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Multi-helicoidal Euclidean Submanifolds of Constant Sectional Curvature

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Abstract. We classify n-dimensional multi-helicoidal submanifolds of nonzero constant sectional curvature and cohomogeneity one in the Euclidean space $\mathbb{R}^{2n-1}$, that is, n-dimensional submanifolds of nonzero constant sectional curvature in $\mathbb{R}^{2n-1}$ that are invariant under the action of an $(n-1)$-parameter subgroup of isometries of $\mathbb{R}^{2n-1}$ with no pure translations. This is accomplished by first giving a complete description of all these subgroups and then deriving a multidimensional version of a lemma due to Bour. We also prove that such submanifolds are precisely the ones that correspond to solutions of the generalized sine-Gordon and elliptic sinh-Gordon equations that are invariant by an $(n-1)$-dimensional subgroup of translations of the symmetry group of these equations.

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1. Introduction

The classical correspondence between solutions of the sine-Gordon and elliptic sinh-Gordon equations and surfaces in Euclidean three-space with constant negative and positive gaussian curvature, respectively, was extended to higher dimensions in [1], [13] and [11], [7], respectively, where similar correspondences were obtained between n-dimensional submanifolds

$M^n(c)$ with constant negative or positive sectional curvature in $(2n-1)$-dimensional Euclidean space $\mathbb{R}^{2n-1}$ and solutions of certain nonlinear systems of partial differential equations called the generalized sine-Gordon and elliptic sinh-Gordon equations, respectively (cf. §5 below). These systems will be referred to hereafter as GSGE and GEShGE.

The symmetry groups of local Lie-point transformations of the n-dimensional GSGE and GEShGE were determined in [14] and [8], [9], respectively, for $n \geq 3$. It was shown that they are finite-dimensional and consist only of translations. Moreover, the class $\mathcal{L}$ of all solutions invariant by an $(n-1)$-dimensional translation subgroup was explicitly described.

As pointed out in [2], it is in general a nontrivial problem to determine the submanifolds associated to a particular class of solutions. For the special subclass of $\mathcal{L}$ consisting of solutions that depend on a single variable, this was done in [12] and [4] (see also [7] for more general results), where the submanifolds were shown to be multi-rotational submanifolds with curves as profiles. The general class of submanifolds associated to elements of $\mathcal{L}$ was studied in [2]. It was shown, among other things, that the submanifolds carry a foliation by flat hypersurfaces, which are foliated themselves by curves with constant Frenet curvatures in the ambient space. However, a classification has not been achieved.

In this paper we prove that these submanifolds are precisely the multi-helicoidal n-dimensional submanifolds of nonzero constant sectional curvature and cohomogeneity one in $\mathbb{R}^{2n-1}$, that is, n-dimensional submanifolds of nonzero constant sectional curvature that are invariant under the action of an $(n-1)$-parameter subgroup of isometries of $\mathbb{R}^{2n-1}$ with no pure translations (see §2 for the precise definitions). Moreover, after providing a complete description of these subgroups, we are able to give explicit parametrizations of all such submanifolds. Our main tool is a multi-dimensional version of a lemma due to Bour ([3]; cf. also [6], pp. 129–130 and [5]), which is of independent interest and should have other applications.

We point out that the aforementioned results in [2] were actually derived for submanifolds of constant sectional curvature in arbitrary pseudo-Riemannian space forms. On the other hand, our proof that solutions in $\mathcal{L}$ correspond to multi-helicoidal submanifolds of cohomogeneity one (cf. Theorem 7 below) extends to this more general setting with minor changes. However, classifying all $(n-1)$-parameter subgroups of arbitrary pseudo-Riemannian space forms and deriving the corresponding Bour's-type lemmas would require a lengthy case-by-case study which we do not undertake here.

2. $(n-1)$-parameter subgroups of $\mathbb{ISO}(\mathbb{R}^{2n-1})$

A k-parameter subgroup of isometries of $\mathbb{R}^m$ is a continuous homomorphism $G : (\mathbb{R}^k, +) \to \mathbb{ISO}(\mathbb{R}^m)$ into the isometry group of $\mathbb{R}^m$. A 1-parameter subgroup of isometries R is said to be *generated by* G if there is $a = (a_1, \ldots, a_k) \in \mathbb{R}^k$ such that $R(s) = G(sa)$ for any $s \in \mathbb{R}$. We say that G *has no pure translations* if no one-parameter subgroup R generated by G is a pure translation, that is, given by $R(s)(x) = x + sv$ for some $v \in \mathbb{R}^m$ and all $x \in \mathbb{R}^m, s \in \mathbb{R}$.

Let $\mathbb{R}^{2n-1}$ be identified with the affine hyperplane

$$\mathbb{R}^{2n-1} = \{(x_1, \ldots, x_{2n}) \in \mathbb{R}^{2n}; x_{2n} = 1\}.$$

Denote

$$R(\theta, k) = \begin{pmatrix} \cos k\theta & \sin k\theta \\ -\sin k\theta & \cos k\theta \end{pmatrix}, \quad L(\phi, h) = \begin{pmatrix} 1 & h\phi \\ 0 & 1 \end{pmatrix}$$

and consider the $(n-1)$-parameter subgroup F of $\mathbb{ISO}(\mathbb{R}^{2n-1})$ given by

$$F(\phi) = F_1(\phi_1) \circ \ldots \circ F_{n-1}(\phi_{n-1}),$$

where $\phi = (\phi_1, \ldots, \phi_{n-1}) \in \mathbb{R}^{n-1}$ and $F_i(\phi_i) \in \mathbb{ISO}(\mathbb{R}^{2n-1})$, $1 \leq i \leq n-1$, is represented by the $2n \times 2n$ matrix $(R_i^1, \ldots, R_i^{n-1}, L_i)$ with 2×2 diagonal blocks

$$R_i^j = \begin{cases} R(\phi_i, k_i), & j = i, \\ 0, & j \neq i \end{cases}, \qquad L_i = L(\phi_i, h_i), \quad k_i, h_i \in \mathbb{R}, \quad k_i \neq 0.$$

The action of F has a simple description in terms of cylindrical coordinates $r_1, \theta_1, \ldots, r_{n-1}, \theta_{n-1}, z$ in $\mathbb{R}^{2n-1}$, which are related to cartesian coordinates by

$$(x_1, x_2, \ldots, x_{2n-3}, x_{2n-2}, x_{2n-1}) = (r_1 \exp i\theta_1, \ldots, r_{n-1} \exp i\theta_{n-1}, z).$$

In fact, the orbit of a point $P = (r_1, \theta_1, \ldots, r_{n-1}, \theta_{n-1}, z)$ under F is the $(n-1)$-dimensional submanifold of $\mathbb{R}^{2n-1}$ parametrized by

$$F(\phi)(P) = (r_1, \theta_1 + k_1\phi_1, \ldots, r_{n-1}, \theta_{n-1} + k_{n-1}\phi_{n-1}, z + \sum_{i=1}^{n-1} h_i\phi_i)$$

with flat induced metric

$$ds^2 = \sum_{i=1}^{n-1} (k_i r_i^2 + h_i^2) d\phi_i^2 + \sum_{i \neq j} h_i h_j d\phi_i d\phi_j.$$

Our first result shows that F is essentially the only $(n-1)$-parameter subgroup of $\mathbb{ISO}(\mathbb{R}^{2n-1})$ with no pure translations.

Theorem 1. *Let G be an $(n-1)$-parameter subgroup of isometries of $\mathbb{R}^{2n-1}$ with no pure translations. Then, there is $H \in \mathbb{O}(2n-1)$ and $B \in \mathbb{GL}(\mathbb{R}^{n-1})$ such that $G(\phi) = H^{-1} \circ F(B\phi) \circ H$ for any $\phi \in \mathbb{R}^{n-1}$.*

Proof. Denote by $\mathbb{I}$ the component of the identity of $\mathbb{ISO}(\mathbb{R}^{2n-1})$ and by $\mathcal{I}$ the Lie algebra of $\mathbb{I}$. Identify $\mathcal{I}$ with the Lie algebra of the $2n \times 2n$-matrices

$$\left\{ \begin{pmatrix} & u_1 \\ A & \vdots \\ & u_{2n-1} \\ 0 & 0 \end{pmatrix}, A^t = -A, \quad u_1, ..., u_{2n-1} \in \mathbb{R} \right\}$$

acting (as Killing fields) in $\mathbb{R}^{2n-1}$ by

$$((0, x), X) \mapsto X(0, x)^t$$

for $x \in \mathbb{R}^{2n-1}$ and $X \in \mathcal{I}$. Then, for $X, Y \in \mathcal{I}$ the Lie bracket $[\ ,\ ]$ of $\mathcal{I}$ is given by $[X, Y] = XY - YX$. It is easy to prove that $X \in \mathcal{I}$ is induced by a pure translation if and only if X is nilpotent, that is, the endomorphism $ad_X(Z) = [X, Z]$, $Z \in \mathcal{I}$, is nilpotent.

Let G_i be the 1-parameter subgroup generated by G given by $G_i(s) := G(se_i)$, where $e_1, \ldots, e_{n-1}$ is the canonical basis of $\mathbb{R}^{n-1}$. Then $G_i(s) = \exp sX_i$ for some $X_i \in \mathcal{I}$, where $\exp\colon \mathcal{I} \to \mathbb{I}$ is the exponential map. Since $G_i(s) \circ G_j(t) = G_j(t) \circ G_i(s)$ for all $s, t \in \mathbb{R}$, it follows that $[X_i, X_j] = 0$ for $1 \le i, j \le n-1$. Let Λ be the commutative Lie subalgebra of $\mathcal{I}$ spanned by $X_1, \ldots, X_{n-1}$. By the Jordan-Chevalley decomposition theorem (Proposition of [10], §4.2), we may write $X_i = S_i + N_i$, where N_i is nilpotent and S_i is semisimple, that is, the operator ad_{S_i} is diagonalizable over $\mathbb{C}$. We observe that $S_1, \ldots, S_{n-1}$ are linearly independent vectors, otherwise G would contain a pure translation, contrary to the hypothesis. Since any endomorphism commuting with X_i commutes with S_i and N_i, it follows that the Lie algebra $\mathcal{K}$ spanned by $S_1, \ldots, S_{n-1}$ is commutative. Moreover, $\mathcal{K}$ is a Cartan subalgebra of $\mathcal{I}$, because $\dim \mathcal{K} = n-1$.

Denote by E_i, $i = 1, \ldots, n-1$, the skew-symmetric matrix of $\mathcal{I}$ having 1 at the $(2i-1, 2i)$ entry, -1 at the $(2i, 2i-1)$ entry and 0 at the other entries. We observe that each E_i is semisimple. Let $\mathcal{H}$ be the commutative $(n-1)$-dimensional Lie subalgebra of $\mathcal{I}$ spanned by $E_1, \ldots, E_{n-1}$. Since $\mathcal{I}$ is a semisimple Lie algebra of rank $n-1$ which has only one Cartan subalgebra up to conjugation, there is $H \in \mathbb{I}$ such that $H\mathcal{K}H^{-1} = \mathcal{H}$. For any given i, it follows that HN_iH^{-1} commutes with all E_j. Some matrix computations then show that $HN_iH^{-1} = a_iE$ for some a_i, where $E \in \mathcal{I}$ has 1 at the $(n-1, n)$ entry and 0 at the other ones. Thus

$$H\Lambda H^{-1} \subset \operatorname{span}\{E_1, \ldots, E_{n-1}, E\}.$$

One may find a basis $R_1, \ldots, R_{n-1}$ of $H\Lambda H^{-1}$ such that

$$R_i = k_iE_i + h_iE$$

for some k_i, $h_i \in \mathbb{R}$, $1 \le i \le n-1$. Let $A = (a_{ij}) \in \mathbb{GL}(\mathbb{R}^{n-1})$ be given by

$$\sum_{j=1}^{n-1} a_{ij} HX_jH^{-1} = R_i.$$

Set $\mu_j = \sum_i a_{ij}\phi_i$ for $\phi = (\phi_1, \ldots, \phi_{n-1}) \in \mathbb{R}^{n-1}$. Then,

$$\begin{aligned} G(A\phi) &= G(\mu_1, \ldots, \mu_{n-1}) = G_1(\mu_1) \circ \ldots \circ G_{n-1}(\mu_{n-1}) \\ &= \exp(\mu_1 X_1) \circ \ldots \circ \exp(\mu_{n-1}X_{n-1}) = \exp(\sum_j \mu_j X_j), \end{aligned}$$

hence

$$\begin{aligned} H \circ G(A\phi) \circ H^{-1} &= \exp(\sum_j \mu_j HX_jH^{-1}) = \exp(\sum_j (\sum_i a_{ij}\phi_i) HX_jH^{-1}) \\ &= \exp(\sum_i \phi_i (\sum_j a_{ij} HX_jH^{-1})) = \exp(\sum_i \phi_i R_i) \\ &= F_1(\phi_1) \circ \ldots \circ F_{n-1}(\phi_{n-1}) = F(\phi), \end{aligned}$$

and the conclusion follows by setting $A = B^{-1}$. □

We say that an isometric immersion $f\colon M^n \to \mathbb{R}^{2n-1}$ is a *multi-helicoidal submanifold of cohomogeneity one* if it is invariant under the action of an $(n-1)$-parameter subgroup G of $\mathbb{ISO}(\mathbb{R}^{2n-1})$, that is, there exists an $(n-1)$-parameter subgroup T of $\mathbb{ISO}(\mathbb{M}^n)$ such that

$$G(\phi) \circ f = f \circ T(\phi), \quad \text{for any } \phi \in \mathbb{R}^{n-1}.$$

An isometric immersion g: $M^n \to \mathbb{R}^{2n-1}$ is said to be *congruent* to f if there exists $H \in \mathbb{ISO}(\mathbb{R}^{2n-1})$ such that $g = H \circ f$.

Corollary 2. *Any multi-helicoidal submanifold f: $M^n \to \mathbb{R}^{2n-1}$ of cohomogeneity one is congruent to a submanifold that is invariant under the action of F.*

Proof. Let G and T be $(n-1)$-parameter subgroups of $\mathbb{ISO}(\mathbb{R}^{2n-1})$ and $\mathbb{ISO}(\mathbb{M}^n)$, respectively, such that $G(\phi) \circ f = f \circ T(\phi)$ for any $\phi \in \mathbb{R}^{n-1}$. By Theorem 1, there is $H \in \mathbb{O}(2n-1)$ and $A \in \mathbb{GL}(\mathbb{R}^{n-1})$ such that $G(A\phi) = H^{-1} \circ F(\phi) \circ H$ for any $\phi \in \mathbb{R}^{n-1}$. Hence,

$$F(\phi) \circ (H \circ f) = (H \circ f) \circ (T \circ A)(\phi),$$

thus $H \circ f$ is invariant under F. □

3. A Bour's-type lemma

A parametrization $X(s, t_1, \ldots, t_{n-1})$ of a multi-helicoidal submanifold of cohomogeneity one is said to be *natural* if the coordinate hypersurfaces $s = s_0 \in \mathbb{R}$ are orbits of F and the induced metric has the form

$$d\sigma^2 = ds^2 + \sum_{i=1}^{n-1} U_i(s)^2 dt_i^2 + \sum_{i \neq j} h_i h_j dt_i dt_j. \tag{1}$$

We now prove the extension of Bour's lemma referred to in the introduction.

Lemma 3. 1) *Every multi-helicoidal submanifold M^n of cohomogeneity one in $\mathbb{R}^{2n-1}$ has locally a natural parametrization.*
2) *Suppose that $U_1(s), \ldots, U_{n-1}(s)$ and $h_1, \ldots, h_{n-1} \in \mathbb{R}$ satisfy $U_i^2 > h_i^2$, $1 \leq i \leq n-1$, and let $\lambda_i = \lambda_i(s)$ be defined by*

$$\lambda_i = \sqrt{U_i^2 - h_i^2}$$

if $n \geq 4$, and by

$$\lambda_1 = m\sqrt{U_1^2 - h_1^2}, \quad \lambda_2 = \frac{1}{m}\sqrt{U_2^2 - h_2^2}, \;\; m \neq 0,$$

if $n = 3$. Suppose further that $\sum_{i=1}^{n-1} (\lambda_i')^2 \leq 1$ everywhere and define

$$\lambda_n(s) = \int_0^s \psi(\tau)\xi(\tau)\, d\tau,$$

where

$$\psi(s) = \sqrt{1 - \sum_{i=1}^{n-1} (\lambda_i')^2} \quad \text{and} \quad \xi(s) = \sqrt{1 + \sum_{i=1}^{n-1} \frac{h_i^2}{\lambda_i^2}}.$$

Finally, define $\phi_i = \phi_i(s, t_i)$, $1 \leq i \leq n-1$, by

$$\phi_i = t_i - h_i \int_0^s \frac{\psi(\tau)}{\lambda_i^2(\tau)\xi(\tau)}\, d\tau$$

if $n \geq 4$ *and*

$$\phi_1 = mt_1 - h_1 \int_0^s \frac{\psi(\tau)}{\lambda_1^2(\tau)\xi(\tau)}\, d\tau, \quad \phi_2 = \frac{1}{m} t_2 - h_2 \int_0^s \frac{\psi(\tau)}{\lambda_2^2(\tau)\xi(\tau)}\, d\tau$$

if $n = 3$. *Then*

$$X(s, t_1, \dots, t_{n-1}) = (\lambda_1, \phi_1, \dots, \lambda_{n-1}, \phi_{n-1}, \lambda_n + \sum_{i=1}^{n-1} h_i \phi_i) \tag{2}$$

is a natural parametrization of a multi-helicoidal submanifold of cohomogeneity one in $\mathbb{R}^{2n-1}$ *with induced metric given by* (1).

Proof. 1) Let the intersection of M^n with the subspace

$$\mathbb{R}^n = \{(r_1, \theta_1, \dots, r_{n-1}, \theta_{n-1}, z); \theta_1 = \dots = \theta_{n-1} = 0\}$$

be locally parametrized by the curve $\lambda\colon (-\epsilon, \epsilon) \to \mathbb{R}^n$, $\lambda = (\lambda_1, \dots, \lambda_n)$, with $\lambda_i(\rho) > 0$ for all $\rho \in (-\epsilon, \epsilon)$, $1 \leq i \leq n$. Then, a local parametrization of M^n is

$$X(\rho, \phi_1, \dots, \phi_{n-1}) = (\lambda_1(\rho), \phi_1, \dots, \lambda_{n-1}(\rho), \phi_{n-1}, \lambda_n(\rho) + \sum_{i=1}^{n-1} h_i \phi_i),$$

where we assumed $k_i = 1$ for all $1 \leq i \leq n-1$ after a change of coordinates. The metric induced by X is

$$d\sigma^2 = \sum_{i=1}^{n} (\lambda_i')^2 d\rho^2 + \sum_{i=1}^{n-1} (\lambda_i^2 + h_i^2) d\phi_i^2 + 2\lambda_n' \sum_{i=1}^{n-1} h_i d\rho d\phi_i + \sum_{i \neq j} h_i h_j d\phi_i d\phi_j,$$

where the prime denotes derivative with respect to ρ. Let $t_1, \dots, t_{n-1}$ be locally defined by

$$dt_i = d\phi_i - \lambda_n' f_i d\rho,$$

where the functions $f_i = f_i(\rho)$ are to be determined. Then

$$\begin{gathered} d\sigma^2 = \left(\textstyle\sum_{i=1}^{n} (\lambda_i')^2 + (\lambda_n')^2 \sum_{i=1}^{n-1} f_i(h_i + g_i)\right) d\rho^2 + \sum_{i=1}^{n-1} (\lambda_i^2 + h_i^2) dt_i^2 \\ +2\lambda_n' \textstyle\sum_{i=1}^{n-1} g_i d\rho dt_i + \sum_{i \neq j} h_i h_j dt_i dt_j, \end{gathered}$$

where

$$g_i = f_i(\lambda_i^2 + h_i^2) + h_i + \sum_{j \neq i} h_i h_j f_j.$$

Let $A = A(\rho)$ be the $(n-1) \times (n-1)$-matrix with entries

$$\begin{cases} A_{ii} = (\lambda_i^2 + h_i^2) \\ A_{ij} = h_i h_j, \ i \neq j. \end{cases}$$

Since

$$\det A = \Pi_{i=1}^{n-1}\lambda_i^2 + \sum_{i=1}^{n-1} h_i^2 \Pi_{j\neq i}^{n-1}\lambda_j^2 > 0,$$

the linear system $Af = -h$, where $h = (h_1, \dots, h_{n-1})^t$, has a solution $f = (f_1, \dots, f_{n-1})^t$. Therefore, the $f_i' s$ can be chosen so that $g_i = 0$ for all $1 \le i \le n-1$. Explicitly, an easy computation shows that

$$f_i = \frac{-h_i}{\det A}\Pi_{j\neq i}^{n-1}\lambda_j^2 = -\frac{h_i}{\lambda_i^2\left(1+\sum_{j=1}^{n-1}\frac{h_j^2}{\lambda_j^2}\right)}. \tag{3}$$

Now observe that

$$\sum_{i=1}^{n}(\lambda_i')^2 + (\lambda_n')^2\sum_{i=1}^{n-1} h_i f_i = \sum_{i=1}^{n}(\lambda_i')^2 + \frac{(\lambda_n')^2}{\det A}\Pi_{i=1}^{n-1}\lambda_i^2 > 0,$$

hence a function $s = s(\rho)$ is locally well-defined by

$$ds^2 = \left(\sum_{i=1}^{n}(\lambda_i')^2 + (\lambda_n')^2\sum_{i=1}^{n-1} h_i f_i\right)d\rho^2 = \sum_{i=1}^{n-1} d\lambda_i^2 + \frac{\Pi_{i=1}^{n-1}\lambda_i^2}{\det A}d\lambda_n^2. \tag{4}$$

From

$$\frac{\partial(s, t_1, \dots, t_{n-1})}{\partial(\rho, \phi_1, \dots, \phi_{n-1})} = \sqrt{\sum_{i=1}^{n}(\lambda_i')^2 + (\lambda_n')^2\sum_{i=1}^{n-1} h_i f_i},$$

we have that $s, t_1, \dots, t_{n-1}$ define locally a system of coordinates. Let

$$\rho = \rho(s, t_1, \dots, t_{n-1}), \quad \phi_i = \phi_i(s, t_1, \dots, t_{n-1})$$

be the coordinate change. Since $\partial s/\partial\phi_i = 0$ for all $1 \le i \le n-1$, the chain rule gives $\partial\rho/\partial t_i = 0$ for all $1 \le i \le n-1$. Therefore $\rho = \rho(s)$ and, denoting $U_i^2(s) = \lambda_i^2(\rho(s)) + h_i^2$, we conclude that

$$X(s, t_1, \dots, t_{n-1}) = X(\rho(s), \phi_1(s, t_1, \dots, t_{n-1}), \dots, \phi_{n-1}(s, t_1, \dots, t_{n-1}))$$

is a natural parametrization of M^n.

2) We look for functions λ_i and ϕ_i of $s, t_1, \dots, t_{n-1}$ satisfying

$$ds^2 = \sum_{i=1}^{n-1} d\lambda_i^2 + \frac{1}{1+\sum_{j=1}^{n-1}\frac{h_j^2}{\lambda_j^2}}d\lambda_n^2, \tag{5}$$

$$U_i dt_i = \sqrt{\lambda_i^2 + h_i^2}\,(d\phi_i - f_i d\lambda_n), \quad 1 \le i \le n-1, \tag{6}$$

and

$$dt_i dt_j = (d\phi_i - f_i d\lambda_n)(d\phi_j - f_j d\lambda_n), \tag{7}$$

where f_i is given by (3). Equation (5) implies that $\lambda_i = \lambda_i(s)$ for all $1 \le i \le n$ and that

$$(\lambda_n')^2 = \left(1 - \sum_{i=1}^{n-1}(\lambda_i')^2\right)\left(1 + \sum_{i=1}^{n-1}\frac{h_i^2}{\lambda_i^2}\right). \tag{8}$$

Equations (6) and (7) yield

$$U_i = \sqrt{\lambda_i^2 + h_i^2}, \quad \frac{\partial \phi_i}{\partial t_j} = \delta_{ij}, \quad 1 \le i, j \le n-1,$$

for $n \ge 4$ and

$$U_1 = m\sqrt{\lambda_1^2 + h_1^2}, \quad U_2 = \frac{1}{m}\sqrt{\lambda_2^2 + h_2^2},$$

$$\frac{\partial \phi_1}{\partial t_2} = \frac{\partial \phi_2}{\partial t_1} = 0, \quad \frac{\partial \phi_1}{\partial t_1} = m, \quad \frac{\partial \phi_2}{\partial t_2} = \frac{1}{m}$$

for some $m \neq 0$ if $n = 3$. In both cases,

$$\frac{\partial \phi_i}{\partial s} = -\frac{h_i}{\lambda_i^2}\sqrt{\frac{1 - \sum_{j=1}^{n-1}(\lambda_j')^2}{1 + \sum_{j=1}^{n-1}\frac{h_j^2}{\lambda_j^2}}},$$

and the proof follows. □

Remarks 4. 1) It follows from Lemma 3 that the orbits of a multi-helicoidal submanifold M^n of cohomogeneity one in $\mathbb{R}^{2n-1}$ provide a foliation of M^n by flat geodesically parallel hypersurfaces. Moreover, any such hypersurface is foliated itself by curves with constant Frenet curvatures in the ambient space, namely, the orbits of the 1-parameter subgroups generated by F. These are the properties that were shown in [2] to be satisfied by n-dimensional submanifolds in $\mathbb{R}^{2n-1}$ which are associated to solutions of the GSGE and GEShGE that are invariant by an $(n-1)$-dimensional translation subgroup of their symmetry groups. They follow immediately from Theorem 7 below, according to which such submanifolds are precisely the multi-helicoidal submanifolds of nonzero constant sectional curvature and cohomogeneity one in $\mathbb{R}^{2n-1}$.

2) Suppose that G is an $(n-1)$-parameter subgroup of isometries of $\mathbb{R}^{2n-1}$ that contains a pure translation, say, $G(sa)(x) = x + sv$ for some vectors $a \in \mathbb{R}^{n-1}, v \in \mathbb{R}^{2n-1}$ and all $x \in \mathbb{R}^{2n-1}$, $s \in \mathbb{R}$. Then, it is easily seen that any submanifold M^n that is invariant under the action of G is isometric to an open subset of a Riemannian product $M^{n-1} \times \mathbb{R}$, the one-dimensional leaves of the product foliation correspondent to the $\mathbb{R}$-factor being immersed as straight lines in $\mathbb{R}^{2n-1}$ parallel to v.

4. Multi-helicoidal submanifolds of constant curvature

Our aim in this section is to classify n-dimensional multi-helicoidal submanifolds of cohomogeneity one and nonzero constant sectional curvature in $\mathbb{R}^{2n-1}$. This follows by putting together Lemma 3 and the following result.

Lemma 5. *Assume that the metric*

$$d\sigma^2 = ds^2 + \sum_{i=1}^{n-1} U_i(s)^2 dt_i^2 + \sum_{i\neq j} h_i h_j dt_i dt_j \tag{9}$$

has constant sectional curvature $c \neq 0$.

1) If $n \geq 4$, *then* $c < 0$, *at most one of the* h_i *is nonzero and, up to a coordinate change* $s \to \pm s + s_0$, $U_i(s) = \mu_i e^{\sqrt{-c}s}$, *where* $\mu_i \in \mathbb{R}$, $1 \leq i \leq n-1$, *satisfy* $\sum_{i=1}^{n-1} \mu_i^2 = 1$.

2) If $n = 3$, *then, up to a coordinate change* $s \to \pm s + s_0$, *one of the following possibilities holds:*

a) $c < 0$, $h_1 h_2 = 0$ *and* $U_i(s) = \mu_i e^{\sqrt{-c}s}$, *where* $\mu_1, \mu_2 \in \mathbb{R}$ *satisfy* $\mu_1^2 + \mu_2^2 = 1$.

b) $h_1 h_2 = 0$ *and* $U_1(s) = \mu_1 \phi(ks)$, $U_2(s) = \mu_2 \phi'(ks)$, *where* $\mu_1, \mu_2 \in \mathbb{R}$, $k = \sqrt{|c|}$, $\phi(s) = \cosh s$ *or* $\sinh s$ *if* $c < 0$ *and* $\phi(s) = \cos s$ *or* $\sin s$ *if* $c > 0$.

c) $h_1 h_2 \neq 0$ *and*

$$U_1^2 = B\phi(2ks) + D, \quad U_2^2 = a(B\phi(2ks) - D),$$

where $B^2 > D^2$, $a = h_1^2 h_2^2/(B^2 - D^2)$, $\phi(s) = \cosh s$ *if* $c < 0$ *and* $\phi(s) = \cos s$ *or* $\sin s$ *if* $c > 0$.

Proof. Set $g_{ij} = \langle \partial/\partial t_i, \partial/\partial t_j \rangle$, $1 \leq i, j \leq n-1$, where inner products are taken in the metric $d\sigma^2$. Thus, $g_{ii} = U_i^2$ and $g_{ij} = h_i h_j$ for $i \neq j$. We first show that $d\sigma^2$ having constant sectional curvature c is equivalent to the system of equations

$$\left.\begin{array}{ll} \text{i)} & 2g''_{jj} - (g'_{jj})^2 g^{jj} + 4cg_{jj} = 0, \ 1 \leq j \leq n-1, \\ \text{ii)} & g'_{ii} g'_{jj} + 4c(g_{ii} g_{jj} - h_i^2 h_j^2) = 0, \ 1 \leq i \neq j \leq n-1, \end{array}\right\} \tag{10}$$

where (g^{ij}) denotes the inverse matrix of (g_{ij}).

The sectional curvature $K(\frac{\partial}{\partial s}, \frac{\partial}{\partial t_j})$ along the plane spanned by $\frac{\partial}{\partial s}, \frac{\partial}{\partial t_j}$ is given by

$$\begin{aligned} K\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t_j}\right) g_{jj} &= \Big\langle \nabla_{\frac{\partial}{\partial t_j}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial s} - \nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t_j}} \frac{\partial}{\partial s}, \frac{\partial}{\partial t_j} \Big\rangle \\ &= \|\nabla_{\frac{\partial}{\partial t_j}} \frac{\partial}{\partial s}\|^2 - \frac{1}{2} \frac{\partial^2}{\partial s^2} \Big\langle \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_j} \Big\rangle. \end{aligned} \tag{11}$$

One can easily check that

$$\nabla_{\frac{\partial}{\partial t_j}} \frac{\partial}{\partial s} = \frac{1}{2} g'_{jj} \sum_{k=1}^{n-1} g^{kj} \frac{\partial}{\partial t_k}, \tag{12}$$

hence the first term on the right-hand-side of (11) equals

$$\frac{(g'_{jj})^2}{4} \left[\sum_{k=1}^{n-1} (g^{kj})^2 g_{kk} + \sum_{i\neq k}^{n-1} g^{kj} g^{ij} g_{ki} \right].$$

The expression between brackets is easily seen to be equal to g^{jj}, hence $K(\frac{\partial}{\partial s}, \frac{\partial}{\partial t_j}) = c$ if and only if equation (10) i) holds.

A similar computation shows that the sectional curvature $K(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j})$ along the plane spanned by $\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}$ is given by

$$K\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}\right)(g_{ii}g_{jj} - g_{ij}^2) = -\frac{1}{4}g'_{ii}g'_{jj},$$

hence $K(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j}) = c$ is equivalent to (10) ii).

Assume first that $n \geq 4$. Since M^n has constant sectional curvature $c \neq 0$ and the coordinate hypersurfaces $s = s_0 \in \mathbb{R}$ are flat, they must be umbilic in M^n and $c < 0$. Hence

$$\langle \nabla_{\frac{\partial}{\partial t_i}} \frac{\partial}{\partial t_i}, \frac{\partial}{\partial s} \rangle = \sqrt{-c}\, g_{ii}, \quad 1 \leq i \leq n-1, \tag{13}$$

up to a sign. By (12), the left-hand-side of (13) is equal to $-(1/2)g'_{ii}$, thus

$$g'_{ii} = -2\sqrt{-c}\, g_{ii}, \quad 1 \leq i \leq n-1.$$

Replacing into (10) ii) yields $h_i h_j = 0$ for all $1 \leq i \neq j \leq n-1$, and part 1) follows easily.

Assume now that $n = 3$. Then equations (10) reduce to

$$\left.\begin{array}{ll} \text{i)} & 2g''_{11} - \dfrac{(g'_{11})^2 g_{22}}{g_{11}g_{22} - h_1^2 h_2^2} + 4cg_{11} = 0, \\ \text{ii)} & 2g''_{22} - \dfrac{(g'_{22})^2 g_{11}}{g_{11}g_{22} - h_1^2 h_2^2} + 4cg_{22} = 0, \\ \text{iii)} & g'_{11}g'_{22} + 4c(g_{11}g_{22} - h_1^2 h_2^2) = 0. \end{array}\right\} \tag{14}$$

Notice that the last equation implies that g'_{11} and g'_{22} are nowhere vanishing. Plugging it into the others yields

$$g'_{22}g''_{11} = -2c(g_{11}g_{22})' = g''_{22}g'_{11}, \tag{15}$$

which implies $(g'_{11}/g'_{22})' = 0$. Thus, there exist $a, b \in \mathbb{R}$, $a \neq 0$, such that

$$g_{22} = ag_{11} + b. \tag{16}$$

From (16) and (15) we get

$$g''_{11} + 4cg_{11} + \frac{2cb}{a} = 0.$$

Set $D = -b/2a$. Then $g_{11}(s) = B\psi(2ks) + D$, $B \neq 0$, $k = \sqrt{|c|}$, and $g_{22}(s) = a(B\psi(2ks) - D)$, where, after a coordinate change $s \to \pm s + s_0$, we may assume that $\psi(s) = \cos s$ or $\sin s$ if $c > 0$ and $\phi(s) = \cosh s$, $\sinh s$ or e^s if $c < 0$. Replacing into (14) iii) gives

$$\psi^2(2ks) + \epsilon(\psi'^2(2ks))^2 = \frac{1}{B^2}\left(D^2 + \frac{h_1^2 h_2^2}{a}\right), \tag{17}$$

where $\epsilon = c/|c|$. Then one of the following possibilities holds:

i) $D = h_1h_2 = 0$; then $c < 0$, $\psi(s) = e^s$ and a) holds.

ii) $h_1h_2 = 0 \neq D$; then $B^2 = D^2$ and $\psi(s) = \cosh s$ if $c < 0$, which gives rise to case b).

iii) $h_1h_2 \neq 0$; then the right-hand-side of (17) equals 1, which implies that $B^2 > D^2$, $a = h_1^2h_2^2/(B^2 - D^2)$ and $\phi(s) = \cosh s$ if $c < 0$. Hence c) holds. □

Therefore, any n-dimensional multi-helicoidal submanifold $M^n(c)$ of cohomogeneity one and nonzero constant sectional curvature c in $\mathbb{R}^{2n-1}$ can be parametrized in terms of cylindrical coordinates in $\mathbb{R}^{2n-1}$ by (2), where λ_i, ϕ_i are given by Lemma 3 in terms of parameters h_i and functions U_i as in Lemma 5. For instance, if $n \geq 4$ then $M^n(c)$ has a natural parametrization

$$X(s, t_1, \ldots, t_{n-1}) = (\lambda_1(s), \phi_1(t_1, s), \lambda_2(s), t_2, \ldots, \lambda_{n-1}(s), t_{n-1}, \lambda_n(s) + h\phi_1),$$

where $\lambda_1(s) = \sqrt{\mu_1^2 e^{2ks} - h^2}$, $\lambda_i(s) = \mu_i e^{ks}$, $2 \leq i \leq n-1$, $k = \sqrt{-c}$, $\sum_{i=1}^{n-1} \mu_i^2 = 1$, $\phi_1 = t_1 - \frac{h}{\mu_1} \int_0^s e^{-k\tau} G(\tau) d\tau$, $\lambda_n(s) = \mu_1 \int_0^s e^{k\tau} G(\tau) d\tau$ and

$$G(s) = \frac{\sqrt{c\mu_1^2 e^{4ks} + [(1 + ch^2)\mu_1^2 - ch^2]e^{2ks} - h^2}}{\mu_1^2 e^{2ks} - h^2}.$$

For $h = 0$, it reduces to the classical Schur's n-dimensional pseudo-sphere of constant sectional curvature c.

The submanifold $M^n(c)$ is isometric to an open subset of hyperbolic space $\mathbb{H}^n(c)$ bounded by two concentric horospheres. More precisely, Euclidean space $\mathbb{R}^n$ endowed with the metric $d\sigma^2 = ds^2 + \sum_{i=1}^{n-1} \mu_i^2 e^{2ks} dt_i^2$, $k = \sqrt{-c}$, $\sum_{i=1}^{n-1} \mu_i^2 = 1$, is a model of $\mathbb{H}^n(c)$ in which the coordinate hypersurfaces $s = s_0 \in \mathbb{R}$ are horospheres with common center Ω, the s-coordinate curves being the orthogonal unit-speed geodesics through Ω. The translations $T(\phi)$, $\phi \in \mathbb{R}^{n-1}$, that leave the horospheres $s = s_0$ invariant form an $(n-1)$-parameter subgroup of isometries of $(\mathbb{R}^n, d\sigma^2)$ such that $F(\phi) \circ X = X \circ T(\phi)$. Hence, X sends each horosphere $s = s_0$, s_0 ranging on a certain open interval, onto an orbit of F.

Similarly, it is not difficult to check that the three-dimensional multi-helicoidal submanifolds of constant sectional curvature $c < 0$ (respectively, $c > 0$) for which the functions U_1, U_2 are given as in part 2b) or 2c) of Lemma 5 are isometric to open subsets of hyperbolic space $\mathbb{H}^3(c)$ (respectively, Euclidean sphere $\mathbb{S}^3(c)$) bounded by two tubes over a common geodesic γ. Each intermediate tube over γ is represented by a coordinate surface $s = s_0$, which is sent by X onto an orbit of F. The s-coordinate curves are the unit-speed geodesics orthogonal to the family of geodesically parallel tubes. In particular, this clarifies all the assertions in Theorem 3.1 of [2].

5. The GSGE and GEShGE

We denote by $\mathbb{O}^{2n}(c)$ either the hyperbolic space $\mathbb{H}^{2n}(c)$ or the Lorentzian space form $\mathbb{L}^{2n}(c)$ of constant sectional curvature c, according to $c < 0$ or $c > 0$, respectively. Recall that the *index of relative nullity* of a submanifold at a point x is the dimension of the kernel of its second fundamental form α at x, whereas its *first normal space* at x is the subspace of the normal space at x spanned by the image of α. The following result was proved in [7].

Theorem 6. *Let $M^n(c) \subset \mathbb{O}^{2n}(c)$ be a simply connected submanifold with flat normal bundle, vanishing index of relative nullity and nondegenerate first normal bundle. Then $M^n(c)$ admits a global principal parametrization $X: U \subset \mathbb{R}^n \to \mathbb{O}^{2n}(c)$ with induced metric*

$$ds^2 = \sum_i v_i^2\, du_i^2, \ v_i > 0, \tag{18}$$

and a smooth orthonormal normal frame $\xi_1, \dots, \xi_n$ such that its second fundamental form and normal connection satisfy

$$\alpha(\frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_j}) = v_i \delta_{ij} \xi_i, \quad \nabla^{\perp}_{\frac{\partial}{\partial u_i}} \xi_j = h_{ij} \xi_i, \tag{19}$$

where $h_{ij} = (1/v_i)\partial v_j / \partial u_i$. Moreover, the pair (v, h), where $v = (v_1, \dots, v_n)$ and $h = (h_{ij})$, satisfies the completely integrable system of PDEs

$$(\mathrm{I}) \quad \begin{cases} \text{i)} \quad \dfrac{\partial v_i}{\partial u_j} h_{ji} v_j, & \text{ii)} \quad \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + \displaystyle\sum_k h_{ki} h_{kj} + c v_i v_j = 0, \\ \text{iii)} \quad \dfrac{\partial h_{ik}}{\partial u_j} = h_{ij} h_{jk}, & \text{iv)} \quad \epsilon_i \dfrac{\partial h_{ij}}{\partial u_j} + \epsilon_j \dfrac{\partial h_{ji}}{\partial u_i} + \epsilon_k \displaystyle\sum_k h_{ik} h_{jk} = 0, \end{cases}$$

where $\epsilon_k = \langle \xi_k, \xi_k \rangle$ and $1 \le i \ne j \ne k \ne i \le n$. Conversely, let (v, h) be a solution of (I) *on an open simply connected subset $U \subset \mathbb{R}^n$ such that $v_i > 0$ everywhere, $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $2 \le i \le n$ (respectively, $\epsilon_i = 1$ for $1 \le i \le n$). Then there exists an immersion $f: U \to \mathbb{O}^{2n}(c)$ with flat normal bundle, vanishing index of relative nullity and induced metric $ds^2 = \sum_i v_i^2\, du_i^2$ of constant sectional curvature $c > 0$ (respectively, $c < 0$).*

By embedding Euclidean space $\mathbb{R}^{2n-1}$ as a totally umbilical hypersurface of $\mathbb{O}^{2n}(c)$, the above result was used in [7] to show that simply connected submanifolds $M^n(c)$ of $\mathbb{R}^{2n-1}$, free of weak-umbilics when $c > 0$, are in correspondence with solutions of the system

$$(\mathrm{II}) \quad \begin{cases} \text{i)} \quad \dfrac{\partial v_i}{\partial u_j} h_{ji} v_j, & \text{ii)} \quad \dfrac{\partial h_{ij}}{\partial u_i} + \dfrac{\partial h_{ji}}{\partial u_j} + \displaystyle\sum_k h_{ki} h_{kj} + c v_i v_j = 0, \\ \text{iii)} \quad \dfrac{\partial h_{ik}}{\partial u_j} = h_{ij} h_{jk}, & \sum_{i=1}^n \epsilon_i v_i^2 = -1/c, \end{cases}$$

which is either the GSGE or the GEShGE, according to $c < 0$ or $c > 0$, respectively. Recall from [11] that a point $x \in M^n(c)$ is said to be *weak-umbilic* if there is a unit normal vector ζ at x such that $A_\zeta = \sqrt{c}\, I$, where A_ζ denotes the shape operator in the direction of ζ.

It was shown in [14] and [8], [9] that all solutions of the GSGE or the GEShGE, respectively, that are invariant by an $(n-1)$-dimensional translation subgroup of their symmetry groups have the form

$$v_i = v_i(\xi), \quad h_{ij} = h_{ij}(\xi), \quad \xi = \sum_{i=1}^n a_i u_i. \tag{20}$$

We now prove that the submanifolds that are associated to such solutions are precisely the multi-helicoidal submanifolds of cohomogeneity one.

Theorem 7. *A solution of either the* GSGE *or the* GEShGE (*system* (II)) *is invariant under an* $(n-1)$*-dimensional translation subgroup of its symmetry group if and only if it is associated to a multi-helicoidal submanifold of cohomogeneity one with constant sectional curvature* c *and no weak-umbilics when* $c > 0$.

Proof. Assume first that $M^n(c) \subset \mathbb{R}^{2n-1}$ is a multi-helicoidal submanifold of cohomogeneity one, constant sectional curvature c and free of weak-umbilics when $c > 0$. We may consider $M^n(c)$ isometrically immersed into $\mathbb{O}^{2n}(c)$ by embedding $\mathbb{R}^{2n-1}$ as a totally umbilical hypersurface of $\mathbb{O}^{2n}(c)$. It is easily seen that $M^n(c)$ having no weak-umbilics as a submanifold of $\mathbb{R}^{2n-1}$ is equivalent to the first normal spaces of $M^n(c)$ being everywhere nondegenerate as a submanifold of $\mathbb{O}^{2n}(c)$.

Let $X\colon U \subset \mathbb{R}^n \to \mathbb{O}^{2n}(c)$ be a principal parametrization of $M^n(c)$ given by Theorem 6. Since every isometry of $\mathbb{R}^{2n-1}$, regarded as an umbilical hypersurface of $\mathbb{O}^{2n}(c)$, is the restriction of an isometry of $\mathbb{O}^{2n}(c)$, we have that $M^n(c) \subset \mathbb{O}^{2n}(c)$ is invariant by an $(n-1)$-parameter subgroup of isometries of $\mathbb{O}^{2n}(c)$, which we still denote by F. Endow U with the metric $ds^2 = \sum_i v_i^2\, du_i^2$ induced by X. We will show that the solution (v, h) of system (II), $v = (v_1, \ldots, v_n)$, $h = (h_{ij})$, associated to $M^n(c)$ has the form (20). Let T be the $(n-1)$-parameter subgroup of isometries of (U, ds^2) induced by F, that is,

$$X \circ T(\phi) = F(\phi) \circ X$$

for all $\phi \in \mathbb{R}^{n-1}$. Then, the second fundamental forms of X and $X \circ T(\phi)$ satisfy

$$\alpha_X(T(\phi)(u))(T(\phi)_*X, T(\phi)_*Y) = \alpha_{X\circ T(\phi)}(u)(X, Y) = F(\phi)_*\alpha_X(u)(X, Y)$$

for all $u \in U$ and $X, Y \in T_uU$. Set $\frac{\partial}{\partial u_i} = v_i X_i$, $1 \le i \le n$. Then, from

$$\alpha_X(T(\phi)(u))(T(\phi)_*X_i, T(\phi)_*X_j) = F(\phi)_*\alpha_X(u))(X_i, X_j) = 0, \quad i \ne j,$$

it follows easily that $X_i \circ T(\phi) = T(\phi)_*X_i$. We obtain from the first equation in (19) that

$$\begin{aligned} v_i(T(\phi)(u))\xi_i(T(\phi)(u)) &= \alpha_X(T(\phi)(u))(X_i(T(\phi)(u)), X_i(T(\phi)(u))) \\ &= F(\phi)_*\alpha_X(u)(X_i(u), X_i(u)) \\ &= v_i(u)F(\phi)_*\xi_i(u), \end{aligned}$$

which shows that $\xi_i \circ T(\phi) = F(\phi)_*\xi_i$ and $v_i \circ T(\phi) = v_i$. Moreover, from

$$\nabla^\perp_{T(\phi)_*X} F(\phi)_*\xi = F(\phi)_*\nabla^\perp_X \xi,$$

we get using the second equation in (19) that

$$\begin{aligned} h_{ij}(T(\phi)(u)) &= \langle \nabla^\perp_{\frac{\partial}{\partial u_i}(T(\phi)(u))}\xi_j(T(\phi)(u)), \xi_i(T(\phi)(u))\rangle \\ &= \langle \nabla^\perp_{T(\phi)_*\frac{\partial}{\partial u_i}(u)} F(\phi)_*\xi_j(u), F(\phi)_*\xi_i(u)\rangle \\ &= \langle F(\phi)_*\nabla^\perp_{\frac{\partial}{\partial u_i}(u)}\xi_j(u), F(\phi)_*\xi_i(u)\rangle = h_{ij}(u). \end{aligned}$$

Therefore, the $v_i's$ and $h_{ij}'s$ are constant along the orbits of T. Hence, there exist smooth functions $\theta\colon U \to \mathbb{R}$ and $\bar{v}_i, \bar{h}_{ij}\colon \mathbb{R} \to \mathbb{R}$ such that

$$v_i = \bar{v}_i \circ \theta, \quad h_{ij} = \bar{h}_{ij} \circ \theta, \;\; 1 \le i \ne j \le n.$$

Since

$$\bar{h}_{ij} \circ \theta = h_{ij} = \frac{1}{v_i}\frac{\partial v_j}{\partial u_i} = \frac{\bar{v}_j' \circ \theta}{\bar{v}_i \circ \theta}\theta_{u_i},$$

there exist smooth functions $f_i\colon \mathbb{R} \to \mathbb{R}$, $1 \le i \le n$, such that

$$\theta_{u_i} = f_i \circ \theta.$$

The integrability conditions of the above equations yield

$$f_i' f_j = f_i f_j', \quad 1 \le i \ne j \le n.$$

We can assume $f_1 \ne 0$. Then, there exist constants $\lambda_2, \ldots, \lambda_n$ such that $f_i = \lambda_i f_1$, $2 \le i \le n$. Thus,

$$\left(\frac{\partial}{\partial u_i} - \lambda_i \frac{\partial}{\partial u_1}\right)\theta = 0, \;\; 2 \le i \le n.$$

Setting $\xi = u_1 + \sum_{i=2}^{n} \lambda_i u_i$, we conclude that $v_i = v_i(\xi), \quad h_{ij} = h_{ij}(\xi)$.

Conversely, assume that $M^n(c) \subset \mathbb{R}^{2n-1}$ is associated to a solution of system (II) of the form (20). As before, consider $M^n(c)$ as a submanifold of $\mathbb{O}^{2n}(c)$ and let $X\colon U \to \mathbb{O}^{2n}(c)$ be a principal parametrization of $M^n(c)$ as in Theorem 6 with induced metric given by (18), where we may assume

$$U = \{u \in \mathbb{R}^n \mid b_1 < \xi < b_2\}, \;\; b_1, b_2 \in \mathbb{R}.$$

Define the $(n-1)$-parameter group of translations T on U by

$$T(\phi)(u) = u + \sum_{i=1}^{n-1} \phi_i Y_i,$$

where $\phi = (\phi_1, \ldots, \phi_{n-1})$ and $Y_1, \ldots, Y_{n-1}$ is an arbitrary basis of the hyperplane $\xi = 0$. Since $T(\phi)_* \frac{\partial}{\partial u_i}(u) = \frac{\partial}{\partial u_i}(T(\phi)(u))$ and the $v_i's$ are constant along the orbits $\xi = \xi_0 \in (b_1, b_2)$ of T, each $T(\phi)$ is an isometry of (U, ds^2). We claim that there exist isometries $G(\phi)$ of $\mathbb{O}^{2n}(c)$ such that

$$X \circ T(\phi) = G(\phi) \circ X. \tag{21}$$

Define a vector bundle isometry $\mathcal{T}(\phi)$ between the normal bundles of X and $X \circ T(\phi)$ by setting $\mathcal{T}(\phi)(\xi_i) = \xi_i \circ T(\phi)$, $1 \le i \le n$, where $\xi_1, \ldots, \xi_n$ is the orthonormal normal frame given by Theorem 6. Then, we have that

$$\begin{aligned}\alpha_{X\circ T(\phi)}(X_i, X_j) &= \alpha_X(T(\phi)_* X_i, T(\phi)_* X_j) = \alpha_X(X_i \circ T(\phi), X_j \circ T(\phi))\\ &= v_i \circ T(\phi)\delta_{ij}\xi_i \circ T(\phi) = \mathcal{T}(\phi)\alpha_X(X_i, X_j).\end{aligned}$$

Moreover,

$$\begin{aligned}\langle \nabla^{\perp}_{X_i \circ T(\phi)} \mathcal{T}(\phi)(\xi_j), \mathcal{T}(\phi)(\xi_i)\rangle = h_{ij} \circ T(\phi) = h_{ij} = \langle \nabla^{\perp}_{X_i} \xi_j, \xi_i \rangle = \\ \langle \mathcal{T}(\phi)(\nabla^{\perp}_{X_i} \xi_j), \mathcal{T}(\phi)(\xi_i)\rangle,\end{aligned} \tag{22}$$

hence $\nabla^{\perp}_{X_i \circ T(\phi)} \mathcal{T}(\phi)(\xi_j) = \mathcal{T}(\phi)(\nabla^{\perp}_{X_i} \xi_j)$ for all $1 \leq i \neq j \leq n$. Thus, the vector bundle isometry $\mathcal{T}(\phi)$ preserves the second fundamental forms and normal connections of X and $X \circ T(\phi)$. The claim now follows from the fundamental theorem of submanifolds.

Let $\bar{G}(\phi)$ denote the restriction of $G(\phi)$ to $\mathbb{R}^{2n-1}$ and let $\bar{X}$ be the parametrization of $M^n(c)$ as a submanifold of $\mathbb{R}^{2n-1}$ induced by X. Then $\bar{G}(\phi) \circ \bar{X} = \bar{X} \circ T(\phi)$, which implies that

$$\bar{G}(\phi_1 + \phi_2) \circ \bar{X} = \bar{G}(\phi_1) \circ \bar{X} + \bar{G}(\phi_2) \circ \bar{X} \quad \text{for any } \ \phi_1, \phi_2 \in \mathbb{R}^{n-1}. \tag{23}$$

Now observe that $X(U)$ cannot be contained in any totally geodesic hypersurface of $\mathbb{O}^{2n}(c)$, since the first normal bundle of X coincides with its normal bundle by the first equation in (19). Hence $\bar{X}(U)$ cannot be contained in any hyperplane of $\mathbb{R}^{2n-1}$. It follows from (23) that $\bar{G}$ is an $(n-1)$-parameter subgroup of $\mathbb{ISO}(\mathbb{R}^{2n-1})$ that leaves $M^n(c)$ invariant. By Remark 4-2), $\bar{G}$ contains no pure translations, since a Riemannian manifold with nonzero constant sectional curvature is irreducible. We conclude that $M^n(c)$ is a multi-helicoidal submanifold of cohomogeneity one. $\square$

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