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On Mixed Multiplicities of Homogeneous Ideals

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1. Introduction

Let $S = \oplus S_{(u,v)}$ be a standard bigraded R-algebra over an Artinian local ring $K = S_{(0,0)}$, i.e. S is generated by finitely many forms of degree $(1,0)$ and $(0,1)$ over K. The Hilbert function of S is defined as

$$H(u,v) := \ell(S_{(u,v)}),$$

where ℓ denotes the length of the underlying K-module. Van der Waerden [12] proved that if K is a field, then $H(u,v)$ is given by a polynomial

$$P(u,v) = \sum_{i+j \le \dim S - 2} a_{ij} \binom{u}{i} \binom{v}{j}$$

for large u and v, where a_{ij} are integers. This has been extended to the Artinian case by Bhattacharya [1].

A bihomogeneous prime ideal $\mathfrak{p}$ of S is called relevant if $\mathfrak{p}$ does not contain $S_{(1,0)}$ and $S_{(0,1)}$. Let $\mathrm{BiProj}(S)$ denote the set of the relevant bihomogeneous prime ideal of S. The relevant dimension of S is defined as

$$\operatorname{rdim} S := \max\{\dim S/\mathfrak{p} \mid \mathfrak{p} \in \mathrm{BiProj}(S)\}.$$

As shown by D. Katz, S. Mandal and J. K. Verma [4], $\deg P(u,v) = \operatorname{rdim} S - 2$. The numbers a_{ij} with $i + j = \operatorname{rdim} S - 2$ are called the mixed multiplicities of S.

Let $(R, \mathfrak{m})$ be a local ring of positive dimension d and I an ideal of R . We can associate with I the Rees algebra $R[It] = \underset{i \ge 0}{\oplus} I^i t^i$. Let $M := (\mathfrak{m}, It)$ be the maximal graded ideal of $R[It]$. The associated graded ring $gr_M R[It] := \underset{n \ge 0}{\oplus} M^n/M^{n+1}$ has a natural bigrading with

$$(gr_M R[It])_{(u,v)} = \mathfrak{m}^u I^v / \mathfrak{m}^{u+1} I^v.$$

As shown by Bhattacharya [1], the numerical function $\dim_k \mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v$ is given by a polynomial in u and v for all large values of u and v. Let s be the degree of this polynomial and write the terms of total degree s as

$$\sum_{i+j=s} \frac{a_{ij}}{i!j!} u^i v^j$$

where a_{ij} are non negative integers.

Teissier and Risler [9] linked these numbers to the Milnor numbers of general hyperplane sections of complex analytic hypersurfaces with isolated singularities. They called the number a_{ij} a mixed multiplicity of the pair $(\mathfrak{m}, I)$ and denoted it by $e_{ij}(\mathfrak{m}|I)$. The multiplicity of the Rees algebra $R[It]$ and of the extended Rees algebra $R[It, t^{-1}]$ can be expressed in terms of the mixed multiplicities as follows:

$$e(R[It]) = \sum_{i+j=d-1} e_{ij}(\mathfrak{m}|I)$$

if I has positive height and, if $I \subsetneq \mathfrak{m}^2$,

$$e(R[It, t^{-1}] = e(R) + \sum_{i+j=d-1} e_{ij}(\mathfrak{m}|I).$$

See [11, Theorem (3.1)], [5, Proof of (3.7)] for more details. Though we have a well developed theory on mixed multiplicities when I is an $\mathfrak{m}$-primary ideal [9], [6], there have been few cases where the mixed multiplicities can be computed in terms of well-known invariants of $\mathfrak{m}$ and I when I is not an $\mathfrak{m}$-primary ideal.

In this paper we study the case $R = \underset{n \geq 0}{\oplus} R_n$ is a standard graded algebra over a field $k = R_0$, $\mathfrak{m} = \underset{n>0}{\oplus} R_n$ and I a homogeneous ideal of R. Note that we can define the mixed multiplicities $e_{ij}(\mathfrak{m}|I)$ as in the local case and that the above formulas for the multiplicities of the Rees algebras can be proved similarly.

Let $x_1, \ldots, x_n$ be a sequence of homogeneous elements in R with $\deg x_1 \leq \ldots \leq \deg x_n$. Let I denote the ideal $(x_1, \ldots, x_n)$. The multiplicity of the Rees algebra $R[It]$ was computed by Herzog, Trung, and Ulrich [2] when $x_1, \ldots, x_n$ is a d-sequence and by Trung [10] when $x_1, \ldots, x_n$ is a subsystem of parameters which is filter-regular. They used a technique which is similar to that of Gröbner bases and which does not involve mixed multiplicities. Using this technique Raghavan and Verma [7] were able to compute the mixed multiplicities $e_{ij}(\mathfrak{m}|I)$ when $x_1, \ldots, x_n$ is a d-sequence. However, their method is a bit complicated and can not be applied to study the case I is generated by a subsystem of parameters.

In Section 2 of this paper we will use a simpler argument to compute the mixed multiplicities $e_{ij}(\mathfrak{m}|I)$ when $x_1, \ldots, x_n$ is a d-sequence. Let $I_i = (x_1, \ldots, x_{i-1}) : x_i$, $i = 1, \ldots, n$, $d_1 = \dim R/I_1$ and $r = \max\{i \mid \dim R/I_i = d_1 - i + 1\}$. We obtain the formula

$$e_{id_1-i-1}(\mathfrak{m}|I) = \begin{cases} 0 & \text{if } 0 \leq i \leq d_1 - r - 1, \\ e(R/I_{d_1-i}) & \text{if } d_1 - r \leq i \leq d_1 - 1. \end{cases}$$

We point out that this formula is more precise than that of Raghavan and Verma.

In Section 3 we will use the same argument to compute the mixed multiplicities $e_{ij}(\mathfrak{m}|I)$ when $x_1, \dots, x_n$ is a subsystems of homogeneous parameters which is filter-regular with respect to I. Put $\deg x_i = a_i$. We obtain the formula

$$e_{id-i-1}(\mathfrak{m}|I) = \begin{cases} 0 & \text{if } 0 \le i \le d-n-1, \\ a_1 \dots a_{d-i-1} e(R) & \text{if } d-n \le i \le d-1. \end{cases}$$

This formula was posed as a problem in [10]. It is worth to mention that the condition $x_1, \dots, x_n$ is a filter-regular sequence with $\deg x_1 \le \dots \le \deg x_n$ is automatically satisfied in a generalized Cohen-Macaulay ring or if I is generated by elements of the same degree.

We do not know whether there is a compact formula for $e_{ij}(\mathfrak{m}|I)$ in the above cases when the degrees of $x_1, \dots, x_n$ are not increasing.

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2. Mixed multiplicities of ideals generated by *d*-sequences

Let $R = \underset{n \ge 0}{\oplus} R_n$ be a standard graded algebra over a field $k = R_0$ and $\mathfrak{m} = \underset{n>0}{\oplus} R_n$. Let $x_1, \dots, x_n$ be a sequence of homogeneous elements of R and $I = (x_1, \dots, x_n)$.

Let A denote the polynomial ring $R[T_1, \dots, T_n]$. If we map T_i to $x_i t$, $i = 1, \dots, n$, we get a representation of the Rees algebra

$$R[It] \cong A/J,$$

where J is the ideal of A generated by the forms vanishing at $x_1, \dots, x_n$. For all $h = (a_0, \dots, a_n) \in \mathbb{N}^{n+1}$ put

$$A_h := R_{a_0} T_1^{a_1} \dots T_n^{a_n}.$$

Then $A = \underset{h \in \mathbb{N}^{n+1}}{\oplus} A_h$, that is, A is an $\mathbb{N}^{n+1}$-graded ring. Note that $(\mathfrak{m}, T_1, \dots, T_n)$ is the maximal graded ideal of A. Define the following degree-lexicographic order on $\mathbb{N}^{n+1}$:

$$(a_0, a_1, \dots, a_n) < (b_0, b_1, \dots, b_n)$$

if the first non-zero component from the left side of

$$\Big(\sum_{i=0}^{n} a_i - \sum_{i=0}^{n} b_i,\ a_0 - b_0,\ \dots,\ a_n - b_n\Big)$$

is negative. Then $<$ is a terms order on $\mathbb{N}^{n+1}$. Set

$$F_h A := \underset{h' \ge h}{\oplus} A_{h'}.$$

It is clear that $F = \{F_h A\}_{h \in \mathbb{N}^{n+1}}$ is a filtration of A. The filtration F imposes a filtration on A/J which we also denote by F.

For every polynomial $f \in A$, we denote by f^* the initial term of f, i.e. $f^* = f_{h'}$ if $f = \sum_{h \in \mathbb{N}^{n+1}} f_h$ and $h' := \min\{h | f_h \neq 0\}$. Let J^* denote the ideal of A generated by all elements $f^*, f \in J$. Then

$$gr_F(A/J) \cong A/J^*.$$

The $\mathbb{N}^{n+1}$-graded structure imposes a bigrading on A with

$$A_{(u,v)} = \bigoplus_{\alpha_1+\ldots+\alpha_n=v} A_{(u,\alpha_1,\ldots,\alpha_n)}$$

for all $(u,v) \in \mathbb{N}^2$. Since J^* is an $\mathbb{N}^{n+1}$-bigraded ideal of A, J^* is also a bigraded ideal of A. Hence A/J^* is a bigraded algebra over k with respect to the bigrading induced from A.

Now we shall see that the Bhattacharya function of $(\mathfrak{m}, I)$ coincides with the Hilbert function of A/J^*.

Lemma 2.1. *For all $(u,v) \in \mathbb{N}^2$ we have*

$$\dim_k(\mathfrak{m}^u I^v/\mathfrak{m}^{u+1}I^v) = \dim_k(A/J^*)_{(u,v)}.$$

Proof. We know that

$$\mathfrak{m}^u I^v/\mathfrak{m}^{u+1}I^v = (gr_M R[It])_{(u,v)}.$$

Let $\mathfrak{M} = (\mathfrak{m}, T_1, \ldots, T_n)$ be the maximal graded ideal of A. Then

$$gr_M R[It] \cong gr_{\mathfrak{M}}(A/J).$$

The bigrading on $gr_M R[It]$ imposes a bigrading on $gr_{\mathfrak{M}}(A/J)$ with

$$gr_{\mathfrak{M}}(A/J)_{(u,v)} =$$

$$\Big(\bigoplus_{\substack{\alpha_0 \geq u \\ \alpha_1+\ldots+\alpha_n \geq v}} A_{(\alpha_0,\ldots,\alpha_n)} + J\Big)\Big/\Big(\bigoplus_{\substack{\alpha_0 \geq u \\ \alpha_1+\ldots+\alpha_n \geq v+1}} A_{(\alpha_0,\ldots,\alpha_n)} + \bigoplus_{\substack{\alpha_0 \geq u+1 \\ \alpha_1+\ldots+\alpha_n \geq v}} A_{(\alpha_0,\ldots,\alpha_n)} + J\Big)$$

$$\cong \underset{\alpha_1+\ldots+\alpha_n=v}{A}{}_{(u,\alpha_1,\ldots,\alpha_n)} + J/J.$$

Using the filtration F on A/J we can decompose the latter module into a series of graded pieces of the associated ring $gr_F(A/J) \cong A/J^*$ and we obtain

$$\begin{aligned}\dim_k gr_{\mathfrak{M}}(A/J)_{(u,v)} &= \sum_{\alpha_1+\ldots+\alpha_n=v} \dim_k (A/J^*)_{(u,\alpha_1,\ldots,\alpha_n)} \\ &= \dim_k \bigoplus_{\alpha_1+\ldots+\alpha_n=v} (A/J^*)_{(u,\alpha_1,\ldots,\alpha_n)} \\ &= \dim_k (A/J^*)_{(u,v)}.\end{aligned}$$

□

According to Lemma 2.1 we can use the Hilbert function of A/J^* to compute the mixed multiplicities $e_i(\mathfrak{m}|I)$. Herzog-Trung-Ulrich [2] computed J^* explicitly when $x_1, \ldots, x_n$ is a d-sequence of homogeneous elements with increasing degrees. Recall that $x_1, \ldots, x_n$ is said to be a d-sequence if

(1) $x_i \notin (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$,

(2) $(x_1, \ldots, x_i) : x_{i+1}x_k = (x_1, \ldots, x_i) : x_k$ for all $k \geq i+1$ and all $i \geq 0$.

Lemma 2.2. [2, Lemma 1.2] *Let* $x_1, \dots, x_n$ *be a homogeneous d-sequence of* R *with* $\deg x_1 \le \dots \le \deg x_n$. *Then*

$$J^* = (I_1T_1, \dots, I_nT_n),$$

where $I_j := (x_1, \dots, x_{j-1}) : x_j$ *for* $1 \le j \le n$.

Now we come to the main result of this section.

Theorem 2.3. *Let* I *be an ideal generated by a homogeneous d-sequence* $x_1, \dots, x_n$ *of* R *with* $\deg x_1 \le \dots \le \deg x_n$. *Let* $I_0 = 0$, $I_i = (x_1, \dots, x_{i-1}) : x_i$, $i = 1, \dots, n$ *and* $d_1 = \dim R/I_1$. *Then the degree of the Hilbert polynomial of* $gr_M R[It]$ *is* $d_1 - 1$ *and*

$$e_{id_1-i-1}(m|I) = \begin{cases} 0 & \text{if } 0 \le i \le d_1 - r - 1, \\ e(R/I_{d_1-i}) & \text{if } d_1 - r \le i \le d_1 - 1, \end{cases}$$

where $r = \max\{i | \dim R/I_i = d_1 - i + 1\}$.

Proof. We will use an idea from [8, Theorem 3.7] to estimate the coefficients of the terms of the total degree of $H_{A/J^*}(u, v)$. For this we will compute the function

$$H_{A/J^*}(\alpha_0, \dots, \alpha_n) = \dim_k (A/J^*)_{(\alpha_0, \dots, \alpha_n)}.$$

Any element $f \in J^*$ with $\deg f = (\alpha_0, \dots, \alpha_n)$ is of the form $yT_1^{\alpha_1} \dots T_n^{\alpha_n}$ with $y \in (I_i)_{\alpha_0}$ for some $i = 1, \dots, n$ with $\alpha_i \neq 0$. Since $I_1, \dots, I_n$ is an increasing sequence of ideals, we get

$$J^*_{(\alpha_0, \dots, \alpha_n)} = (I_{m(\alpha_1, \dots, \alpha_n)})_{\alpha_0} T_1^{\alpha_1} \dots T_n^{\alpha_n}$$

where $m(\alpha_1, \dots, \alpha_n) := \max\{i | \alpha_i \neq 0\}$. Therefore

$$H_{A/J^*}(\alpha_0, \dots, \alpha_n) = \dim_k (R/I_{m(\alpha_1, \dots, \alpha_n)})_{\alpha_0} = H_{R/I_{m(\alpha_1, \dots, \alpha_n)}}(\alpha_0)$$

if $(\alpha_1, \dots, \alpha_n) \neq 0$. From this we get the Hilbert function of A/J^* as a bigraded algebra:

$$\begin{aligned} H_{A/J^*}(u, v) &= \sum_{\alpha_1 + \dots + \alpha_n = v} H_{R/I_{m(\alpha_1, \dots, \alpha_n)}}(u) \\ &= \sum_{i=1}^{n} \binom{v + i - 2}{i - 1} H_{R/I_i}(u), \end{aligned}$$

where the latter equality follows from the fact that the number of vectors $(\alpha_1, \dots, \alpha_n)$ with $\alpha_1 + \dots + \alpha_n = v$ and $m(\alpha_1, \dots, \alpha_n) = i$ is given by $\binom{v+i-2}{i-1}$.

Put $d_i = \dim R/I_i$. Then

$$H_{R/I_i}(u) = \frac{e(R/I_i)}{(d_i - 1)!} u^{d_i - 1} + \text{terms of lower degree.}$$

Therefore,

$$H_{A/J^*}(u, v) = \sum_{i=1}^{n} \left[\frac{e(R/I_i)}{(d_i - 1)!(i - 1)!} u^{d_i - 1} v^{i-1} + \text{terms of total degree} < d_i + i - 2 \right].$$

Since x_{i-1} is a non-zerodivisor modulo I_{i-1} we have

$$d_i = \dim R/I_i \le \dim R/(I_{i-1}, x_{i-1}) = \dim R/I_{i-1} - 1 = d_{i-1} - 1.$$

From this it follows that $d_i < d_{i-1} < \ldots < d_1$. Hence $d_1 - 1$ is the total degree of $H_{A/J^*}(u,v)$. By the assumption, $d_i = d_1 - i + 1$ if $1 \le i \le r$, and $d_i < d_1 - i + 1$ if $r < i \le n$. Hence from the above formula for $H_{A/J^*}(u,v)$ we obtain

$$e_{id_1-i-1}(m|I) = \begin{cases} 0 & \text{if } 0 \le i \le d_1 - r - 1, \\ e(R/I_{d_1-i}) & \text{if } d_1 - r \le i \le d_1 - 1. \end{cases} \qquad \square$$

Remark. Raghavan and Verma [7] already computed the bigraded Hilbert series $gr_{\mathfrak{M}}R[It]$. From this they get the formula

$$e_{ij}(\mathfrak{m}|I) = e_i(R/I_{j+1}) - e_i(R/I_{j+2}) \quad (i + j = s),$$

where for a standard graded algebra B over a field the symbol $e_i(B)$ denotes the i-th coefficient of the Hilbert polynomial $P_B(u)$ of B, i.e. $P_B(u) = \sum_{i\ge 0} e_i(B)\binom{u+i}{i}$. This formula is not explicit as our formula in Theorem 2.3.

Examples 2.4. It is known that the sequence $x_1, \ldots, x_n$ is a d-sequence in the following cases (see [3]). Hence we can use Theorem 2.3 to compute the mixed multiplicities.

(1) *Regular sequence.* Let I be generated by an R-sequence $x_1, \ldots, x_n$ of homogeneous elements with $\deg x_i = a_i, a_1 \le \ldots \le a_n$. Since $e(R/I_i) = a_1 \ldots a_{i-1} e(R)$, we have

$$e_{id-i-1}(m|I) = a_1 \ldots a_{i-1} e(R), 1 \le i \le n.$$

(2) *Subsystem of parameters of Buchsbaum rings.* Let R be a graded Buchsbaum ring and I an ideal of R generated by a subsequence $x_1 \ldots, x_n$ of a homogeneous system of parameters of R with $\deg x_i = a_i$, $a_1 \le \ldots \le a_n$. By [2, Corollary 1.5] $e(R/I_i) = a_1 \ldots a_{i-1} e(R)$. Hence

$$e_{id-i-1}(m|I) = a_1 \ldots a_{i-1} e(R), 1 \le i \le n.$$

(3) *Almost complete intersection.* Let R be a Gorenstein ring and $I = (x_1, \ldots, x_n)$ a homogeneous almost complete intersection of R of height $n - 1 > 0$ which satisfies the following conditions:

(i) $x_1, \ldots, x_{n-1}$ is a regular sequence,
(ii) $a_1 \le \ldots \le a_n$, $a_i = \deg x_i$,
(iii) R/I is Cohen-Macaulay,
(iv) $IR_P = (x_1, \ldots, x_{n-1})_P$ for all minimal prime ideals P of I.

Note that $e(R/I_i) = a_1 \ldots a_{i-1} e(R)$ for $i = 1, \ldots, n-1$, and $e(R/I_n) = a_1 \ldots a_{n-1}$ $e(R) - e(R/I)$ because $(x_1, \ldots, x_{n-1}) = I_n \cap I$. Then we obtain

$$e_{id-i-1}(m|I) = \begin{cases} a_1 \ldots a_{i-1} e(R) & \text{if } 1 \le i \le n-1, \\ a_1 \ldots a_{n-1} e(R) - e(R/I) & \text{if } i = n. \end{cases}$$

3. Mixed multiplicities of subsystems of parameters

Let $R = \bigoplus_{n\geq 0} R_n$ be a standard graded algebra over a field k, $\mathfrak{m} = \bigoplus_{n>0} R_n$ and $I = (x_1, \dots, x_n)$ a homogeneous ideal of R. Assume that $R[It] \cong A/J$, where $A = R[T_1, \dots, T_n]$. As we have seen in Section 2, A has a natural $\mathbb{N}^{n+1}$-graded structure. The degree lexicographical order on $\mathbb{N}^{n+1}$ induces a filtration F on $R[It]$. We may write

$$gr_F R[It] \cong A/J^*,$$

where J^* is the ideal generated by the initial elements of J. Moreover, A/J^* is a bigraded algebra with respect to the bigrading induced from A. By Lemma 2.1, the Bhattacharya function $\ell(\mathfrak{m}^u I^v/\mathfrak{m}^{u+1} I^v)$ coincides with the Hilbert function of A/J^*.

We shall see that the ideal J^* can be estimated if I is generated by a filter-regular sequence. Recall that a sequence $x_1, \dots, x_n$ of elements of R is called filter-regular with respect to I if $x_i \notin P$ for all associated prime ideals $P \not\supseteq I$ of $(x_1, \dots, x_{i-1})$, $i = 1, \dots, n$ (see e.g. [10]). For $i = 1, \dots, n$ we set

$$J_i := \cup_{m=1}^{\infty} (x_1, \dots, x_{i-1}) : I^m.$$

Note that J_i is equal to the intersection of all primary components of $(x_1, \dots, x_{i-1})$ whose associated prime ideals do not contain I.

Lemma 3.1. [10, Lemma 3.1] *Let I be generated by a filter-regular sequence $x_1, \dots, x_n$ with respect to I with $\deg x_1 \leq \dots \leq \deg x_n$. Let $P := (J_1 T_1, \dots, J_n T_n)$. Then*

$$J^* \subseteq P.$$

Set $I_i := (x_1, \dots, x_{i-1})R$, $i = 1, \dots, n$, and

$$L := (I_2 T_2, \dots, I_n T_n).$$

Since L is the ideal generated by the initial forms of the relations $x_i T_j - x_j T_i$ we have

$$L \subseteq J^*.$$

If I is generated by a subsystem of parameters with increasing degrees which is filter-regular, we can use Lemma 2.1 to show that the mixed multiplicities of A/J^* and A/L are the same. Note that every subsystem of parameters of R is filter-regular if R is a generalized Cohen-Macaulay ring.

Proposition 3.2. *Let I be a homogeneous ideal generated by a subsystem of parameters $x_1, \dots, x_n$ which is a filter-regular sequence with $\deg x_1 \leq \dots \leq \deg x_n$. Then the mixed multiplicities of A/J^* and A/L are equal.*

To prove Proposition 3.2 we shall need the following observation on the additivity of mixed multiplicities.

Lemma 3.3. *Let S be a standard bigraded algebra with $0 = Q_1 \cap \ldots \cap Q_s \cap Q$, where $Q_1, \ldots, Q_s$ are the relevant primary components of highest dimension. Then*

$$e_j(S) = \sum_{i=1}^{s} e_j(S/Q_i).$$

Proof. We use induction on s. If $s = 1$, from the exact sequence

$$0 \to S = S/Q_1 \cap Q \to S/Q_1 \oplus S/Q \to S/Q_1 + Q \to 0$$

we get

$$H_S(u,v) = H_{S/Q_1}(u,v) + H_{S/Q}(u,v) - H_{S/Q_1+Q}(u,v).$$

Since $\operatorname{rdim} S/Q_1 > \operatorname{rdim} S/Q \geq \operatorname{rdim} S/Q_1 + Q$,

$$e_j(S) = e_j(S/Q_1).$$

If $s > 1$, put $P = Q_2 \cap \ldots \cap Q_s \cap Q$. From the exact sequence

$$0 \to S = S/Q_1 \cap P \to S/Q_1 \oplus S/P \to S/Q_1 + P \to 0$$

we get

$$H_S(u,v) = H_{S/Q_1}(u,v) + H_{S/P}(u,v) - H_{S/Q_1+P}(u,v).$$

Since the associated prime ideals of P are not contained in the associated prime ideal of Q_1, $\operatorname{rdim} S/Q_1 + P < \operatorname{rdim} S/Q_1 = \operatorname{rdim} S/P = \operatorname{rdim} S$. Hence

$$e_j(S) = e_j(S/Q_1) + e_j(S/P).$$

By induction we may assume that

$$e_j(S/P) = \sum_{i=2}^{s} e_j(S/Q_i).$$

Therefore,

$$e_j(S) = \sum_{i=1}^{s} e_j(S/Q_i).$$ □

Proof of Proposition 3.2. By Lemma 3.3 we only need to show that the relevant primary components of highest dimension of J^* and L are equal. The ideal L has the decomposition

$$L = \cap_{i=1}^{n} (I_i, T_{i+1}, \ldots, T_n).$$

It is clear that every relevant primary component of highest dimension of L must be of the form $(\mathfrak{q}, T_{i+1}, \ldots, T_n)$ for some primary component $\mathfrak{q}$ of I_i with

$$\dim R/\mathfrak{q} = \dim R/I_i = \dim R - i + 1,\ i = 1, \ldots, n.$$

Let $\mathfrak{p}$ denote the associated prime ideal of $\mathfrak{q}$. Then $I \not\subseteq \mathfrak{p}$ because $\dim R/I < \dim R/\mathfrak{p}$. Therefore

$$J_i R_{\mathfrak{p}} = (\cup_{m=1}^{\infty} I_i : I^m) R_{\mathfrak{p}} = I_i R_{\mathfrak{p}}.$$

From this we deduce that $\mathfrak{q} \supseteq J_i$. On the other hand, the ideal $P = (J_1T_1, \ldots , J_nT_n)$ has the following decomposition

$$P = \cap_{i=1}^{n} (J_i, T_{i+1}, \ldots , T_n) \cap (T_1, \ldots , T_n).$$

So P is contained in all relevant primary components of highest dimension of L. But $L \subseteq J^* \subseteq P$ by Lemma 3.1. Therefore, the relevant primary components of highest dimension of L and J^* must be equal. □

Now we will compute the mixed multiplicities of A/L and therefore the mixed multiplicities $e_{ij}(\mathfrak{m}|I)$.

Lemma 3.4. [10, Lemma 1.6] *Let x be a homogeneous filter-regular element with respect to an ideal I of R with $\operatorname{ht} I \geq 2$, set $a := \deg x$. Then*

$$e(R/(x)) = ae(R).$$

Theorem 3.5. *Let I be a homogeneous ideal of R generated by a subsystem of parameters $x_1, \ldots , x_n$ which is a filter-regular sequence with respect to I with $\deg x_1 = a_1 \leq \ldots \leq \deg x_n = a_n$. Then*

$$e_{id-i-1}(\mathfrak{m}|I) = \begin{cases} 0 & \text{if } 0 \leq i \leq d-n-1, \\ a_1 \ldots a_{d-i-1} e(R) & \text{if } d-n \leq i \leq d-1. \end{cases}$$

Proof. By Lemma 2.1 and Lemma 3.2, $e_{ij}(A/J^*) = e_{ij}(A/L)$. Therefore we only need to compute the mixed multiplicities of A/L. As in the proof of Theorem 2.3 we have

$$H_{A/L}(u, v) = \sum_{\alpha_1+\ldots+\alpha_n=v} H_{R/I_{m(\alpha_1,\ldots,\alpha_n)}}(u) = \sum_{i=1}^{n} \binom{v+i-2}{i-1} H_{R/I_i}(u).$$

Since $\dim R/I_i = d - i + 1$,

$$H_{R/I_i}(u) = \frac{e(R/I_i)}{(d-i)!} u^{d-i} + \text{terms of lower degree.}$$

Using Lemma 3.4 we can easily show that

$$e(R/I_i) = a_1 \ldots a_{i-1} e(R).$$

Thus

$$H_{A/L}(u, v) = \sum_{i=1}^{n} \frac{a_1 \ldots a_{i-1} e(R)}{(d-i)!(i-1)!} u^{d-i} v^{i-1} + \text{terms of total degree} < d-1.$$

From this it follows show that

$$e_{id-i-1}(\mathfrak{m}|I) = \begin{cases} 0 & \text{if } 0 \leq i \leq d-n-1, \\ a_1 \dots a_{d-i-1} e(R) & \text{if } d-n \leq i \leq d-1. \end{cases} \qquad \square$$

Remark. The formula of Theorem 3.5 was posed as a problem in [10, Remark of Th. 3.3].

Using the characterization of the multiplicity of the Rees algebra $R[It]$ and the extended Rees algebra $R[It, t^{-1}]$ we immediately obtain the following result which was proved in [10] by a different method.

Corollary 3.6. [10, Corollary 3.6 and Corollary 4.4] *Let I be as in Theorem 3.5. Then*

$$e(R[It]) = \Big(1 + \sum_{i=1}^{n-1} a_1 \dots a_i\Big) e(R),$$

$$e(R[It, t^{-1}]) = \Big(1 + \sum_{i=l}^{n-1} a_1 \dots a_i\Big) e(R),$$

where l is the largest integer for which $a_l = 1$ ($l = 0$ and $a_1 \dots a_l = 1$ if $a_i > 1$ for all $i = 1, \dots, n$).

References

[1] Bhattacharya, P. B.: *The Hilbert function of two ideals.* Math. Proc. Cambridge Philos. Soc. **53** (1957), 568–575.

[2] Herzog, J.; Trung, N. V.; Ulrich, B.: *On the multiplicity of blow-up rings of ideals generated by d-sequences.* J. Pure Appl. Algebra **80** (1992), 273–297.

[3] Huneke, C.: *The theory of d-sequences and powers of ideals.* Adv. Math. **46** (1982), 249–279.

[4] Katz, D.; Mandal, S.; Verma, J. K.: *Hilbert functions of bigraded algebras.* To appear in Proceedings of the workshop on commutative algebra, Trieste 1992.

[5] Katz, D.; Verma, J.: *Extended Rees algebras and mixed multiplicities.* Math. Z. **202** (1989), 111–128.

[6] Rees, D.: *Generalisation of reduction and mixed multiplicities.* J. London Math. Soc. **29** (1984), 397–414.

[7] Raghavan, K. N.; Verma, J. K.: *Mixed Hilbert coefficients of homogeneous d-sequences and quadratic sequences.* J. Algebra **195** (1997), 211–232.

[8] Simis, A.; Trung, N. V.; Valla, G.: *The diagonal subalgebra of a blow-up algebra.* J. Pure Appl. Algebra **125** (1998), 305–328.

[9] Teissier, B.: *Cycles évanescent, sections planes, et conditions de Whitney.* Singularités à Cargése 1972, Astérisque **7-8** (1973), 285–362.

[10] Trung, N. V.: *Filter-regular sequences and multiplicities of blow-up rings of ideals of the principal class.* J. Math. Kyoto Univ. **333** (1993), 665–683.

[11] Verma, J. K.: *Rees algebras and mixed multiplicities.* Proc. Amer. Math. Soc. **104** (1988), 1036–1045.

[12] van der Waerden, B. L.: *On Hilbert's function, series of composition of ideals and a generalization of the theorem of Bezout.* Proc. K. Akad. Wet. Amsterdam **31** (1928), 749–770.

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