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Deformation to the Normal Cone, Rationality and the Stückrad-Vogel Cycle

Dedicated to the memory of Wolfgang Vogel

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Abstract. The Stückrad-Vogel cycle of refined intersection theory collects together the irreducible components of the intersection cycle, together with their multiplicities, needed for a version of Bezout's Theorem applicable to the case of possibly improper intersection (see [10, 25]). The invariance of the Stückrad-Vogel cycle under deformation to the normal cone forms the cornerstone of van Gastel's (geometrical) proof of the coincidence of the refined intersection theories of Fulton-MacPherson and Stückrad-Vogel (see [12]). Later on, Achilles and Manaresi [2, 3] gave an algebraic proof of a local version of this invariance first at a rational and subsequently at a general component of the cycle. Their approach used a detailed analysis of the behaviour of filter-regular sequences in associated graded rings. In related work, van Gastel [12] showed the coincidence of the distinguished varieties of Fulton [10] with the rational components of the Stückrad-Vogel cycle. Subsequently, Achilles and Manaresi [1] obtained a local version of the fact that a rational component was necessarily distinguished in a slightly weak sense (see the introduction to Section 3), using an algebraic approach involving analytic spread.
The first aim of this paper is to present a natural and alternative independent approach to the work of Achilles and Manaresi [2, 3] by giving an algebraic proof of the

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general fact of the invariance of the Stückrad-Vogel cycle under deformation to the normal cone, using extended Rees rings as a natural deformation space (see Section 2), and by refining and generalizing somewhat their basic theorem [1, (2.2)] which discusses the connections between distinguished and rational components. Indeed a possibly non-reduced version of this generalization is given, using previous work of O'Carroll and Qureshi [20] on tensor products of fields (see Section 3). Section 4 is devoted to an example which illustrates a paradoxical aspect of our work.
The other aim of this paper is to revisit a classical result of Northcott and Rees [18] which essentially states that the correct number of sufficiently general elements in an ideal of a local ring generates a minimal reduction; this is proved using the rather technical machinery of 'resultant systems' (plural) (cf. [16]). The same approach is employed by Achilles and Manaresi [1] in their work on rational components of the Stückrad-Vogel cycle. In the final section we sketch a clean geometrical proof of this classical result, using the old notion of 'Cayley form' (singular) (also called 'ground form' or 'associated form': see [14, 24]), which we hope is of interest.
For general background, see [6, 11].

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1. Preliminaries

1.1. The Stückrad-Vogel intersection cycle

Let $\mathbf{P}^n := \mathbf{P}^n_K$ denote n-dimensional projective space over an algebraically closed field K. Let X, Y be pure-dimensional closed subschemes of $\mathbf{P}^n$ without embedded components. (From the point of view of intersection theory, pure-dimensionality is no real restriction, although – as we shall see – it is not really needed.) We denote by $I(X)$, respectively $I(Y)$, the largest (homogeneous) defining ideal of X, respectively Y, in the ring $R_x := K[x_0, \dots, x_n]$. We introduce a second copy $R_y := K[y_0, \dots, y_n]$ of R_x and denote by $I(Y)'$ the ideal in R_y corresponding to $I(Y)$.

Next consider the polynomial ring $R := K[x_0, \dots, x_n, y_0, \dots, y_n]$ and the ideal $\mathfrak{c} := (x_0 - y_0, \dots, x_n - y_n)$ defining the diagonal Δ in $\mathbf{P}^{2n+1}_K$. The *ruled join* $J := J(X, Y)$ is the subscheme of $\mathbf{P}^{2n+1}_K :=$ Proj R defined by the ideal $I(X) \cdot R + I(Y)' \cdot R$. The original $\mathbf{P}^n$ is naturally isomorphic to the diagonal Δ in $\mathbf{P}^{2n+1}$ defined by the vanishing of the ideal $\mathfrak{c}$ and one has the isomorphism of schemes $X \cap Y \simeq J(X, Y) \cap \Delta$.

In order to discuss generic linear forms, we introduce $(n+1)^2$ independent variables u_{ij} over K for $i, j = 0, \dots, n$. Let $\bar{K}$ denote the algebraic closure of $K(u_{00}, \dots, u_{nn})$ and let $\bar{X} = X \times_K \bar{K}$, $\bar{Y} = Y \times_K \bar{K}$ and $\bar{J} = J \times_K \bar{K}$. For $i = 0, \dots, n$ the linear forms $\ell'_i := \sum_{j=0}^{n} u_{ij} \cdot (x_j - y_j)$ in $\bar{R} := R \otimes_K \bar{K}$ define hyperplanes $L_0, \dots, L_n$ with

$$\begin{aligned} &L_0 \cap \ldots \cap L_n = \bar{\Delta} := \Delta \times_K \bar{K} \text{ ; or equivalently} \\ &(\ell'_0, \dots, \ell'_n) \cdot \bar{R} = \mathfrak{c} \cdot \bar{R} = I(\bar{\Delta}) \text{ .} \end{aligned}$$

We say that a (closed) subscheme of $\mathbf{P}^{2n+1}_{\bar{K}}$ is K-*rational* if it is defined over the ground field K.

Note that $\bar{X} \cap \bar{Y} = \bar{J} \cap \bar{\Delta}$. Letting $\mathcal{O}_J$ (respectively $\mathcal{O}_{\bar{J}}$) denote the coordinate ring of the affine cone over J (respectively $\bar{J}$), set $D = \mathfrak{c} \cdot \mathcal{O}_J$ (respectively $\bar{D} = \mathfrak{c} \cdot \mathcal{O}_{\bar{J}}$), so that D and $\bar{D}$ are defined by the 'diagonal'. Then a variety in $\mathbf{P}^n_{\bar{K}}$ is said to be a *distinguished subvariety of the intersection* $\bar{X} \cap \bar{Y}$ (in the sense of Fulton [10]) if its defining ideal arises from a prime ideal in $\mathcal{O}_{\bar{J}}/\bar{D}$ which is the restriction of a minimal prime ideal of the associated graded ring $G := G(\mathcal{O}_{\bar{J}}, \bar{D}) := \bigoplus_{m \geq 0} \bar{D}^m/\bar{D}^{m+1}$. More generally, a closed subscheme of $\mathbf{P}^n_{\bar{K}}$ is said to be *distinguished* (in the same context) if its defining ideal arises from a primary ideal in $\mathcal{O}_{\bar{J}}/\bar{D}$ which is the restriction of an isolated primary component of the zero ideal 0_G in G. We define the analogous notions of distinguished subvariety (resp. distinguished subscheme) of the intersection $X \cap Y$ by removing all overbars in the corresponding definitions for the case of the intersection $\bar{X} \cap \bar{Y}$.

The collection $\mathcal{C}(X, Y)$ of subvarieties of $\bar{X} \cap \bar{Y}$ which comprise the Stückrad-Vogel intersection cycle can be described as follows.

Let $\mathcal{U}(\dots)$ denote the intersection of all the highest-dimensional associated primary ideals of an ideal $\dots$; let

$$e := \dim(X \cap Y) - \dim X - \dim Y + n$$

be the excess dimension of the intersection.

Step 1: The first step in the Stückrad-Vogel algorithm considers

$$\mathcal{U}(I(X) \cdot \bar{R} + I(Y)' \cdot \bar{R}) = \mathcal{U}(J(X, Y) \cdot \bar{R}) = \mathfrak{a}_0 \cap \mathfrak{b}_0$$

where $\mathfrak{b}_0$ is the intersection of all primary ideals whose associated prime ideals do not contain $I(\bar{\Delta})$ and $\mathfrak{a}_0$ is the intersection of the remaining primary ideals (if any) whose prime radicals do contain $I(\bar{\Delta})$ (with $\mathfrak{b}_0 = \bar{R}$, resp. $\mathfrak{a}_0 = \bar{R}$, if there are no such corresponding primes); if $\mathfrak{p}_i$, $1 \leq i \leq t_0$, are the associated primes of $\mathfrak{a}_0$, then, by reduction to the diagonal, the $\mathfrak{p}_i$ correspond to irreducible components $C_{0,i}$ (if any) of $\bar{X} \cap \bar{Y}$ of dimension $\dim X + \dim Y + 1$; moreover, the intersection number $j(X, Y; C_{0,i})$ is defined to be the length of $(\bar{R}/\mathfrak{a}_0)_{\mathfrak{p}_i}$. If $\mathfrak{b}_0 = \bar{R}$ (the trivial case) then the algorithm stops. (Note that the $\mathcal{U}$-process is not really needed at this stage, since $I(X) \cdot \bar{R} + I(Y)' \cdot \bar{R}$ is unmixed; see [25, 19].)

Step 2: If $\mathfrak{b}_0 \neq \bar{R}$, the second step considers

$$\mathcal{U}(\mathfrak{b}_0 + \ell'_0 \cdot \bar{R}) = \mathfrak{a}_1 \cap \mathfrak{b}_1$$

where $\mathfrak{a}_1$ and $\mathfrak{b}_1$ are constructed as in Step 1, etc.

Note that $\dim J = \dim \bar{J} = \dim \mathfrak{a}_0 = \dim \mathfrak{b}_0$ (unless $\mathfrak{a}_0 = \bar{R}$ or $\mathfrak{b}_0 = \bar{R}$). If $\mathfrak{b}_0 \neq \bar{R}$ then a given prime divisor $\mathfrak{p}$ of $\mathfrak{b}_0$ is defined over K (see Section 1.4 below), so that if $\ell'_0 \in \mathfrak{p}$ then $\{\ell'_0, \dots, \ell'_n\} \subseteq \mathfrak{p}$ – a contradiction. Hence if $\mathfrak{b}_0 \neq \bar{R}$ then

$$\dim(\mathfrak{b}_0 + \ell'_0 \cdot \bar{R}) = \dim \mathfrak{b}_0 - 1 ,$$

and so on.

The collection $\mathcal{C}(X, Y)$ is comprised of the subvarieties of $\bar{X} \cap \bar{Y}$ defined by the various associated primes of the $\mathfrak{a}_i$ on reduction to the diagonal.

Note also that the first $\mathfrak{a}_j \neq \bar{R}$ occurs when $j = n - e + 1$; we then obtain the highest-dimensional irreducible components of $\bar{X} \cap \bar{Y}$. In any case $\mathfrak{b}_{n+1} = \bar{R}$, since all of $\ell'_0, \ldots, \ell'_n$ have been used by this stage.

For further details and background, and for more recent developments, see [25, 12, 7].

Remarks. 1. We will find it useful to employ the notation

$$\mathcal{U}_{\mathfrak{b}}(I(X) \cdot \bar{R} + I(Y)' \cdot \bar{R}) := \mathfrak{b}_0 \ , \quad \mathcal{U}_{\mathfrak{b}}(\mathfrak{b}_0 + \ell'_0 \cdot \bar{R}) := \mathfrak{b}_1 \ , \text{ etc.}$$

2. Some variations and mild generalizations of the above are possible, which could then be used to give analogous versions of the results of this paper. Basically, what we need for the algorithm are degree one homogeneous elements $\ell_0, \ldots, \ell_n$ in a graded algebra finitely generated in degree one over a suitable field (in the above, $\ell_i = x_i - y_i$): quite general local versions are possible, as are more immediate local versions obtained by localising a graded version of the algorithm. See [2, 4, 7, 21], for example (note that the rôles of the $\mathfrak{a}_i$ and $\mathfrak{b}_i$ are interchanged in [2, 5, 25]). Note also that we do not need to assume pure-dimensionality a priori.

3. Geometrically speaking, we first take into the collection $\mathcal{C}(X,Y)$ those reduced top-dimensional components (if any) of the ruled join which lie in the diagonal; we then intersect the residual scheme with the first generic hyperplane and repeat the process, etc., stopping when the intersection all falls inside the diagonal. Reduction to the diagonal brings the elements of the collection back into the projective space $\mathbf{P}^n_{\bar{K}}$. The j-multiplicity of an element C of the collection is the length at C of its non-reduced structure. Thus, in geometrical terms, the collection $\mathcal{C}(X,Y)$, together with the j-multiplicities, corresponds to the fundamental cycles of those components of the various intersections which fall inside the diagonal (see [12, 7]); this concentration on fundamental cycles is of course entirely natural in the context of intersection theory (see [10]).

1.2. Extended Rees rings

Let I be an ideal of the noetherian ring A and let t be an indeterminate over A. Then the *extended Rees ring of A with respect to I* is the ring

$$\mathcal{R} := \mathcal{R}(A,I) := \bigoplus_{m \in \mathbf{Z}} I^m \cdot t^m \quad (\subseteq A[t,t^{-1}]) \ ,$$

where $I^m = A$ for $m \leq 0$.

Let J be an ideal of A. We define a corresponding ideal J^* of $\mathcal{R}$ as follows:

$$J^* := \bigoplus_{m \in \mathbf{Z}} (J \cap I^m) \cdot t^m \ .$$

For detailed properties of these objects see [13, 22]. The properties needed in this paper are easily deduced from the following:

1. The natural epimorphism from $\mathcal{R}(A,I)$ onto $\mathcal{R}(A/J, I+J/J)$ induces an isomorphism $\mathcal{R}(A,I)/J^* \simeq \mathcal{R}(A/J, I+J/J)$.

2. $[J^*]_0 = J$.
3. $I \subseteq J$ if and only if $I \cdot t \subseteq J^*$.
4. For ideals J_1 and J_2 of A, $J_1 \subseteq J_2$ if and only if $J_1^* \subseteq J_2^*$, and $(J_1 \cap J_2)^* = J_1^* \cap J_2^*$. In particular, $\mathfrak{q}$ is a primary ideal of A with prime radical $\mathfrak{p}$ if and only if $\mathfrak{q}^*$ is a primary ideal of $\mathcal{R}$ with prime radical $\mathfrak{p}^*$, and in this case $\lg(\mathfrak{q})$ equals $\lg(\mathfrak{q}^*)$; moreover, if A is finite dimensional, $\dim \mathfrak{q}^* = \dim \mathfrak{q} + 1$; finally, if $0_A = \bigcap \mathfrak{q}$ is a minimal primary decomposition then $0_{\mathcal{R}} = \bigcap \mathfrak{q}^*$ is also a minimal primary decomposition (see [22, Section 1]).

   Note also that $A^* = \mathcal{R}$.

The element $u := t^{-1}$ is a non-zero-divisor in $\mathcal{R}$ (since it acts as an automorphism of the over-ring $A[t, t^{-1}]$) and

$$\mathcal{R}/u \cdot \mathcal{R} \simeq G := \bigoplus_{m \geq 0} I^m / I^{m+1}$$

where $G =: G(A, I)$ is the associated graded ring of A with respect to I.

1.3. Superficial elements

Now let I be a non-zero ideal of a noetherian local ring A with infinite residue field. Then (see [17, (22.1)]) I contains a *superficial element*, y say, (*of order one*) in the sense that the leading form y^* of y in the associated graded ring G is of order 1 and avoids the relevant prime divisors of G; it then follows that there exists a positive integer c such that $(I^n : y \cdot A) \cap I^c = I^{n-1}$ when $n > c$. Note that, on localising at a prime $\mathfrak{p}$ of A such that $I \subseteq \mathfrak{p}$, a superficial element remains superficial (though the order may increase).

Suppose further that I contains a non-zero-divisor. Then the argument of [17, (22.1)] shows that y as above can be chosen to be a non-zero-divisor. Moreover, an Artin-Rees result [17, (3.12)] gives a positive integer d such that $I^n : y \cdot A \subseteq I^{n-d}$ when $n > d$. In this case therefore

$$I^n : y \cdot A = I^{n-1}\ , \quad n \gg 0\ ,$$

or equivalently,

$$y \cdot A \cap I^n = y \cdot I^{n-1}\ , \quad n \gg 0\ .$$

Suppose further that A is one-dimensional. Then $A/y \cdot A$ is Artinian, so that $I^n \subseteq y \cdot A$, for $n \gg 0$, whence

$$I^n = y \cdot I^{n-1}\ , \quad n \gg 0$$

in this case; i.e., y is a reduction of I.

Remark. Since the data involved in the Stückrad-Vogel algorithm involves (homogeneous) primary ideals, their lengths and their prime radicals, we can move to a local ring by a suitable localisation while preserving all the data of interest in a given context.

1.4. Tensoring over a field

Let I be an ideal of an algebra B over a field k; let $k \subseteq k'$ be a field extension. Denote $B \otimes_k k'$ by B' and the ideal $I \otimes_k k'$ by I'. Note that $B'/I' \simeq (B/I) \otimes_k k'$.

Suppose that k is its own separable algebraic closure in k' (the extension $k \subseteq k'$ is then said to be *quasi maximally algebraic* (*q.m.a.* for short) [26, Vol. I] or, alternatively, *primary* [15, 20]). If I is a primary ideal, then applying [20, Proposition 4] to B/I, I' is also primary.

Suppose, further, that $k \subseteq k'$ is a separable extension and that I is a prime ideal. Then, applying the proof of [26, Corollary 2, p. 198] to $K' := \text{q.f.}(B/I)$ – the quotient field of B/I – and noting that we have $B'/I' \hookrightarrow K' \otimes_k k'$ in the obvious way, we deduce that I' is again a prime ideal.

In particular, suppose that k is algebraically closed. Consider a field extension $k \subseteq k'$; let Q be a primary ideal of B with prime radical P. Then $Q \cdot B'$ is a primary ideal with prime radical $P \cdot B'$. By base extension, $B \subseteq B'$ is faithfully flat, so finite intersections of ideals are preserved; hence primary decompositions behave as expected in the extension $B \subseteq B'$.

For details and applications of this theory, see [26, 15, 20].

2. Deformation to the normal cone via the extended Rees ring

It is well-known that the extended Rees ring provides a deformation space for deformation to the normal cone: in simple algebraic term, using the notation of Section 1.2,

$$\mathcal{R}/(u-1)\cdot\mathcal{R} \simeq A\ ;\ \ \mathcal{R}/u\cdot\mathcal{R} \simeq G := \bigoplus_{m\geq 0} I^m/I^{m+1}\ .$$

(See [6, 10], for example.) In fact, for an ideal J of A, $J^* := \bigoplus_{m\in\mathbf{Z}} (J \cap I^m)\cdot t^m$ is the defining ideal of the strict transform of J (see [13]).

Recall that deformation to the normal cone is the cornerstone of the Fulton-MacPherson approach to intersection theory; see [11] for an excellent, well-motivated exposition. Further, we recall that the invariance of the Stückrad-Vogel cycle under deformation to the normal cone is the basis of van Gastel's result on the coincidence of the Fulton-MacPherson and the Stückrad-Vogel intersection theories; see [12, 7].

We now show that using the extended Rees ring as a deformation space provides an explicit, natural and simple explanation of the invariance of the Stückrad-Vogel cycle under deformation to the normal cone. This result is due to van Gastel [12, Theorem 3.2] using sophisticated geometrical methods. For a sophisticated algebraic approach, using techniques from [8], see [7]. On the other hand, Achilles and Manaresi [2] give a local version of this result at a rational component, using their so-called 'super-reductions' (but without using Rees rings). Subsequently, and independently, they extended their result to the case of an arbitrary component, using the properties of filter-regular sequences; see [3].

Let $(A, \mathbf{m})$ be a noetherian local ring containing an algebraically closed field K, let $\ell_0, \ldots, \ell_n$ be elements of $\mathbf{m}$, let $\{u_{ij} \mid 0 \leq i, j \leq n\}$ be indeterminates over A and let $\bar{K}$ denote the algebraic closure of $K(u_{00}, \ldots, u_{nn})$; suppose further that $\bar{A} := A \otimes_K \bar{K}$ is noetherian. (This is the case in the geometrical situation we are basically dealing with; see Section 1.1 and 1.4, and especially the former's closing remark.) Then $\mathbf{m} \cdot \bar{A}$ is a prime ideal (see Section 1.4); let (B, M) denote the local ring $\bar{A}_{\mathbf{m} \cdot \bar{A}}$.

Fix $\ell_0, \dots, \ell_n$ in $\mathbf{m}$; for $i = 0, \dots, n$, let $\ell_i' = \sum_{j=0}^{n} u_{ij} \cdot \ell_j \in B$ and let $D = (\ell_0', \dots, \ell_n') \cdot B$; then $D = (\ell_0', \dots, \ell_n') \cdot B = (\ell_0, \dots, \ell_n) \cdot B$. To avoid trivialities we assume that $D \neq 0$. Note that each ℓ_i' is of order one. For if some $\ell_i' \in D^2$, then $\{\ell_0', \dots, \ell_n'\} \subseteq D^2$ (since D^2 is defined over A). Hence $D = 0$, a contradiction.

Consider the extended Rees ring $\mathcal{R} := \mathcal{R}(B, D)$, the associated graded ring $G := G(B, D)$ (see Section 1.2) and the elements $Z_j := \ell_j \cdot t$ in $[\mathcal{R}]_1$, $0 \leq j \leq n$, and $Z_i' := \ell_i' \cdot t = \sum_{j=0}^{n} u_{ij} \cdot Z_j$ in $[\mathcal{R}]_1$, $0 \leq i \leq n$. Let z_j (respectively z_i') denote the canonical image of Z_j (respectively Z_i') in $[G]_1$; then $z_i' = \sum_{j=0}^{n} u_{ij} \cdot z_j$, $0 \leq i \leq n$.

We wish to examine the corresponding Stückrad-Vogel algorithms in B, $\mathcal{R}$ and G, respectively.

2.1. The algorithm in B and $\mathcal{R}$

Denote by $\mathfrak{a}_0, \mathfrak{b}_0; \mathfrak{a}_1, \mathfrak{b}_1; \dots$ (respectively $\tilde{\mathfrak{a}}_0, \tilde{\mathfrak{b}}_0; \tilde{\mathfrak{a}}_1, \tilde{\mathfrak{b}}_1; \dots$) the ideals in B (respectively $\mathcal{R}$) produced by the Stückrad-Vogel algorithm (see Section 1.1). By the discussion in Section 1.2, $\tilde{\mathfrak{a}}_0 = \mathfrak{a}_0^*$, $\tilde{\mathfrak{b}}_0 = \mathfrak{b}_0^*$ and the corresponding j-multiplicities coincide. Passing to $\mathcal{R}/\mathfrak{b}_0^* = \mathcal{R}(B/\mathfrak{b}_0, D + \mathfrak{b}_0/\mathfrak{b}_0)$ we may suppose now that $\mathfrak{b}_0 = 0_B$, $\tilde{\mathfrak{b}}_0 = 0_{\mathcal{R}}$. Note that B and $\mathcal{R}$ are now equidimensional and ht $D > 0$.

The next step of the algorithm in $\mathcal{R}$ considers $(Z_0') = (\ell_0' \cdot t)$. Now $(\ell_0' \cdot t)_u = ((\ell_0')^*)_u$ in $\mathcal{R}_u$ and, of course, $(\ell_0' \cdot t) \subseteq (\ell_0')^*$. In view of Section 1.2, our aim is to show that a top-dimensional prime divisor of $(\ell_0' \cdot t)$ does not contain u: it then follows that $\mathcal{U}((\ell_0' \cdot t)) = \mathcal{U}((\ell_0')^*)$. So suppose that $\mathfrak{q}$ is a top-dimensional primary component of $(\ell_0' \cdot t)$ with $u \in \sqrt{\mathfrak{q}}$. Then

$$\dim_{\mathcal{R}} \mathfrak{q} \leq \dim_{\mathcal{R}}(\ell_0' \cdot t, u) = \dim_G(z_0') \ .$$

Let $\mathfrak{p}$ denote the image of $\sqrt{\mathfrak{q}}$ in G. Now $\mathfrak{p}$ is a top-dimensional prime divisor of G. Hence, by Section 1.4, $\mathfrak{p}$ is defined over K and it follows that $\{z_0', \dots, z_n'\} \subseteq \mathfrak{p}$, so that $\mathfrak{p}$ is irrelevant. Hence

$$\dim \mathfrak{p} \leq \dim B/D < \dim B = \dim G \ ,$$

which is a contradiction.

[For future reference, we note explicitly that we have in fact shown that if $\mathfrak{p}'$ is a prime divisor of (u) which is relevant, then

$$\ell_0' \cdot t \notin \mathfrak{p}' \ . \tag{$*$}$$

That is, ℓ_0' is a superficial element of D of order one; ℓ_0' is also a non-zero-divisor since $\mathfrak{b}_0 = 0$ and ht $D > 0$.]

Hence $\tilde{\mathfrak{a}}_1 = \mathfrak{a}_1^*$, $\tilde{\mathfrak{b}}_1 = \mathfrak{b}_1^*$ and the corresponding j-multiplicities coincide (see Section 1.2). Now pass to

$$\mathcal{R}/\mathfrak{b}_1^* = \mathcal{R}(B/\mathfrak{b}_1, D + \mathfrak{b}_1/\mathfrak{b}_1)$$

and so on. So to sum up, we have :

Theorem 1. *The Stückrad-Vogel cycles (with respect to D) in B and* $\mathcal{R}(B,D)$ *are in one-one correspondence, viz*

$$\mathfrak{a}_i^* = \tilde{\mathfrak{a}}_i\ ,\quad \mathfrak{b}_i^* = \tilde{\mathfrak{b}}_i\ ,\quad i = 0,1,\dots\ ,$$

and the corresponding j-multiplicities coincide. Moreover $\mathcal{U}_{\mathfrak{b}}(z_0',\dots,z_{i-1}') = \tilde{\mathfrak{b}}_i$, $i = 0,1,\dots$.

2.2. The algorithm in $\mathcal{R}$ and G

Denote by $\hat{\mathfrak{a}}_0, \hat{\mathfrak{b}}_0; \hat{\mathfrak{a}}_1, \hat{\mathfrak{b}}_1; \dots$ the ideals in G produced by the Stückrad-Vogel algorithm.
Let $\mathfrak{p}$ be a prime ideal in G. Then $\mathfrak{p} = P/(u)$ for some prime P in $\mathcal{R}$ with $u \in P$.
Suppose firstly that $\mathfrak{p}$ is top-dimensional, with $G_+ := \bigoplus_{m>0} D^m/D^{m+1} \subseteq \mathfrak{p}$; i.e. $\mathfrak{p}$ is a prime divisor of $\hat{\mathfrak{a}}_0$. Then $\mathcal{R}_+ := \bigoplus_{m>0} D^m \subseteq P$ and $D \subseteq Q := [P]_0$, with $\mathcal{R}/P = B/Q$; hence

$$\dim Q = \dim P = \dim B\ ,$$

i.e. Q is a top-dimensional prime in B. Clearly then Q is a prime divisor of $\mathfrak{a}_0$ and $\mathfrak{p} = (Q/D) \oplus G_+$. Conversely, given such a Q, the last equation defines a corresponding prime $\mathfrak{p}$ in G which is a prime divisor of $\hat{\mathfrak{a}}_0$. Localising at $B\backslash Q$ and relabelling B_Q, etc., as B, etc., we get that Q and D are nilpotent so that $\mathfrak{p}$ is nilpotent; hence 0_G is primary to the unique maximal ideal $\mathfrak{p}$ in G. Moreover,

$$\begin{aligned}\lg_B(0_B) &= \sum_{i\geq 0} \lg_B(D^i/D^{i+1}) \\ &= \sum_{i\geq 0} \lg_{G_{\mathfrak{p}}}(G_+^i/G_+^{i+1})_{\mathfrak{p}} \\ &= \lg_G(0_G)\ .\end{aligned}$$

Hence the corresponding j-multiplicities coincide.
The following notation is useful for the remainder of this section. Let $^- : \mathcal{R} \longrightarrow G$ denote the natural epimorphism and (as essentially noted in Section 1.1) for an ideal I of B (respectively $\mathcal{R}$, G) let $\mathcal{U}_{\mathfrak{b}}(I)$ denote the intersection of those top-dimensional primary components of I whose radicals do not contain D (respectively $\mathcal{R}_+$, G_+); $\mathcal{U}_{\mathfrak{a}}(I)$ has then the obvious meaning.
Returning to our original notation now, suppose next that the prime $\mathfrak{p}$ in G is relevant and top-dimensional. Then the corresponding prime P in $\mathcal{R}$ is relevant, one-codimensional and $u \in P$. Consider the decomposition

$$0_{\mathcal{R}} = \tilde{\mathfrak{a}}_0 \cap \tilde{\mathfrak{b}}_0 \cap \tilde{\mathfrak{c}}_0$$

where $\tilde{\mathfrak{c}}_0$ is the intersection of the smaller-dimensional primary components. Since P is relevant and $\mathcal{R}_P$ has depth one, we deduce that $0_{\mathcal{R}_P} = (\tilde{\mathfrak{b}}_0)_P$ so that $0_{G_{\mathfrak{p}}} = \left(\overline{\tilde{\mathfrak{b}}_0}\right)_{\mathfrak{p}}$. This last equation implies that $\overline{\tilde{\mathfrak{b}}_0} \subseteq \hat{\mathfrak{b}}_0$ and, in fact, $\hat{\mathfrak{b}}_0 = \mathcal{U}(\overline{\tilde{\mathfrak{b}}_0})$. Since $\tilde{\mathfrak{b}}_0 = \mathfrak{b}_0^*$ we have

$$\begin{aligned}G/\overline{\tilde{\mathfrak{b}}}_0 = \mathcal{R}/\mathfrak{b}_0^* + (u) &= \mathcal{R}(B/\mathfrak{b}_0, D + \mathfrak{b}_0/\mathfrak{b}_0)/(u) \\ &= G(B/\mathfrak{b}_0, D + \mathfrak{b}_0/\mathfrak{b}_0)\ .\end{aligned}$$

Hence we may pass to the case $\mathfrak{b}_0 = 0$ (and ht $D > 0$); note that $\hat{\mathfrak{b}}_0$ is invariant under this base-change.

Now consider the next stage of the algorithm in G, where we 'cut' with z'_0. Since ℓ'_0 is superficial in D and a non-zero-divisor, we deduce from Section 1.3 that $[(z'_0)]_n = [\overline{(\ell'_0)^*}]_n$, $n \gg 0$, (recall that $\ell'_0 \cdot D^{n-1} = (\ell'_0) \cap D^n$, $n \gg 0$). Hence $(z'_0)_{\mathfrak{p}} = (\overline{(\ell'_0)^*})_{\mathfrak{p}}$, for all homogeneous relevant primes $\mathfrak{p}$ in G. Considering top-dimensional such primes containing z'_0, we deduce that

$$\mathcal{U}_{\mathfrak{b}}((z'_0)) = \mathcal{U}_{\mathfrak{b}}(\overline{(\ell'_0)^*}) .$$

Hence

$$\begin{aligned} \hat{\mathfrak{b}}_1 &= \mathcal{U}_{\mathfrak{b}}(\hat{\mathfrak{b}}_0 + (z'_0)) \\ &= \mathcal{U}_{\mathfrak{b}}((z'_0)) && \text{by a 'depth-one' argument as before, since } z'_0 \text{ avoids relevant} \\ & && \text{associated primes of } G, \\ &= \mathcal{U}_{\mathfrak{b}}(\overline{(\ell'_0)^*}) && \text{by the above.} \end{aligned}$$

So, as regards the $\mathfrak{b}$-cycle, we can factor out ℓ'_0 and proceed to the next step (replacing the ground field K by the algebraic closure of $K(u_{00}, \ldots, u_{0n})$ and localising in a suitable way).

Turning finally to the $\mathfrak{a}$-cycle, let $\mathfrak{p}$ be a minimal prime divisor of $\hat{\mathfrak{b}}_0 + (z'_0)$ such that $G_+ \subseteq \mathfrak{p}$; as discussed above, $\mathfrak{p}$ corresponds to a prime ideal Q of B with $D \subseteq Q$; here Q is of codimension one. Localise at Q and relabel B_Q as B, etc. Note that ℓ'_0 is still superficial, and still of order one: this easily follows from the fact that ℓ'_0 remains a non-zero-divisor and the equation $\ell'_0 \cdot D^{n-1} = (\ell'_0) \cap D^n$, $n \gg 0$ remains valid under localisation. Moreover B is now 1-dimensional. Then ℓ'_0 is a minimal reduction of D (see Section 1.3), so z'_0 is a minimal reduction of G_+. Hence

$$\begin{aligned} \lg(G/\hat{\mathfrak{b}}_0 + (z'_0)) &= e(z'_0, G/\hat{\mathfrak{b}}_0) \\ &= e(z'_0, G) && \text{by the Associativity Law} \\ &= e(G_+, G) \\ &= e(D, B) && \text{since } \lg_G(G/G_+^m) = \lg_B(B/D^m) \\ &= e(\ell'_0, B) \\ &= \lg(B/(\ell'_0)) , \end{aligned}$$

so the $\mathfrak{a}$-cycles coincide, etc.

Thus we have (setting, in short, $G_D(\mathfrak{b}_i, B) := \bigoplus_{m \geq 0} ((\mathfrak{b}_i \cap D^m) + D^{m+1})/D^{m+1} = \overline{\mathfrak{b}_i^*}$, $i = 0, 1, \ldots$ as in [2]):

Theorem 2. 1. *There is a one-one correspondence between the minimal prime divisors $\mathfrak{p}$ of $\hat{\mathfrak{a}}_i$ and the minimal prime divisors P of $\tilde{\mathfrak{a}}_i$ via*

$$\mathfrak{p} = [P]_0/D \oplus G_+$$

and the corresponding j-multiplicities coincide.

2. *For all relevant primes $\mathfrak{p}$ in $G(B, D)$, with $\bar{\ }: \mathcal{R}(B, D) \longrightarrow G(B, D)$ the canonical epimorphism, we have*

$$\hat{\mathfrak{b}}_i \cdot G_{\mathfrak{p}} = (z'_0, \ldots, z'_{i-1}) \cdot G_{\mathfrak{p}} = (\overline{(\ell'_0, \ldots, \ell'_{i-1})^*})_{\mathfrak{p}} = G_D(\mathfrak{b}_i, B)_{\mathfrak{p}} , \quad i = 0, 1, \ldots .$$

3. Rationality

Returning to the terminology of Section 1.1, van Gastel [12, Proposition 3.9], using a geometrical approach, showed explicitly that a K-rational component of the Stückrad-Vogel cycle is necessarily a distinguished variety of the intersection $X \cap Y$ in the sense of Fulton [10]. Subsequently Achilles and Manaresi [1] obtained a local version of [12, Proposition 3.9] but at the level of the intersection $\bar{X} \cap \bar{Y}$, using the theory of analytic spread (see [1, (2.3) *iii*)]).

Here we refine and explicate these results by showing algebraically that distinguished varieties of the intersection $\bar{X} \cap \bar{Y}$ are in one-one correspondence with distinguished varieties of the intersection $X \cap Y$ and so are necessarily K-rational; in fact we give a non-reduced generalization as well. Our basic technique is the use of flatness, especially in the case of tensor products over fields (see Section 1.4); for a previous application of these ideas in intersection theory, see [19]. For another consideration of van Gastel's result, see [7, 2.4.13].

We first employ a general approach.

Proposition 3. *Let K be an algebraically closed field, A a finitely generated K-algebra and $K \subseteq L$ a field extension; let $B := A \otimes_K L$. Consider a minimal prime ideal P of B and let Q be the isolated P-primary component of 0_B. Let $\mathfrak{p} := P \cap A$ and $\mathfrak{q} := Q \cap A$. Then $\mathfrak{p}$ is a minimal prime of A, and $\mathfrak{q}$ is the isolated $\mathfrak{p}$-primary component of 0_A, $P = \mathfrak{p} \cdot B$, $Q = \mathfrak{q} \cdot B$ and $\lg_A(\mathfrak{q}) = \lg_B(Q)$.*
Conversely, any isolated primary component of 0_A arises in this way.

Proof. By base change, B is a (faithfully) flat extension of A; thus the extension $A \subseteq B$ satisfies the "Going Down" property. Hence $\mathfrak{p}$ is a minimal prime. As noted in Section 1.4, $\mathfrak{p} \cdot B$ is a prime ideal. Since P is minimal, we deduce that $\mathfrak{p} \cdot B = P$. (Note that this argument needs no noetherian hypothesis.) As mentioned in Section 1.4, $\mathfrak{q} \cdot B$ is a primary ideal. Since $\mathfrak{q} \cdot B \subseteq P$ and P is minimal, we deduce that $\sqrt{\mathfrak{q} \cdot B} = P$. Noting that $\mathfrak{q} \cdot B \subseteq Q$, it follows from the uniqueness of isolated primary components that $\mathfrak{q} \cdot B = Q$.

Consider the faithfully flat extension $A_\mathfrak{p} \subseteq B_P$. Then $\mathfrak{q}_\mathfrak{p} \subseteq Q_P \cap A_\mathfrak{p} = 0_{B_P} \cap A_\mathfrak{p} = 0_{A_\mathfrak{p}}$, and hence $\mathfrak{q}$ is indeed the isolated $\mathfrak{p}$-primary component of 0_A.

Moreover the extension $(A/\mathfrak{q})_\mathfrak{p} \longrightarrow (B/Q)_P$ is faithfully flat and the extension functor preserves residue fields, since $\mathfrak{p} \cdot B = P$. Hence $\lg_A(\mathfrak{q}) = \lg_B(Q)$.

Finally, let $\mathfrak{p}$ be a minimal prime of A. Then the extended ideal $\mathfrak{p} \cdot B_\mathfrak{p}$ is nilpotent; it follows that there is a minimal prime P of B such that $\mathfrak{p} = P \cap A$. Let $\mathfrak{q}$ be the isolated $\mathfrak{p}$-primary component of 0_A. The argument in the first part of the proof actually establishes that $\mathfrak{q} \cdot B$ is the isolated P-primary component of 0_B, and the full result follows.

Corollary 4. (to the proof) *Let K be an algebraically closed field, A a K-algebra and $K \subseteq L$ a field extension; let $B := A \otimes_K L$. Consider a minimal prime P of B and let $\mathfrak{p} = P \cap A$. Then $\mathfrak{p}$ is a minimal prime of A and $P = \mathfrak{p} \cdot B$. Moreover, every minimal prime of A arises in this way.*

Remarks. 1. It suffices in the above result to assume that the field extension $K \subseteq L$ is separable and q.m.a. (see Section 1.4).

2. In the case where A is finitely generated, the Corollary follows directly from the Proposition on replacing A by $A/\mathfrak{p}$ at the appropriate point in the argument.

We now apply these results in the following context, using in the main the terminology and notation of Section 1.1.

Let X and Y be pure-dimensional closed subschemes of $\mathbf{P}^n_K$, where K is an algebraically closed field, let $\mathcal{O}_J$ be the coordinate ring of the affine cone over the ruled join J in $\mathbf{P}^{2n+1}_K$ and let D be the image in $\mathcal{O}_J$ of the vanishing ideal of the diagonal $\Delta \subseteq \mathbf{P}^{2n+1}_K$. Denote by L the algebraic closure of $K(u_{00}, \dots, u_{nn})$, where $\{u_{ij} \mid 0 \le i, j \le n\}$ are indeterminates over K.

Set $A = G(\mathcal{O}_J, D)$ and $B = A \otimes_K L$; so $B = G(\mathcal{O}_{\bar{J}}, \bar{D})$. Then A is graded with $K \subseteq [A]_0$, and B is graded with $L \subseteq [B]_0$; indeed B has the grading induced from A by the action of $- \otimes_K L$. Let P be a minimal prime of B, with Q the isolated P-primary component of 0_B. Set $\mathfrak{p} := P \cap A$, $\mathfrak{q} := Q \cap A$. Then P, Q, $\mathfrak{p}$ and $\mathfrak{q}$ are homogeneous and it follows immediately from the Proposition that

$$[P]_0 = [\mathfrak{p}]_0 \cdot [B]_0 \ , \quad [Q]_0 = [\mathfrak{q}]_0 \cdot [B]_0 \ .$$

Clearly then, $[P]_0$ and $[Q]_0$ are K-rational.

Thus, noting that $\bar{\Delta}$ is K-rational and recalling the definition of distinguished closed subschemes, etc., of $X \cap Y$ and $\bar{X} \cap \bar{Y}$ from Section 1.1, we have :

Theorem 5. *The distinguished closed subschemes (resp. distinguished subvarieties) in $\mathbf{P}^n_{\bar{K}}$ of the intersection $\bar{X} \cap \bar{Y}$ are in one-one correspondence with the distinguished closed subschemes (resp. distinguished subvarieties) in $\mathbf{P}^n_K$ of the intersection $X \cap Y$ via the base change $K \longrightarrow \bar{K}$ giving extension and contraction of defining ideals; in particular, the former are necessarily K-rational.*

Remarks. **1.** Following the above notation, suppose that $P_1, \dots, P_r$ are the minimal primes P' in B such that $[P']_0 = [P]_0$: denote the corresponding primary components of 0_B by $Q_1, \dots, Q_r$ and let $\bar{Q} = Q_1 \cap \dots \cap Q_r$. Then it follows from the above that $\bar{Q}$ is also K-rational (as is $[Q]_0$), and this continues to be the case on localisation at $[P]_0$. This situation was considered in [9, 2.9 and 3.12] from the slightly coarser viewpoint of fundamental cycles, where it was shown that the j-multiplicity of the distinguished component of the Stückrad-Vogel cycle defined by $[P]_0$ is given by the degree of the projective scheme Proj $((B/\bar{Q})_{[P]_0})$ (using our notation, where we regard $B/\bar{Q}$ as an $\mathcal{O}_{\bar{J}}/\bar{D}$-module). The fundamental cycle of Proj $((B/\bar{Q})_{[P]_0})$ was also considered, at least implicitly, by R. Achilles and M. Manaresi [2, (3.11), (1.2) and (1.3)]; it is in fact the fundamental cycle of Proj $((B/[P]_0^t B)_{[P]_0})$, t sufficiently large, whose degree is the μ-multiplicity of [2].

2. Again we could formulate a non-noetherian graded version if we confine ourselves purely to prime ideals.

3. We can apply the argument above to a localisation of A by a multiplicatively closed set of elements of degree zero.

4. Thus in [1, Theorem (2.2)], $C \in \mathcal{C}(X, Y)$ being K-rational need not be an a priori hypothesis but could instead be added to [1, Theorem (2.2) *i*)].

Moreover, applying [2, Proposition (1.5)] and its proof to the faithfully flat extension $[A]_{0[\mathfrak{p}]_0} \longrightarrow [B]_{0[P]_0}$, we deduce from [2, Corollary (3.11) *ii*)] that the intersection multiplicity $j(X,Y;C)$ can be calculated from data in $[A]_{0[\mathfrak{p}]_0}$, i.e. from rational data. (This could also be interpreted in terms of the proposition above, in view of the proof of [2, Theorem (3.8)].)

However, simple examples show that the (not necessarily reduced) component C' of the intersection cycle such that $C'_{red} = C$ need not be rational. We hope to explore this paradox further.

4. An Example: The algorithm localised in non-rational components

Despite the fact that the Stückrad-Vogel algorithm is not dependent on the order of the elements ℓ'_i (cf. the notation in Section 1.1), since any permutation of the $\ell'_0, \dots, \ell'_n$ creates an isomorphism of the rings, components and non-reduced parts involved, after localising at a non-rational component the picture is different. By the proof of Theorem 5, we have that the analytic spread of the diagonal is less than the dimension d of this local ring. This indicates that the first d general linear combinations of $\ell_0, \dots, \ell_n$ may not be sufficiently general anymore. The possibly simplest example of this sort given by is the self-intersection of a smooth conic in $\mathbf{P}^2$. Here we follow Section 1.1, with $n = 2$ and $X = Y$, and assume that char $K \neq 2$.

Let X be the conic defined by $x_0^2 - x_1^2 - x_2^2 = 0$. The self-intersection of X is well-studied (see, for instance, [25]) and we know that the intersection cycle consists of X and two non-rational points on X. Let P be one of these points. We consider $A := \mathcal{O}_{J(\bar{X},\bar{X}),J(\bar{P},\bar{P})\cap\bar{\Delta}}$, where $\bar{X} := X \times_K \bar{K}$. By the above, we know that the diagonal $D \cdot A$ has analytic spread 2. In fact, it can be shown that $D \cdot A$ is generated by two elements, which can be chosen from $\ell'_0, \ell'_1, \ell'_2$. Furthermore, by our algorithm, $D \cdot A$ is not generated by ℓ'_0, ℓ'_1, yet it is easy to see that the diagonal is generated by ℓ'_0, ℓ'_2 and by ℓ'_1, ℓ'_2. Therefore we cannot change the order of the elements ℓ'_i in our algorithm if we localise in a non-rational component of the intersection cycle.

5. Rationality and Cayley forms

In [1], Achilles and Manaresi's approach to the rational component of the intersection cycle centres round a classical theorem of Northcott and Rees [18] which states that if the analytic spread of an ideal I in a local ring is t, then t 'generic' elements of I yield a minimal reduction of I; see [16, Theorem 14.14], where the restriction to the case I primary to the maximal ideal is not essential. Each of these references uses 'resultant systems' (plural) (cf. [1]). Here we note the folklore connection with an older geometrical notion, that of a *single* 'Cayley' form [14] or 'associated' form [24]; as the latter reference shows, there is a strong interrelationship connecting such a form, Noether normalisations and generic linear projections. The fact that these last two are heavily involved in the theory of analytic spread is well-known, see [23, 13].

More explicitly, we have :

Theorem 6. ([18]) *Let $(A, \mathbf{m}, k)$ be a noetherian local ring with infinite residue field k. Let*

$I = (x_1, \ldots, x_s)$ be an ideal of A with analytic spread r, say. Then if $y_i = \sum_{j=1}^{s} a_{ij} \cdot x_j$, $1 \le i \le r$, are r sufficiently general elements of I, it follows that the ideal $J := (y_1, \ldots, y_r)$ is a minimal reduction of I.

Proof. (Sketch) Consider the special fibre ring $F := \bigoplus_{n \ge 0} I^n/\mathbf{m} \cdot I^n$. Since we wish to deduce that F is a noetherian $k[\bar{y}_1, \ldots, \bar{y}_r]$-module (with $\bar{y}_i$ the image in $[F]_1$ of y_i, $1 \le i \le r$), we may assume that k is algebraically closed. We may also pass to F_{red}, which we relabel as F. Then F can be viewed as the coordinate ring of the affine cone over a closed reduced projective subscheme X of dimension $r-1$ in $\mathbf{P}_k^{s-1}$. Let $F_X(\xi^{(1)}, \ldots, \xi^{(r)})$ be the associated (Cayley) form on $\mathbf{P}_k^{s-1\vee} \times \ldots \times \mathbf{P}_k^{s-1\vee}$ (r factors, '$\vee$' denoting the dual space): recall that $F_X((a_{11}, \ldots, a_{1s}), \ldots, (a_{r1}, \ldots, a_{rs})) = 0$ if and only if X and the hyperplanes defined by the equations $y_i := \sum_{j=1}^{s} a_{ij} \cdot x_j = 0$, $1 \le i \le r$, have a non-empty intersection in $\mathbf{P}_k^{s-1}$. (Thus F_X is the resultant of the defining equations of X and these r linear equations; clearly $F_X = \prod F_{X_i}$, where the X_i are the irreducible components of X and F_{X_i} is the classical Cayley form of the variety X_i.)

Then the elements $\bar{y}_i$ in $[F]_1$ give rise to a Noether normalisation $k[\bar{y}_1, \ldots, \bar{y}_r] \subseteq F$ if and only if $(\bar{y}_1, \ldots, \bar{y}_r) \cdot F$ is F_+-primary, that is, if and only if

$$F_X((a_{11}, \ldots, a_{1s}), \ldots, (a_{r1}, \ldots, a_{rs})) \neq 0 .$$

(Note that F_X is defined over k; this then explains the meaning of 'sufficiently general'.) Since, as noted above, such normalisations correspond to (minimal) reductions of I and since the original field k was infinite, the result now follows.

Remark: In [1], it is then reasonable that generic linear combinations are indeed 'sufficiently' general in the above sense.

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