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On a Covering Problem in the Plane

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Abstract. We make a remark to a covering problem stated by V. Boltyanski, H. Martini and P. S. Soltan.

At the end of their interesting book ,,Excursions into combinatorial geometry" [1], V. Boltyanski, H. Martini and P. S. Soltan present a list of problems. The latest is the following:

Problem. Let Q be a class of figures in the plane $\mathbb{R}^2$. We say that a convex figure $W \subset \mathbb{R}^2$ is a *covering set* for the class Q if any figure M from the class Q can be covered by a congruent copy of W. If no convex figure $W' \subset W$ is a covering set for the class Q, then we say that W is a *minimal, convex covering set* for the class Q.

We consider the following classes of figures in the plane:
Q_1 is the class of all subsets of diameter 1;
Q_2 is the class of all compact, convex subsets of diameter 1;
Q_3 is the class of all figures of constant width 1;
Q_4 is the class of all Reuleaux polygons of width 1;
Q_5 is the class of all Reuleaux pentagons of width 1.

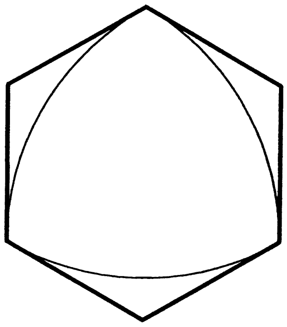
It is easy to show that the classes Q_1, Q_2, Q_3, Q_4 are equivalent in the following sense: any minimal, convex covering set for one of these classes is at the same time a minimal, convex covering set for each other class. The problem is to decide whether also the class Q_5 is equivalent (in this sense) to the previous four classes.

It is very easy to show, that the answer of this question as it is formulated is "no!". The following theorem holds:

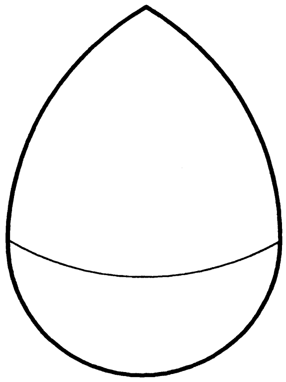
Theorem. *There are minimal covering sets for all planar sets of constant width δ, except for closed Reuleaux triangles.*

This condition is simply to state by looking at some well-known covering sets. For instance let D_1 be J. Pál's hexagon and let D_2 be the cover which H. G. Eggleston described in 1963. (This is the union of a circle and a Reuleaux triangle of diameter δ arranged in such a way, that two corners of the Reuleaux triangle are situated on the periphery of the circle [2].)

In D_1 a Reuleaux triangle can be embedded in two positions only and in D_2 it can be embeded even in only one way (see the figure).



D_1



D_2

If we denote the set of the corners of D_i by C_i, then $D_i \backslash C_i$ can not cover a closed Reuleaux triangle of the given breadth. On the other hand one can cover each planar set of constant width δ different from the Reuleaux triangles with the sets $D_i \backslash C_i$. Only the Reuleaux triangles have corners with a projection cone of the measure $\frac{2\pi}{3}$. Clearly, by Zorn's lemma it follows, that $D_i \backslash C_i$ must contain a minimal set with the same covering properties.

These remarks show that the problem must be reformulated as a covering problem for *open* sets of constant width.

References

[1] Boltyanski, V.; Martini, H.; Soltan, P. S.: *Excursions into combinatorial geometry.* Springer Verlag Berlin Heidelberg New York 1997.

[2] Eggleston, H. G.: *Minimal universal covers in E^n.* Israel J. Math. **1** (1963), 149–155.

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