so W contains nontrivial hyperbolic elements. This contradicts the fact that $W \subset \mathfrak{m} + \operatorname{unip} \mathfrak{p}_\beta$ does not contain nontrivial hyperbolic elements. □

References

[1] Adams, S.: *Locally free actions on Lorentz manifolds.* Geometric and Functional Analysis **10**(3) (2000), 453–515.

[2] Adams, S.: *Transitive actions on Lorentz manifolds with noncompact stabilizer.* Preprint 1999.

[3] Adams, S.: *Nontame Lorentz actions of simple Lie groups.* Preprint 2000.

[4] Adams, S.; Stuck, G.: *The isometry group of a compact Lorentz manifold. I, II.* Invent. Math. **129**(2) (1997), 239–261, 263–287.

[5] Borel, A.; Tits, J.: *Groupes réductifs.* Inst. Hautes Études Sci. Publ. Math. **27** (1965), 55–150.

[6] Borel, A.; Tits, J.: *Eléments unipotents et sous-groupes paraboliques de groupes réductifs I.* Invent. Math. **12** (1971), 95–104.

[7] Hochschild, G. P.: *Basic Theory of Algebraic Groups and Lie Algebras.* Springer, New York 1981.

[8] Jacobson, N.: *Lie Algebras.* Dover, New York 1962.

[9] Kowalsky, N.: *Actions of non-compact simple groups on Lorentz manifolds and other geometric manifolds.* Ph.D. Thesis, University of Chicago 1994.

[10] Kowalsky, N.: *Actions of non-compact simple groups on Lorentz manifolds.* C. R. Acad. Sci. Paris **321**, Série I, (1995), 595–599.

[11] Kowalsky, N.: *Noncompact simple automorphism groups of Lorentz manifolds and other geometric manifolds.* Ann. of Math. **144**(3) (1996), 611–640.

[12] Oh, H.; Witte, D.: *Compact Clifford-Klein forms of homogeneous spaces of* $\mathrm{SO}(2,n)$. Geometriae Dedicata, to appear.

[13] Onishchik, A. L.; Vinberg, E. B.: *Lie Groups and Algebraic Groups.* Springer, New York 1990.

[14] Zeghib, A.: *Sur les espaces-temps homogènes.* In: I. Rivin, C. Rourke, and C. Series, eds., *The Epstein birthday schrift.* Geom. Topol. Monogr. **1** (1998), paper 26, 551–576 (electronic).

[15] Zeghib, A.: *The identity component of the isometry group of a compact Lorentz manifold.* Duke Math. J. **92**(2) (1998), 321–333.

[16] Zimmer, R. J.: *On the automorphism group of a compact Lorentz manifold and other geometric manifolds.* Invent. Math. **83**(3) (1986), 411–424.

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