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Decomposition of Isometric Immersions between Warped Products

Michael Schaaf

Mathematisches Institut der Universität zu Köln
Weyertal 86-90, D-50931 Köln, Germany

Abstract. A decomposition theorem for isometric immersions between two arbitrary warped products is derived. Thereby the well known decomposition theorems of Moore and Nölker are generalized.

Introduction

One is often interested in the decomposition of mathematical objects as a first step for a classification. The decomposition of riemannian manifolds into riemannian products is well described by the theorem of de Rham, see [7]. Moore proved in [4, Lemma] the decomposition of an isometric immersion $f : \prod_{i=0}^{k} M_i \to \mathbb{R}^n$ from a riemannian product into the euclidean n-space $\mathbb{R}^n$, which is based on the fact that there exist many product decompositions $\mathbb{R}^n \simeq \mathbb{R}^{n_0} \times \cdots \times \mathbb{R}^{n_k}$. This theorem was an essential step in Ferus' classification of symmetric submanifolds of $\mathbb{R}^n$, see [3]. The real space forms $N^n_\varkappa$ of constant curvature $\varkappa \neq 0$ do not admit local representations by riemannian products. In order to generalize Ferus' result to $N^n_\varkappa$ Backes and Reckziegel introduced extrinsic products $\Phi : \prod_{i=0}^{k} N_i \to N^n_\varkappa$, special isometric immersions of higher codimension, see [1]; and Takeuchi made use of a generalization of Moore's lemma to the Lorentz space, see [10, Lemma 1.4]. But Nölker proved in [6, Theorem 7] that the best substitute for the product representations of $\mathbb{R}^n$ are the local representations of $N^n_\varkappa$ by warped products (see Definition 2). Herewith, he could generalize Moore's result to isometric immersions of warped products, see [6, Theorem 16]:

Theorem. *Let $f : M \to N^n_\varkappa$ be an isometric immersion of a connected warped product $M = M_0 \times_\rho \prod_{i=1}^{k} M_i$ into a real space form $N^n_\varkappa$ of constant curvature $\varkappa$, and suppose that the second fundamental form of f satisfies the condition $h|E_i \times_M E_j \equiv 0$ for $i \neq j$, where* *the E_i are the subbundles of the tangent bundle TM tangential to the factors M_i. Then there exist an isometry $\Phi : N_0 \times_\sigma \prod_{i=1}^k N_i \to G$ from a warped product $N_0 \times_\sigma \prod_{i=1}^k N_i$ onto an open, dense subset $G \subset N_\varkappa^n$ and isometric immersions $f_i : M_i \to N_i$ such that*

$$f = \Phi \circ (f_0 \times \ldots \times f_k) .$$

Using this result, for instance, Dillen and Nölker could classify normally flat semi-parallel submanifolds of real space forms, see [2].

This article is devoted to generalizations of Nölker's result, where the ambient space is an arbitrary pseudoriemannian manifold. To formulate them we make use of the notions of WP-nets and net morphisms, the first of which has been introduced in [5] and specializes the notion of nets.

Definition 1. 1. *A family $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ of totally integrable subbundles E_i of the tangent bundle TM of a manifold M is called a net on M if and only if TM is the direct sum $\bigoplus_{i=0}^k E_i$. The foliation induced by E_i shall be denoted by $L_i^{\mathcal{E}}$ and its leaf through some point $p \in M$ by $L_i^{\mathcal{E}}(p)$. If in this situation M is pseudoriemannian, each subbundle E_i is non-degenerate, $E_i \perp E_j$ for $i \neq j$ and moreover for every $i = 1, \ldots, k$ the subbundle E_i is spherical[1] and its orthogonal complement $E_i^\perp$ is autoparallel[2] then the net $\mathcal{E}$ is called a WP-net on M and $(M, \mathcal{E})$ a WP-netted manifold.*

2. *A C^∞-map $f : M \to N$ is called a net morphism with respect to some nets $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on M and $\mathcal{F} = (F_i)_{i=0,\ldots,k}$ on N if and only if $f_* E_i(p) \subset F_i(f(p))$ for every $p \in M$, which is equivalent to the fact that f maps every leaf of the foliation $L_i^{\mathcal{E}}$ differentiably into a leaf of the foliation $L_i^{\mathcal{F}}$ $(i = 0, \ldots, k)$.*

It was proved in [5, Proposition 4] that the product net of a warped product is a WP-net, and in [5, Corollary 1], conversely, that a WP-netted manifold $(M, \mathcal{E})$ is locally a warped product, that means for every $p \in M$ there exist an open neighbourhood U of p in M, a warped product $N = N_0 \times_\rho \prod_{i=1}^k N_i$ and an isometry $\Phi : N \to U$ which is a net morphism with respect to the product net of N and $\mathcal{E}$. Therefore, isometric net morphisms between WP-netted manifolds can locally be decomposed in the sense of the following Proposition 1, which motivates us to derive conditions for their existence.

In general the occurance of WP-nets cannot be guaranteed; for instance on an irreducible simply connected riemannian symmetric space of non-constant curvature there exist no WP-nets, see [9, Theorem 15]. Hence, to get a chance that Nölker's theorem keeps valid for arbitrary pseudoriemannian manifolds N instead of $N_\varkappa^n$, we have to suppose the existence of a WP-net on N, which is adapted to the immersion f at least at one point. In fact, this is the crucial point as is proved by the following main results of this article.

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[1] A non-degenerate subbundle E of TM is said to be spherical if and only if there exists a section $H^E \in \Gamma(E^\perp)$ with the following properties: for every $X, Y \in \Gamma(E)$ we have $(\nabla_X Y)^{E^\perp} = \langle X, Y \rangle \cdot H^E$ and $(\nabla_X H)^{E^\perp} = 0$, where $(\cdot)^{E^\perp}$ denotes the orthogonal projection $TM \to E^\perp$; H^E is called the mean curvature normal of E.

[2] A non-degenerate subbundle E of TM is said to be autoparallel if and only if $\nabla_X Y \in \Gamma(E)$ for every $X, Y \in \Gamma(E)$, that means if and only it is spherical with vanishing mean curvature normal.

Theorem 1. *Let* $M = M_0 \times_\rho \prod_{i=1}^k M_i$ *be a connected warped product with product net* $\mathcal{E} = (E_i)_{i=0,\dots,k}$, N *a pseudoriemannian manifold, which is equipped with a WP-net* $\mathcal{F} = (F_i)_{i=0,\dots,k}$, *and* $\bar{p} \in M$ *some point. Furthermore, let* $f : M \to N$ *be an isometric immersion satisfying the following conditions*

$$h|E_i \times_M E_j \equiv 0 \quad \text{for } i \neq j \,, \tag{D}$$

$$f(L_i^{\mathcal{E}}(\bar{p})) \subset L_i^{\mathcal{F}}(f(\bar{p})) \quad \text{for every } i = 0, \dots, k \,. \tag{I}$$

Then f *is a net morphism, and therefore* f *can locally be decomposed in the manner of Proposition* 1.

Combining Theorem 1 with Proposition 1 we immediately obtain:

Corollary 1. *Let* $f : M \to N$ *be an isometric immersion between two warped products* $M = M_0 \times_\rho \prod_{i=1}^k M_i$ *and* $N = N_0 \times_\sigma \prod_{i=1}^k N_i$. *If* M *is connected,* f *satisfies the conditions* (D) *and* (I) *of Theorem* 1 *with respect to the product nets* $\mathcal{E}$ *and* $\mathcal{F}$ *of* M *and* N, *respectively, and some point* $\bar{p} \in M$ *and if the representations of* M *and* N *as warped products are normalized with respect to* $\bar{p}$ *and its image* $f(\bar{p})$ *(see Definition* 2*), respectively, then there exist isometric immersions* $f_i : M_i \to N_i$ *for* $i = 0, \dots, k$ *such that*

$$f = f_0 \times \cdots \times f_k \,.$$

As Nölker's examples show, the domain of a WP-net often is only an open subset of the considered manifold. The following result enables us to handle such situations in general.

Theorem 2. *Let* N *be a pseudoriemannian manifold,* $\Phi : N_0 \times_\sigma \prod_{i=1}^k N_i \to G$ *an isometry from a warped product onto an open subset* G *of* N *and* $\mathcal{F}$ *the WP-net on* G *induced by* Φ. *Furthermore, let* $M := M_0 \times_\rho \prod_{i=1}^k M_i$ *be a connected warped product with product net* $\mathcal{E}$ *and* $f : M \to N$ *an isometric immersion satisfying the conditions* (D) *and* (I) *of Theorem* 1 *with respect to the given nets* $\mathcal{E}$ *and* $\mathcal{F}$ *and some point* $\bar{p} \in M$. *Then the image* $f(M)$ *lies completely in* G*; if moreover the representations of* M *and* G *as warped products are normalized with respect to* $\bar{p}$ *and* $\Phi^{-1}(f(\bar{p}))$, *respectively, then there exist isometric immersions* $f_i : M_i \to N_i$ *for* $i = 0, \dots, k$ *such that*

$$f = \Phi \circ (f_0 \times \cdots \times f_k) \,.$$

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1. Warped product nets and net morphisms

Firstly, we recall the definition of warped products and show some basic results.

Definition 2. *Let* $M_0, \dots, M_k$ *be manifolds,* $M := \prod_{i=0}^k M_i$ *the product and* $\pi_i : M \to M_i$ *the canonical projections* ($i = 0, \dots, k$). *A pseudoriemannian metric* $\langle \cdot, \cdot \rangle$ *on* M *is said to be a warped product metric if there exist pseudoriemannian metrics* $\langle \cdot, \cdot \rangle_0, \dots, \langle \cdot, \cdot \rangle_k$ *on*

$M_0, \dots, M_k$, respectively, and a C^∞-function $\rho := (\rho_1, \dots, \rho_k) : M_0 \to \mathbb{R}^k_+$ such that for every $X, Y \in \mathfrak{X}(M)$ we have

$$\langle X, Y \rangle \;=\; \langle \pi_{0*}X, \pi_{0*}Y \rangle_0 + \sum_{i=1}^{k} (\rho_i \circ \pi_0)^2 \cdot \langle \pi_{i*}X, \pi_{i*}Y \rangle_i \;. \tag{1}$$

In this situation the pseudoriemannian manifold $(M, \langle \cdot, \cdot \rangle)$ is called a warped product; we denote it by $M_0 \times_\rho \prod_{i=1}^k M_i$ and call ρ its warping function. If we fix some point $\bar{p} \in M$, then we can choose the representing data $\langle \cdot, \cdot \rangle_i$ and ρ_i of $\langle \cdot, \cdot \rangle$ in such a way that we have $\rho(\bar{p}_0) = (1, \dots, 1)$; in this situation we say that the representation of M as a warped product is normalized with respect to $\bar{p}$.

Remark 1. In [5, Theorem 2] there are derived sufficient conditions for a WP-netted manifold to be a warped product.

In the proof of the main results we make use of the following rules for the Levi-Civita connection of a WP-netted manifold:

Lemma 1. *Let $\mathcal{E} = (E_0, E_1)$ be a WP-net on a pseudoriemannian manifold M, $H \in \Gamma(E_0)$ the mean curvature normal of E_1 and $TM \to E_i$, $v \mapsto v^i$ the orthogonal projection ($i = 0, 1$). Then the following assertions are true.*

(R1) $\forall X \in \Gamma(E_0),\, Y \in \Gamma(E_1) \;:\; \nabla_X Y \in \Gamma(E_1)$

(R2) $\forall X \in \mathfrak{X}(M),\, Y \in \Gamma(E_1) \;:\; (\nabla_X Y)^0 = \langle X, Y \rangle\, H$

(R3) $\forall X \in \Gamma(E_1),\, Y \in \Gamma(E_0) \;:\; (\nabla_X Y)^1 = -\langle Y, H \rangle\, X$

(R4) *If $\alpha : I \to M$ is an E_1-integral C^∞-curve, $t_0 \in I$ and $X \in \mathfrak{X}_\alpha(M)$ fulfills the equation $\nabla_\partial X = -\langle X, H \circ \alpha \rangle\, \dot{\alpha}$ with $X(t_0) \in E_0(\alpha(t_0))$, then $X \in \Gamma_\alpha(E_0)$.*

(R5) *If $\alpha : I \to M$ is an E_0-integral C^∞-curve, $t_0 \in I$ and $X \in \mathfrak{X}_\alpha(M)$ is a ∇-parallel vector field along α with $X(t_0) \in E_1(\alpha(t_0))$, then $X \in \Gamma_\alpha(E_1)$.*

Proof. We define a linear connection $\widetilde{\nabla}$ on M by putting

$$\widetilde{\nabla}_X Y \;:=\; \nabla_X Y \;-\; \langle X^1, Y^1 \rangle \cdot H + \langle X, H \rangle \cdot Y^1 + \langle Y, H \rangle \cdot X^1 \tag{2}$$

for every $X, Y \in \mathfrak{X}(M)$. In the case that M is the warped product $M = M_0 \times_\rho M_1$, $\widetilde{\nabla}$ is the Levi-Civita connection of the ordinary pseudoriemannian product $M_0 \times M_1$ (see Proposition 1 in [5]). As M locally is always of this form, we obtain that each subbundle E_i is $\widetilde{\nabla}$-parallel. Herewith, the rules (R1)–(R3) follow immediately.

For (R4): As $(\nabla_\partial X^0)^1 \overset{(R3)}{=} -\langle X, H \circ \alpha \rangle\, \dot{\alpha} = (\nabla_\partial X)^1$, we get $(\nabla_\partial X^1)^1 = 0$, that means $\nabla_\partial X^1 \in \Gamma_\alpha(E_0)$. Thus, we have $\nabla_\partial X^1 \overset{(R2)}{=} \langle \dot{\alpha}, X^1 \rangle\, H \circ \alpha$, that means X^1 fulfills some linear differential equation (more exactly: X^1 is a parallel vectorfield along α with respect to the linear connection $\nabla' := \nabla - \langle \cdot, \cdot \rangle\, H$) with $X^1(t_0) = 0$. Therefore, we get $X^1 \equiv 0$, that means $X \in \Gamma(E_0)$.

For (R5): Since $\nabla_\partial X^0 \in \Gamma_\alpha(E_0)$ because of (2) and $\nabla_\partial X^1 \in \Gamma_\alpha(E_1)$ because of (R1), we obtain that also X^0 is parallel along α. As $X^0(t_0) = 0$, we therefore get $X^0 \equiv 0$, that means $X \in \Gamma_\alpha(E_1)$. □

We will list the main properties of isometric net morphisms between warped products in the following proposition. It can locally be applied to isometric net morphisms between arbitrary WP-netted manifolds; thus, it is of interest in combination with Theorem 1.

Proposition 1. *Let* $M := M_0 \times_\rho \prod_{i=1}^k M_i$ *and* $N := N_0 \times_\sigma \prod_{i=1}^k N_i$ *be two warped products with their product nets* $\mathcal{E}$ *and* $\mathcal{F}$ *, respectively. Furthermore, let* $f : M \to N$ *be an isometric immersion, which at the same time is a net morphism with respect to* $\mathcal{E}$ *and* $\mathcal{F}$ *. Then* f *has a representation* $f = f_0 \times \cdots \times f_k$ *with* C^∞*-maps* $f_i : M_i \to N_i$ *and the following assertions are true:*

1. $f_0 : M_0 \to N_0$ *is an isometric immersion.*
2. *For every* $i = 1, \ldots, k$ *the map* $f_i : M_i \to N_i$ *is a homothetical immersion, that means: there exists a constant* $c_i \in \mathbb{R}_+$ *such that* $\langle f_{i*}X, f_{i*}Y \rangle_i = c_i^2 \cdot \langle X, Y \rangle_i$ *for all* X, $Y \in \mathfrak{X}(M_i)$ *. Moreover, we have* $\rho_i \equiv c_i \cdot \sigma_i \circ f_0$ *.*
   If the representations of M *and* N *as warped products are normalized with respect to some point* $\bar{p}$ *and its image* $f(\bar{p})$ *, then* $c_1 = \cdots = c_k = 1$ *; in this way all immersions* f_i *become isometric.*
3. f *satisfies the condition* (D) *with respect to* $\mathcal{E}$ *.*

Proof. As $f(L_i^{\mathcal{E}}(p)) \subset L_i^{\mathcal{F}}(f(p))$ for every $p \in M$, $i = 0, \ldots, k$, it is easy to obtain the representation $f = f_0 \times \cdots \times f_k$ (see the proof of [8, Lemma 1]). The assertions 1. and 2. follow immediately from the special kind of warped product metrics and the fact that f is isometric; and 3. follows from the fact that the second fundamental form of f can be expressed in terms of the second fundamental forms of $f_0, \ldots, f_k$ (see [6, Lemma 12]). □

Remark 2. If $(M, \mathcal{E})$ and $(N, \mathcal{F})$ are WP-netted manifolds and $f : M \to N$ is an isometric net morphism with respect to $\mathcal{E}$ and $\mathcal{F}$, then we can apply locally Proposition 1 and get that f satisfies the condition (D) with respect to $\mathcal{E}$. Hence, condition (D) in Theorem 1 is necessary.

2. Proof of the main results

Proof of Theorem 1. From condition (I) we get

$$f_* E_i(p) \subset F_i(f(p)) \quad \text{for every } p \in L_i^{\mathcal{E}}(\bar{p}), i = 0, \ldots, k. \tag{3}$$

We give a proof by induction. Therefore, we start with the case $k = 1$. Let ∇^M resp. ∇^N denote the Levi-Civita connections of M and N, $H \in \Gamma(E_0)$ resp. $\tilde{H} \in \Gamma(F_0)$ the mean curvature normals of E_1 and F_1, and $TM \to E_i$, $v \mapsto v^i$ resp. $TN \to F_i$, $w \mapsto w^i$ the orthogonal projections ($i = 0, 1$). We show step by step

$$\forall p \in M \ : \ f_* E_1(p) \subset F_1(f(p)), \tag{4}$$

$$\forall p \in L_0^{\mathcal{E}}(\bar{p}) \ : \ \tilde{H}(f(p)) - f_* H(p) \in \perp_p f, \tag{5}$$

$$\forall p \in M \ : \ \tilde{H}(f(p)) - f_* H(p) \in \perp_p f, \tag{6}$$

$$\forall p \in M \ : \ f_* E_0(p) \subset F_0(f(p)) \, . \tag{7}$$

Because of (4) and (7) we then get that f is a net morphism with respect to $\mathcal{E}$ and $\mathcal{F}$.

For (4): Let $p = (p_0, p_1) \in M$ be an arbitrary point. Then the point $q := (\bar{p}_0, p_1)$ satisfies $q \in L_1^{\mathcal{E}}(\bar{p}) \cap L_0^{\mathcal{E}}(p)$. Hence, there exists a C^∞-curve $\alpha : [0,1] \to L_0^{\mathcal{E}}(p)$ such that $\alpha(0) = p$ and $\alpha(1) = q$. For a fixed vector $v \in E_1(p)$ let X be the ∇^M-parallel vector field along α with $X(0) = v$. From rule (R5) (see Lemma 1) we get $X \in \Gamma_\alpha(E_1)$, hence, $f_*X(1) \in F_1(f(q))$ because of (3). Therefore, we have $\nabla^N{}_\partial f_*X = f_*\nabla^M{}_\partial X + h(\dot\alpha, X) = 0$ and $0 = (\nabla^N{}_\partial f_*X)^0 = (\nabla^N{}_\partial (f_*X)^0)^0 + (\nabla^N{}_\partial (f_*X)^1)^0 \overset{(R2)}{=} (\nabla^N{}_\partial (f_*X)^0)^0 + \langle f_*\dot\alpha, (f_*X)^1\rangle\, \tilde H \circ f \circ \alpha$. Since moreover

$$0 = \langle \dot\alpha, X\rangle = \langle f_*\dot\alpha, f_*X\rangle = \langle (f_*\dot\alpha)^0, (f_*X)^0\rangle + \langle (f_*\dot\alpha)^1, (f_*X)^1\rangle,$$

we obtain

$$(\nabla^N{}_\partial (f_*X)^0)^0 = \langle (f_*\dot\alpha)^0, (f_*X)^0\rangle\, \tilde H \circ f \circ \alpha.$$

Thus, $(f_*X)^0$ is a solution of some linear differential equation (more exactly: $(f_*X)^0$ is a parallel section in the vector bundle F_0 along $f\circ\alpha$ with respect to the linear connection ∇' given by $\nabla'_Y Z := (\nabla^N{}_Y Z)^0 - \langle Y, Z\rangle\, \tilde H$ for every $Y \in \mathfrak{X}(N)$, $Z \in \Gamma(F_0)$) with $(f_*X)^0(1) = 0$. Therefore, we get $(f_*X)^0 \equiv 0$, in particular $f_*v \in F_1(f(p))$. Herewith, (4) is proved.

We prepare the proof of (5) by the following

Lemma 2. *Let M, N be two pseudoriemannian manifolds, $L \subset M$, $\tilde L \subset N$ two pseudoriemannian submanifolds and $g : M \to N$ an isometric immersion, such that $g|L$ is a C^∞-map in $\tilde L$. Then for every $p \in L$ the following implication holds true*[3]*:*

$$g_*(\perp_p L) \subset \perp_{g(p)} \tilde L \;\Rightarrow\; \forall u, v \in T_pL : h^{\tilde L}(g_*u, g_*v) - g_*h^L(u,v) \in \perp_p g. \tag{8}$$

Proof of Lemma 2. By assumption there exists an isometric immersion $\tilde g : L \to \tilde L$ such that $(\tilde L \hookrightarrow N) \circ \tilde g = g \circ (L \hookrightarrow M)$, thus $h^{\tilde L}(\tilde g_*u, \tilde g_*v) + h^{\tilde g}(u,v) = h^g(u,v) + g_*h^L(u,v)$ for all $p \in L$ and $u, v \in T_pL$. Therefore, it suffices to conclude from the left hand side of (8) that $h^{\tilde g}(u,v) \in \perp_p g$. Now we can decompose the tangent space $T_{g(p)}N$ as

$$T_{g(p)}N = \underbrace{g_*T_pM}_{=g_*T_pL \oplus g_*\perp_p L} \oplus \perp_p g = \underbrace{T_{\tilde g(p)}\tilde L}_{=\tilde g_*T_pL \oplus \perp_p \tilde g} \oplus \perp_{g(p)} \tilde L\ .$$

Since $g_*T_pL = \tilde g_*T_pL$ we therefore obtain from $g_* \perp_p L \subset \perp_{g(p)} \tilde L$ immediately $\perp_p \tilde g \subset \perp_p g$, thus in particular $h^{\tilde g}(u,v) \in \perp_p g$. □

To show now (5) let $p \in L_0^{\mathcal{E}}(\bar p)$ be an arbitrary point. Then $f_*E_0(p) \subset F_0(f(p))$ because of (3) and $f_*E_1(q) \subset F_1(f(q))$ for every $q \in L_1^{\mathcal{E}}(p)$ because of (4). We apply Lemma 2 with $g = f$, $L := L_1^{\mathcal{E}}(p)$ and $\tilde L := L_1^{\mathcal{F}}(f(p))$; in this situation $\perp_p L = E_0(p)$, $\perp_{f(p)} \tilde L = F_0(f(p))$ and $f|L$ is a C^∞-map in $\tilde L$. Since $h^{\tilde L}(f_*u, f_*v) = \langle f_*u, f_*v\rangle\, \tilde H(f(p)) = \langle u, v\rangle\, \tilde H(f(p))$ and $f_*h^L(u,v) = \langle u,v\rangle\, f_*H(p)$ for every $u, v \in T_pL$, we conclude with $u = v \neq 0$ from Lemma 2: $\tilde H(f(p)) - f_*H(p) \in \perp_p f$.

For (6): Let $p = (p_0, p_1) \in M$ be an arbitrary point. Then the point $q := (p_0, \bar p_1)$ satisfies $q \in L_0^{\mathcal{E}}(\bar p) \cap L_1^{\mathcal{E}}(p)$. Hence, there exists a C^∞-curve $\alpha : [0,1] \to L_1^{\mathcal{E}}(p)$ such that

[3] We denote the normal bundle of an isometric immersion f resp. submanifold L by $\perp f$ resp. $\perp L$.

$\alpha(0) = p$, $\alpha(1) = q$. For an arbitrary vector $v \in E_1(p)$ we have $f_*v \in F_1(f(p))$, thus $\langle \tilde{H}(f(p)), f_*v\rangle = 0$. Moreover $\langle f_*H(q), f_*v\rangle = \langle H(p), v\rangle = 0$. Therefore, it suffices to show $\langle \tilde{H}(f(p)) - f_*H(p), f_*v\rangle = 0$ for every $v \in E_0(p)$. Let $v \in E_0(p)$ be an arbitrary vector and X the vector field along α, which satisfies $\nabla_\partial X = -\langle X, H\circ\alpha\rangle$ and $X(0) = v$. Then we have $X \in \Gamma_\alpha(E_0)$ because of (R4) and ${}^N\!\nabla_\partial f_*X = f_*{}^M\!\nabla_\partial X + h^f(\dot\alpha, X) = -\langle H\circ\alpha, X\rangle\, f_*\dot\alpha$. As $\langle \tilde{H}\circ f\circ\alpha - f_*H\circ\alpha, f_*X\rangle(1) = 0$ because of (5), it suffices to show

$$\partial\langle \tilde{H}\circ f\circ\alpha - f_*H\circ\alpha, f_*X\rangle = 0. \tag{9}$$

Now we regard the splitting

$$\begin{aligned}\partial\langle \tilde{H}\circ f\circ\alpha - f_*H\circ\alpha, f_*X\rangle = &\underbrace{\langle {}^N\!\nabla_\partial(\tilde{H}\circ f\circ\alpha), f_*X\rangle}_{=:A}\\ &- \underbrace{\langle {}^N\!\nabla_\partial(f_*H\circ\alpha), f_*X\rangle}_{=:B} + \underbrace{\langle \tilde{H}\circ f\circ\alpha - f_*H\circ\alpha, {}^N\!\nabla_\partial f_*X\rangle}_{=:C},\end{aligned}$$

with $A,\ B,\ C \in \mathrm{C}^\infty([0,1])$. Since

$$C = \langle \tilde{H}\circ f\circ\alpha - f_*H\circ\alpha, -\langle H\circ\alpha, X\rangle\, f_*\dot\alpha\rangle$$

and α is E_1-integral, thus $f\circ\alpha$ is F_1-integral, we obtain $C \equiv 0$.

Furthermore ${}^N\!\nabla_\partial(f_*H\circ\alpha) = f_*{}^M\!\nabla_\partial(H\circ\alpha) + h(\dot\alpha, H\circ\alpha) = f_*{}^M\!\nabla_\partial(H\circ\alpha)$, thus $B = \langle f_*{}^M\!\nabla_\partial(H\circ\alpha), f_*X\rangle$. Since $H\circ\alpha$ is parallel in E_0, we have ${}^M\!\nabla_\partial(H\circ\alpha) \in \Gamma_\alpha(E_1)$ and hence $B \equiv 0$.

Since $\tilde{H}\circ f\circ\alpha$ is parallel in F_0, we have ${}^N\!\nabla_\partial(\tilde{H}\circ f\circ\alpha) \in \Gamma_{f\circ\alpha}(F_1)$, thus $A = \langle {}^N\!\nabla_\partial(\tilde{H}\circ f\circ\alpha), (f_*X)^1\rangle$. Because of $\langle \tilde{H}\circ f\circ\alpha, (f_*X)^1\rangle \equiv 0$ we obtain

$$\begin{aligned}0 =&\partial\langle \tilde{H}\circ f\circ\alpha, (f_*X)^1\rangle\\ =&\langle {}^N\!\nabla_\partial(\tilde{H}\circ f\circ\alpha), (f_*X)^1\rangle + \langle \tilde{H}\circ f\circ\alpha, {}^N\!\nabla_\partial(f_*X)^1\rangle.\end{aligned} \tag{10}$$

Since moreover $f\circ\alpha$ is F_1-integral, we get

$$({}^N\!\nabla_\partial(f_*X)^1)^0 \overset{(R2)}{=} \langle f_*\dot\alpha, f_*X\rangle\, \tilde{H}\circ f\circ\alpha = 0\ ,$$

and with (10) we conclude immediately $A \equiv 0$. Herewith, (9) and therefore also (6) is proved completely.

For (7): Let $p = (p_0, p_1) \in M$ be an arbitrary point. Then the point $q := (p_0, \bar p_1)$ satisfies $q \in L_0^{\mathcal{E}}(\bar p) \cap L_1^{\mathcal{E}}(p)$. Hence, there exists a C^∞-curve $\alpha : [0,1] \to L_1^{\mathcal{E}}(p)$ such that $\alpha(0) = p$, $\alpha(1) = q$. Let $v \in E_0(p)$ be an arbitrary vector and $X \in \mathfrak{X}_\alpha(M)$ be the vector field along α which satisfies $\nabla_\partial X = -\langle X, H\circ\alpha\rangle$ and $X(0) = v$. Then we have $X \in \Gamma_\alpha(E_0)$ because of (R4). Using (6) we obtain

$$\begin{aligned}{}^N\!\nabla_\partial f_*X &= f_*\nabla_\partial X + h(\dot\alpha, X) = -\langle X, H\circ\alpha\rangle\, f_*\dot\alpha\\ &= -\langle f_*X, f_*H\circ\alpha\rangle\, f_*\dot\alpha = -\langle f_*X, \tilde{H}\circ f\circ\alpha\rangle\, f_*\dot\alpha \in \Gamma_{f\circ\alpha}(F_1)\ .\end{aligned}$$

Since $f_*X(1) \in f_*E_0(q) \subset F_0(f(q))$ because of (3), we therefore obtain from (R4): $f_*X \in \Gamma_{f\circ\alpha}(F_0)$, in particular $f_*v \in F_0(f(p))$.

Herewith, the theorem in the case $k = 1$ is proved completely.

In the proof by induction for the cases $k > 1$ in two situations we make use of the following lemma, which is a certain associativity of the property that a map is a net morphism, see [8, Proposition 2]. For that we define that a net $\mathcal{E} = (E_i)_{i=0,\ldots,k}$ on a manifold M is said to be *locally decomposable* if for every $p \in M$ there exists an open neighbourhood U of p in M and a diffeomorphism $\Phi : U \to N$ onto a product manifold $N = \prod_{i=0}^k N_i$ which is a net morphism with respect to $\mathcal{E}|_U$ and the product net of N; this is satisfied, for instance, if $(M, \mathcal{E})$ is a WP-netted manifold.

Lemma 3. *Let $\hat{M} := \prod_{i=0}^k \hat{M}_i$ be a connected product manifold with product net $\hat{\mathcal{E}} := (\hat{E}_i)_{i=0,\ldots,k}$ and $\hat{p} \in \hat{M}$ some point. Let N be another manifold which is equipped with a net $\hat{\mathcal{F}} = (\hat{F}_i)_{i=0,\ldots,k}$. Suppose further that there exist two locally decomposable nets $\hat{\mathcal{E}}' = (\hat{E}'_j)_{j=0,\ldots,l}$, $\hat{\mathcal{F}}' = (\hat{F}'_j)_{j=0,\ldots,l}$, which are refinements of $\hat{\mathcal{E}}$ and $\hat{\mathcal{F}}$, that means there exists a partition $\{0,\ldots,l\} = \dot{\bigcup}_{i=0}^k I_i$ such that $\hat{E}_i = \oplus_{j\in I_i} \hat{E}'_j$ and $\hat{F}_i = \oplus_{j\in I_i} \hat{F}'_j$ for every $i = 0,\ldots,k$. If in this situation $f : M \to N$ is a net morphism with respect to $\hat{\mathcal{E}}$ and $\hat{\mathcal{F}}$ and moreover*

$$\forall i = 0,\ldots,k,\ p \in L_i^{\hat{\mathcal{E}}}(\hat{p}),\ j \in I_i \ : \ f_* \hat{E}'_j(p) \subset \hat{F}'_j(f(p)) \ , \tag{11}$$

then f is a net morphism with respect to the refinements $\hat{\mathcal{E}}'$ and $\hat{\mathcal{F}}'$.

Assuming that the theorem is true for some $k \geq 1$ let a connected warped product $M = M_0 \times_\rho \prod_{i=1}^{k+1} M_i$ with product net $\mathcal{E} = (E_i)_{i=0,\ldots,k+1}$, a WP-net $\mathcal{F} = (F_i)_{i=0,\ldots,k+1}$ on N and an isometric immersion $f : M \to N$ with the quoted properties be given. Then we define the connected warped product $\tilde{M} := M_0 \times_{(\rho_1,\ldots,\rho_k)} \prod_{i=1}^k M_i$, its product net $\tilde{\mathcal{E}} = (\tilde{E}_i)_{i=0,\ldots,k}$, the point $\tilde{p} := (\bar{p}_0, \ldots, \bar{p}_k)$ and the totally geodesic immersion $\varphi : \tilde{M} \to M$, $(p_0,\ldots,p_k) \mapsto (p_0,\ldots,p_k,\bar{p}_{k+1})$. The second fundamental form $\tilde{h}$ of the isometric immersion $\tilde{f} := f \circ \varphi : \tilde{M} \to N$ is given by $\tilde{h} = h \circ (\varphi_* \times \varphi_*)$. As $\varphi_* \tilde{E}_i(p) = E_i(\varphi(p))$ for every $p \in \tilde{M}$ and $i = 0,\ldots,k$, $\tilde{f}$ satisfies the condition (D) with respect to $\tilde{\mathcal{E}}$. Further, we define the WP-net $\tilde{\mathcal{F}} = (F_0 \oplus F_{k+1}, F_1, \ldots, F_k)$ on N. As $\varphi(L_i^{\tilde{\mathcal{E}}}(\tilde{p})) = L_i^{\mathcal{E}}(\bar{p})$ for every $i = 0,\ldots,k$, $\tilde{f}$ satisfies the condition (I) with respect to $\tilde{p}$ and the nets $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$. Therefore, we can apply the theorem in the case k to $\tilde{f}$ and obtain that $\tilde{f}$ is a net morphism with respect to $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{F}}$. If we define the WP-net $\tilde{\mathcal{E}}' := (\tilde{E}_0, \ldots, \tilde{E}_k, 0)$ on $\tilde{M}$ (where 0 represents the trivial subbundle of $\tilde{M}$), then we can apply Lemma 3 using the partition $\{0,\ldots,k+1\} = \{0,k+1\} \dot{\cup} \{1\} \cdots \dot{\cup} \{k\}$ and the dictionary

Lemma 3	$\hat{M}$	$\hat{p}$	f	$\hat{\mathcal{E}}$	$\hat{\mathcal{F}}$	$\hat{\mathcal{E}}'$	$\hat{\mathcal{F}}'$
Here	$\tilde{M}$	$\tilde{p}$	$\tilde{f}$	$\tilde{\mathcal{E}}$	$\tilde{\mathcal{F}}$	$\tilde{\mathcal{E}}'$	$\mathcal{F}$

and obtain that $\tilde{f}$ is a net morphism with respect to $\tilde{\mathcal{E}}'$ and $\mathcal{F}$; notice that (11) is satisfied in this situation because $\tilde{f}_* \tilde{E}_i(p) = f_* E_i(\varphi(p)) \subset F_i(\tilde{f}(p))$ for every $p \in L_i^{\tilde{\mathcal{E}}}(\tilde{p})$ and $i = 0,\ldots,k$. In particular we now even know that for every point $p \in \tilde{M}$ we have

$$\forall i = 0,\ldots,k : f_* E_i(\varphi(p)) = \tilde{f}_* \tilde{E}_i(p) \subset F_i(\tilde{f}(p)) \ , \tag{12}$$

hence $\tilde{f}_* T_p \tilde{M} \subset F_{k+1}^{\perp}(\tilde{f}(p))$. If now $\mathcal{E}'$ and $\mathcal{F}'$ are the WP-nets on M resp. N given by $\mathcal{E}' = (E_0', E_1') = \big(E_{k+1}^{\perp}, E_{k+1}\big)$ and $\mathcal{F}' = (F_0', F_1') = \big(F_{k+1}^{\perp}, F_{k+1}\big)$, then we therefore conclude $f(L_0^{\mathcal{E}'}(\bar{p})) = \tilde{f}(\tilde{M}) \subset L_0^{\mathcal{F}'}(f(\bar{p}))$. By assumption of the theorem we also have $f(L_1^{\mathcal{E}'}(\bar{p})) = f(L_{k+1}^{\mathcal{E}}(\bar{p})) \subset L_{k+1}^{\mathcal{F}}(f(\bar{p})) = L_1^{\mathcal{F}'}(f(\bar{p}))$. Furthermore, f trivially satisfies also condition (D) with respect to $\bar{p}$ and the net $\mathcal{E}'$, and $\mathcal{E}'$ in fact is the product net of the WP-representation $M = \tilde{M} \times_{\rho'} M_{k+1}$ of M, where ρ' is given by $\rho'(p_0, \dots, p_k) := \rho_{k+1}(p_0)$. Thus, we can apply the theorem in the case $k = 1$ to f and obtain that f is a net morphism with respect to $\mathcal{E}'$ and $\mathcal{F}'$. Since because of (12) and (3) we moreover have $f_* E_i(p) \subset F_i(f(p))$ for every $p \in L_0^{\mathcal{E}'}(\bar{p}) = \varphi(\tilde{M})$, $i = 0, \dots, k$ and $f_* E_{k+1}(p) \subset F_{k+1}(f(p))$ for every $p \in L_1^{\mathcal{E}'}(\bar{p}) = L_{k+1}^{\mathcal{E}}(\bar{p})$, we can apply Lemma 3 again using now the partition $\{0, \dots, k+1\} = \{0, \dots, k\} \dot\cup \{k+1\}$ and the dictionary

Lemma 3	$\hat{M}$	$\hat{p}$	f	$\hat{\mathcal{E}}$	$\hat{\mathcal{F}}$	$\hat{\mathcal{E}}'$	$\hat{\mathcal{F}}'$
Here	$\tilde{M} \times_{\rho'} M_{k+1}$	$\bar{p}$	f	$\mathcal{E}'$	$\mathcal{F}'$	$\mathcal{E}$	$\mathcal{F}$

and obtain that f is a net morphism with respect to $\mathcal{E}$ and $\mathcal{F}$. □

Proof of Theorem 2. The crucial point is the proof of the inclusion $f(M) \subset G$. All other assertions are an immediate consequence of Corollary 1 applied to the map $\Phi^{-1} \circ f$. But this inclusion has already been proved in the setting of arbitrary nets in [8, Proposition 3], if two hypotheses are satisfied: The first hypothesis (i) is identical with condition (I); the second hypothesis (ii) means that for every family $(U_i)_{i=0,\dots,k}$ of connected open subsets $U_i \subset M_i$ with $\bar{p} \in U := \prod_{i=0}^{k} U_i$ and $f(U) \subset G$ the restriction $f|U : U \to G$ is a net morphism with respect to $\mathcal{E}|_U$ and $\mathcal{F}$; this is satisfied in our situation because of Theorem 1. □

References

[1] Backes, E.; Reckziegel,H.: *On symmetric submanifolds of spaces of constant curvature.* Math. Ann. **263** (1984), 419–433.

[2] Dillen, F.; Nölker, S.: *Semi-parallelity, multi-rotation surfaces and the helix property.* J. reine angew. Math. **435** (1993), 33–63.

[3] Ferus, D.: *Produkt-Zerlegungen von Immersionen mit paralleler zweiter Fundamentalform.* Math. Ann. **211** (1974), 1–5.

[4] Moore, J. D.: *On isometric immersions of Riemannian products.* J. Differential Geom. **5** (1971), 159–168.

[5] Meumertzheim, M.; Reckziegel, H.; Schaaf, M.: *Decomposition of twisted and warped product nets.* To appear in Results Math.

[6] Nölker, S.: *Isometric immersions of warped products.* Diff. Geom. Appl. **6** (1996), 1–30.

[7] de Rham, G.: *Sur la réductibilité d'un espace de Riemann.* Comment. Math. Helv. **26** (1952), 328–344.

[8] Reckziegel, H.; Schaaf, M.: *De Rham decomposition of netted manifolds.* Results Math. **35** (1999), 175–191.

[9] Schaaf, M.: *On decomposition theorems in differential geometry with special regard to the theory of warped products.* Doctoral Thesis, Cologne 1998.

[10] Takeuchi, M.: *Parallel submanifolds of space forms.* Manifolds and Liegroups, Papers in honor of Y. Matsushima, Birkhäuser 1981, 429–447.

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