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# On Associated Discriminants for Polynomials in one Variable

*To Vladinir Igorevich Arnold on the occasion of his 60-th birthday*

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**Abstract.** In this note we introduce a family $\Sigma_i,\ i = 0, \ldots, n-2$ of discriminants in the space $\mathcal{P}_n$ of polynomials of degree $n$ in one variable and study some of their algebraic and topological properties following [1], [10] and [6]. The discriminant $\Sigma_i$ consists of all polynomials $p$ such that some nontrivial linear combination $\alpha_0 p + \alpha_1 p' + \cdots + \alpha_i p^{(i)}$ has a zero of multiplicity greater or equal $i+2$. In particular, using the inversion of differential operators with constant coefficients (which induces the nonlinear involution on $\mathcal{P}_n$) we obtain the algebraic isomorphism of $\Sigma_i$ and $\Sigma_{n-2-i}$ for all $i$.

## 0. Introduction

It is well known that every solution to a finite nonperiodic Toda lattice

$$\begin{aligned} \dot{a}_i &= a_i(b_{i+1} - b_i)\ ,\ \dot{b}_i = a_i - a_{i-1} \\ &(i = 0, ..., i\ ;\ a_i, b_i \in \mathbb{R}\ ;\ a_0 a_1 ... a_{n-1} \neq 0\ ;\ a_{-1} = a_n = 0), \end{aligned}$$

can be presented as

$$a_i(x) = \frac{\Delta^y_{i-1}(x)\Delta^y_{i+1}(x)}{(\Delta^y_i(x))^2}\ ,\ b_i(x) = \frac{d}{dx}\ \ln\left(\frac{\Delta^y_i(x)}{\Delta^y_{i-1}(x)}\right)\ ,$$

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where $\Delta_i^y(x)$ is the $i$-th principal minor of the Hankel matrix

$$H_y(x) = \begin{pmatrix} y(x) & \dots & y^{(n)}(x) \\ y'(x) & \dots & y^{(n+1)}(x) \\ \vdots & \ddots & \vdots \\ y^{(n)}(x) & \dots & y^{(2n)}(x) \end{pmatrix},$$

and $y(x)$ is some solution to the linear ordinary differential equation with constant coefficients of order $n+1$ determined by the Toda lattice, see e.g. [2]. Therefore singularities of solutions correspond to the zeros of the determinants of $\Delta_i^y(x)$. They are also intrinsically related with Schubert calculus, see [4],[7].

The subset of solutions for which the $i$-th principal minor $\Delta_i^y(x)$ has a multiple zero for some $x$ is a hypersurface in the space of all solutions. The union of these hypersurfaces separates the space of (real) solutions into domains of solutions with different qualitative behavior.

A similar situation occurs both in the theory of linear Hamiltonian systems and linear ordinary differential equations with one essential difference that instead of the space of solutions one has to consider the space of fundamental systems. Vanishing of principal minors is related, for example, to the index of the trajectory of a Lagrange subspace. Again the space of fundamental solutions contains certain discriminants formed by all fundamental solutions for which at least one of principal minors vanishes with multiplicity $\geq 2$. The study of the stratification of the space of (fundamental) solutions coming from the union of these discriminants is an important open problem even for linear Hamiltonian systems and ordinary differential equations with constant coefficients.

The present paper is an attempt to study some properties of the above discriminants in the space of polynomials, i.e. in the space of solutions to the simplest equation $x^{(n+1)} = 0$. The paper is organized as follows. Section 1 contains some general information on induced and associated discriminants. In Section 2 we present the corresponding Sylvester formula for associated discriminants. Section 3 contains various algebro-topological information, i.e. duality induced by a special nonlinear involution of the space of monic polynomials, resolution of singularities and a natural stratification. Finally, in Section 4 we calculate the cohomology with compact supports for a specific example.

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## 1. Induced discriminants $\tilde{\Sigma}_i$ and generalized multiple zeros

**1.1. Notations.** Given a function $y(x)$ and a positive integer $i$ let us consider $\Delta_i^y(x)$ the determinant of the $i$-th principal subminor of the above matrix $H_y(x)$. Zeros of $\Delta_i^y(x)$ are called $(i-1)$-*generalized roots of* $y$. (We shift the index for the sake of convenience.)

Consider now a family $\mathcal{F}(x,\lambda_1,\dots,\lambda_k)$ of functions in one variable $x$. *The $i$-th induced discriminant of the family $\mathcal{F}$* is the set $\widetilde{\Sigma}_i(\mathcal{F})$ of values of parameters $\lambda_1,\dots,\lambda_k$ for which $\Delta^{\mathcal{F}}_{i+1}$ (considered as a function in $x$ only) has a multiple zero.

*The $i$-th associated discriminant of the family $\mathcal{F}$* is the set $\Sigma_i(\mathcal{F})$ of values of parameters $\lambda_1,\dots,\lambda_k$ for which $\Delta_i$ satisfies the following condition. There exists a nontrivial linear combination $\alpha_0\mathcal{F}+\alpha_1\mathcal{F}'+\dots+\alpha_i\mathcal{F}^{(i)}$ which has a zero of multiplicity $i+2$ at some point $x$.

The relation between the induced discriminants and the associated discriminants is described by the following proposition.

**1.2. Proposition.** *The $i$-th induced discriminant $\widetilde{\Sigma}_i(\mathcal{F})$, $i\geq 1$, of the family $\mathcal{F}$ coincides with the union of two associated discriminants $\Sigma_{i-1}(\mathcal{F})\cup\Sigma_i(\mathcal{F})$ while $\widetilde{\Sigma}_0(\mathcal{F})=\Sigma_0(\mathcal{F})$.*

*Proof.* First of all, $\widetilde{\Sigma}_0=\Sigma_0$ by definition. The proof of the fact $\widetilde{\Sigma}_i(\mathcal{F})=\Sigma_{i-1}(\mathcal{F})\cup\Sigma_i(\mathcal{F})$ for all $i>1$ is based on manipulations with the subminors in the following Hankel matrix of the family $\mathcal{F}(x,\lambda_1,\dots\lambda_k)$

$$H_{\mathcal{F}}(x,\lambda_1,\dots,\lambda_n)=\begin{pmatrix}\mathcal{F}, & \mathcal{F}', & \dots, & \mathcal{F}^{(i+2)}\\ \mathcal{F}', & \mathcal{F}'', & \dots, & \mathcal{F}^{(i+3)}\\ \vdots & \vdots & \ddots & \vdots\\ \mathcal{F}^{(i+2)}, & \mathcal{F}^{(i+3)}, & \dots, & \mathcal{F}^{(2i+4)}\end{pmatrix},$$

where $\mathcal{F}^{(j)}$ denotes the $j$th derivative of $\mathcal{F}$ w.r.t. $x$. Let us denote by $H^{l_1,\dots,l_p}_{m_1,\dots,m_q}$ the submatrix in $H_{\mathcal{F}}$ formed by the rows with numbers $l_1,\dots,l_p$ and the columns with the numbers $m_1,\dots,m_q$. By $\det^{l_1,\dots,l_p}_{m_1,\dots,m_q}$ we denote the determinant of $H^{l_1,\dots,l_p}_{m_1,\dots,m_q}$.

$\widetilde{\Sigma}_i$ consists of all $\phi\in\mathcal{F}$ such that the determinant $\Delta^{\phi}_{i+1}(x)=\det^{1,\dots,i+1}_{1,\dots,i+1}$ has a multiple zero as a function in $x$, i.e. there exists $x$ such that $\Delta^{\mathcal{F}}_{i+1}=(\Delta^{\mathcal{F}}_{i+1})'=0$. Differentiating $\Delta^{\mathcal{F}}_{i+1}$ w.r.t. $x$ one gets that $(\Delta^{\mathcal{F}})'=\det^{1,\dots,i+1}_{1,\dots,i,i+2}$. Thus $\phi\in\widetilde{\Sigma}_i(\mathcal{F})$ if and only if there exists $x$ such that $\det^{1,\dots,i+1}_{1,\dots,i+1}=\det^{1,\dots,i+1}_{1,\dots,i,i+2}=0$.

At the same time $\phi\in\Sigma_{i-1}(\mathcal{F})$ if $\exists x$ such that the submatrix $H^{1,\dots,i+1}_{1,\dots,i}$ has the rank $<i$ and $\phi\in\Sigma_i(\mathcal{F})$ if $\exists x$ such that the submatrix $H^{1,\dots,i+2}_{1,\dots,i+1}$ or equivalently $H^{1,\dots,i+1}_{1,\dots,i+2}$c has the rank $<i+1$.

Obviously, the union $\Sigma_{i-1}(\mathcal{F})\cup\Sigma_i(\mathcal{F})$ is contained in $\widetilde{\Sigma}_i(\mathcal{F})$. Indeed, if $\exists x$ such that $H^{1,\dots,i+1}_{1,\dots,i}$ has the rank $<i$ then for the same $x$ one gets $\det^{1,\dots,i+1}_{1,\dots,i+1}=\det^{1,\dots,i+1}_{1,\dots,i,i+2}=0$. Analogously, if $\exists x$ such that the rank of $H^{1,\dots,i+1}_{1,\dots,i+2}$ is less than $i+1$ then all its $(i+1)\times(i+1)$-determinants vanish and, in particular, $\det^{1,\dots,i+1}_{1,\dots,i+1}=\det^{1,\dots,i+1}_{1,\dots,i,i+2}=0$.

Let us prove the opposite inclusion. Given $\phi$ and $x$ such that $\det^{1,\dots,i+1}_{1,\dots,i+1}=\det^{1,\dots,i+1}_{1,\dots,i,i+2}=0$ we get two cases. If for this $\phi$ and $x$ the submatrix $H^{1,\dots,i+1}_{1,\dots,i}$ has the rank $<i$ then $\phi$ belongs to $\Sigma_{i-2}(\mathcal{F})$ by definition. At the same time if the rank of $H^{1,\dots,i+1}_{1,\dots,i}$ equals $i$ then the system $\det^{1,\dots,i+1}_{1,\dots,i+1}=\det^{1,\dots,i+1}_{1,\dots,i,i+2}=0$ is exactly equivalent to $rk\ H^{1,\dots,i+1}_{1,\dots,i+2}<i+1$. □

**1.3. Generalized discriminants and generalized roots for polynomials.** In what follows we will study the generalized discriminants only for the families of polynomials over the field $\mathbb{C}$. Let $\mathcal{P}_n$ denote the set of all monic polynomials of degree $n$

$$p(x)=x^n+\lambda_1x^{n-1}+\dots+\lambda_n,\ \lambda_i\in\mathbb{C}\ .$$

From now we will omit the indication of the family if $\mathcal{F} = \mathcal{P}_n$.

For $\mathcal{P}_n$ only the first $n$ determinants $\Delta_m$, $m = 1, \dots, n$ are functions depending on parameters $\lambda_1, \dots, \lambda_n$ and $\Delta_{n+1} = (n+1)!^{n+1}$. Thus the only nontrivial induced discriminants of $\mathcal{P}_n$ are $\tilde{\Sigma}_0, \dots, \tilde{\Sigma}_{n-1}$.

Let us denote by $\Sigma_i$ the standard associated discriminant $\Sigma_i(\mathcal{P}_n)$. $\Sigma_0$ is the usual discriminant (also called *the swallowtail*), i.e. the set of polynomials with multiple zeros and the 0-generalized multiple roots are the usual multiple roots. One has $\tilde{\Sigma}_0 = \Sigma_0$, $\tilde{\Sigma}_{n-1} = \Sigma_{n-2}$, $\tilde{\Sigma}_i = \Sigma_{i-1} \cap \Sigma_i$ for $i = 1, \dots, n-2$.

In the last part of this section we show that any polynomial has a finite number of $i$-generalized roots counted with multiplicities and therefore a finite number of pairwise different multiple roots. This result can be also interpreted as the estimation of the complexity of the selfintersection of the discriminant $\Sigma_i$.

**1.3.1. Proposition.** *The number of i-generalized roots (counted with multiplicities) of any polynomial p of degree n is equal to $(i+1)(n-i)$. Thus the number of multiple i-generalized roots of any polynomial of degree n does not exceed $\frac{(i+1)(n-i)}{2}$.*

*Proof.* One can easily see that for any $p(x) \in \mathcal{P}_n$ the degree of $\Delta_{i+1}(p)$ as a function of $x$ is equal to $(i+1)(n-i)$. Let us now consider in the space $\mathcal{P}_{(i+1)(n-i)}$ the following 2 subsets, namely, the $n$-dimensional image $\Delta_{i+1}(\mathcal{P}_n)$ and the usual discriminant $\Sigma_0$ consisting of all polynomials with multiple zeros. It is not hard to show that $\Delta_{i+1}(\mathcal{P}_n) \not\subset \Sigma_0$. Thus a typical polynomial in $\Delta_{i+1}(\mathcal{P}_n)$ has exactly $(i+1)(n-i)$ pairwise different $i$-generalized zeros. The estimation of the number of pairwise different multiple roots follows. □

**1.3.2. Remark.** To calculate the exact value of the maximal number of pairwise different multiple $i$-generalized roots for polynomials of degree $n$ is apparently a very nontrivial problem for all $i > 0$. (For $i = 0$ the obvious answer is $[\frac{n}{2}]$).

The authors have made some calculations for $i = 1$ and small $n$. Recall that in this case the upper bound for the maximal number $\sharp_{max}(i, n)$ of multiple 1-generalized roots equals $n - 1$.

For $i = 1$ and $n = 2, 3, 4, 5, 6$ the number $\sharp_{max}(i, n)$ equals $1, 1, 3, 4, 4$ resp. For the interesting cases $n = 4$ and $n = 5$ the corresponding polynomials $p$ with the maximal number of pairwise different 1-generalized roots are of the form $x^4 + ax$ and $x^5 + ax$ resp.

## 2. Sylvester formula for $\Sigma_i$

**2.1. Projective closure $\widehat{\Sigma}_i$ of $\Sigma_i$.** In order to write an explicit equation for the hypersurface $\Sigma_i$ let us consider a more general homogeneous problem. Let $\mathbb{P}_1 = \mathbb{P}(U)$ be the projective line obtained by projectivization of a 2-dimensional vector space $U$ with a base $\{e_0, e_1\}$ and let $\{t_0, t_1\}$ be the dual base of $U^*$. We denote by $\mathcal{V}_n$ the space of all homogeneous forms of degree $n$ in $(t_0 : t_1)$:

$$\mathcal{V}_n = \{\, p(t) = \sum_{\nu=0}^{n} \lambda_\nu t_0^\nu t_1^{n-\nu} \mid \lambda_\nu \in \mathbb{C} \,\} \,.$$

Let us consider the linear operator $D : \mathcal{V}_n \to \mathcal{V}_n$ sending an element $p$ to $t_0 \dfrac{\partial p}{\partial t_1}$ and denote by $\mathcal{D}_i \subset End_{\mathbb{C}}(\mathcal{V}_n)$ the subspace of all operators of the form

$$\Psi(D) = \psi_0 + \psi_1 D + \cdots + \psi_i D^i, \qquad \psi_j \in \mathbb{C} .$$

**2.1.1. Lemma.** *Projective closure $\widehat{\Sigma}_i \subset \mathbb{P}_n = \mathbb{P}(\mathcal{V}_n)$ of the surface $\Sigma_i$ consists of all forms $p$ such that $\Psi p$ has a zero of multiplicity $\geq i+2$ at some point $u \in \mathbb{P}_1$ for some $\Psi \in \mathbb{P}(\mathcal{D}_i)$.*

*Proof.* The standard affine chart $\{t_0 = 1\}$ on $\mathbb{P}_1 = \mathbb{P}(U)$ is an affine line $\mathbb{A}_1$ with the coordinate $x = t_1/t_0$ and the standard affine chart $\{\lambda_0 = 1\}$ on $\mathbb{P}_n = \mathbb{P}(\mathcal{V}_n)$ coincides with the affine space $\mathcal{P}_n$ of all monic polynomials of order $n$. The operator $D = \dfrac{d}{dx}$ on this space is induced by the operator $D = t_0 \dfrac{\partial}{\partial t_1}$ on the space $\mathcal{V}_n$. It remains to note that for any $p \in \mathcal{P}_n$ and any nontrivial $i$-tuple $\alpha_0, \alpha_1, \dots, \alpha_i$ the polynomial $\alpha_0 p + \alpha_1 p' + \cdots + \alpha_i p^{(i)}$ has no zeros of multiplisity $\geq i+2$ at infinity, because its degree is greater or equal $n-i$. Therefore the affine restriction of the hypersurface $\widehat{\Sigma}_i$ coincides with our original discriminant $\Sigma_i$. □

**2.2. Derivation of determinantal formula.** In order to write an explicit homogeneous equation defining the hypersurface $\widehat{\Sigma}_i$, consider the product

$$\mathbb{P}_i \times \mathbb{P}_1 = \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(U)$$

and denote by $S(d_1, d_2)$ the space of bihomogeneous forms of bidegree $(d_1, d_2)$ on this product. In other words, $S(d_1, d_2)$ is the space of forms

$$F(\Psi, t) = F(\psi_0, \dots, \psi_i; t_0, t_1)$$

on two groups of variables $\Psi = (\psi_0, \dots, \psi_i)$ and $t = (t_0, t_1)$, which are homogeneous of degree $d_1$ in $\Psi$ and homogeneous of degree $d_2$ in $t$. We associate to each $p \in \mathcal{V}_n$ the collection of $i+2$ forms $F_p^0, F_p^1, \dots, F_p^{i+1} \in S(1, n-i-1)$ defined by

$$F_p^\alpha(\Psi, t) = \left(\frac{\partial}{\partial t_0}\right)^\alpha \left(\frac{\partial}{\partial t_1}\right)^{i+1-\alpha} \Psi p(t) , \qquad \alpha = 0,\, 1,\, \dots ,\, i+1.$$

**2.2.1. Lemma.** *A form $\Psi p$ has a zero of multiplicity $\geq i+2$ at a point $t \in \mathbb{P}_1$ if and only if all $i+2$ forms $F_p^\nu$ vanish at the pair $(\Psi, t) \in \mathbb{P}_i \times \mathbb{P}_1$.*

*Proof.* This follows immediately from Taylors formula for $\Psi p$. □

**2.2.2. Resultant hypersurface.** It is shown in [6] (ch. 13, §2) that all collections of $i+2$ forms on $\mathbb{P}_i \times \mathbb{P}_1$, which have a common zero, form an irreducible hypersurface in the projective space of all collections of $i+2$ forms on $\mathbb{P}_i \times \mathbb{P}_1$ of the same bidegree. An irreducible equation of this hypersurface is called *the resultant* of $i+2$ forms of a given bidegree.

In our case the condition that $i+2$ forms $F_p^0, \dots F_p^{i+1}$ have a common zero is equivalent (see [6], p.439) to the fact that the ideal $(F_p^0, \dots F_p^{i+1})$ generated by these forms does not contain the subspace $S(2, 2n-2i-3)$. More precisely, if we consider the linear operator

$$\partial_p : \underbrace{S(1, n-i-2) \oplus \cdots \oplus S(1, n-i-2)}_{i+2} \longrightarrow S(2, 2n-2i-3)$$

induced by multiplication by our forms $F_p^0, \dots F_p^{i+1}$, i. e.

$$\partial_p : (G_0, G_1, \dots, G_{i+1}) \mapsto \sum_{\alpha=0}^{i+1} G_\alpha F_p^\alpha \ ,$$

then the forms $F_p^\nu$ have a common zero on $\mathbb{P}_i \times \mathbb{P}_1$ if and only if this operator is not epimorphic. Since both spaces $S(1, n-i-2)^{\oplus i+2}$ and $S(2, 2n-2i-3)$ have the same dimension equal to $(i+1)(i+2)(n-i-1)$ the last condition takes the form $\det(\partial_p) = 0$ as soon as we choose some bases in both spaces. So, we get

**2.2.3. Proposition.** *The projective closure $\widehat{\Sigma}_i \subset \mathbb{P}_n = \mathbb{P}(\mathcal{V}_n)$ of the discriminant $\Sigma_i \subset \mathcal{P}_n$ is an irreducible hypersurface of degree $(i+1)(i+2)(n-i-1)$.*

*Proof.* As we have seen above, the condition that for given $p \in \mathcal{V}_n$ there exists a nontrivial operator $\Psi \in \mathbb{P}(\mathcal{D}_i)$ such that $\Psi p$ has a root of multiplicity $\geq i+2$ at some point on $\mathbb{P}_1$ is equivalent to the determinantal condition $\det(\partial_p) = 0$, where $\partial_p$ is the square matrix of the size $(i+1)(i+2)(n-i-1)$ the entries of which are equal to some coefficients of $p$ multiplied by appropriate constants. □

**2.3. Some examples of Sylvester formula.** In order to write down the matrix $\partial_p$, let us fix the standard monomial bases in all spaces. Namely, in the space

$$\underbrace{S(1, n-i-2) \oplus \dots \oplus S(1, n-i-2)}_{i+2}$$

we fix the basis $\{\psi_\beta^{(\alpha)} t_0^\gamma t_1^{n-i-2-\gamma}\}$, where $\alpha = 0, \dots, i+1$ enumerates the direct summands, $\gamma = 0, \dots, n-i-2$ indicates the power of $t$, and $\beta = 0, \dots, i$ enumerates the basic vectors of $\mathcal{D}_i^*$ giving the coefficients of $\Psi = \sum \psi_\beta D^\beta$.

Similarly, in the space $S(2, 2n-2i-3)$ we fix the basis $\{\psi_j \psi_k t_0^l t_1^{2n-2i-3-l}\}$. (Note that $\psi_j \psi_k$ and $\psi_k \psi_j$ are two different notations for the same basic vectors; they are convenient and we hope this will not lead to any confusion in the following formulas).

After some efforts one gets:

$$\partial_p(\psi_\beta^{(\alpha)} t_0^\gamma t_1^{n-i-2-\gamma}) = \sum_{\mu,\nu} \zeta_{\alpha\beta\gamma}^{\mu\nu} \cdot \psi_\beta \psi_{\alpha+\mu} t_0^{\gamma+\nu} t_1^{2n-2i-3-(\gamma+\nu)} \ ,$$

where the summation is taken over all $\mu, \nu$ such that

$$\begin{cases} 0 \leq \nu \leq n-i-1 \\ 0 \leq \gamma + \nu \leq 2n-2i-3 \\ -(n-\nu) \leq \mu \leq \nu \\ 0 \leq \alpha + \mu \leq i \end{cases} .$$

The matrix elements $\zeta_{\alpha\beta\gamma}^{\mu\nu}$ are given by

$$\zeta_{\alpha\beta\gamma}^{\mu\nu} = \underbrace{(\nu+1)(\nu+2)\dots(\nu+\alpha)}_{\alpha} \cdot \underbrace{(n-\nu-i)(n-\nu-i+1)\dots(n-\nu+\mu)}_{i+1+\mu} \cdot \lambda_{\nu-\mu} \ ,$$

where the factors in both underbraced groups are consequently increasing positive integers.

**2.3.1. Example.** The usual discriminant corresponds to the case $i = 0$. In this case for a given $p = \lambda_0 t_1^n + \lambda_1 t_0 t_1^{n-1} + \cdots + \lambda_n t_0^n \in \mathcal{V}_n$ we have to construct two forms

$$F_p^0 = \frac{\partial p}{\partial t_0} = \lambda_1 t_1^{n-1} + 2\lambda_2 t_0 t_1^{n-2} + \cdots + n\lambda_n t_0^{n-1}$$

$$F_p^1 = \frac{\partial p}{\partial t_1} = n\lambda_0 t_1^{n-1} + (n-1)\lambda_1 t_0 t_1^{n-2} + \cdots + \lambda_{n-1} t_0^{n-1}$$

and consider the linear operator

$$\partial_p : \mathcal{V}_{n-2} \oplus \mathcal{V}_{n-2} \to \mathcal{V}_{2n-3}$$

sending $(G_0, G_1)$ to $G_0 F_p^0 + G_1 F_p^1$.

The above bases specialize to the standard monomial bases $t_0^\alpha t_1^\beta$ of the spaces in question and the matrix of $\partial_p$ gives the classical Sylvester representation for the usual discriminant of a homogeneous form $p$. For example, we get the well-known formula

$$\det \partial_p = \det \begin{pmatrix} \lambda_1 & 2\lambda_2 & 3\lambda_3 & 0 \\ 0 & \lambda_1 & 2\lambda_2 & 3\lambda_3 \\ 3\lambda_0 & 2\lambda_1 & \lambda_2 & 0 \\ 0 & 3\lambda_0 & 2\lambda_1 & \lambda_2 \end{pmatrix} = 3(4\lambda_1^3\lambda_3 + 4\lambda_0\lambda_2^3 + 27\lambda_0^2\lambda_3^2 - \lambda_1^2\lambda_2^2 - 18\lambda_0\lambda_1\lambda_2\lambda_3)$$

for $n = 3$ and

$$\det \partial_p = \det \begin{pmatrix} \lambda_1 & 2\lambda_2 & 3\lambda_3 & 4\lambda_4 & 0 & 0 \\ 0 & \lambda_1 & 2\lambda_2 & 3\lambda_3 & 4\lambda_4 & 0 \\ 0 & 0 & \lambda_1 & 2\lambda_2 & 3\lambda_3 & 4\lambda_4 \\ 4\lambda_0 & 3\lambda_1 & 2\lambda_2 & \lambda_3 & 0 & 0 \\ 0 & 4\lambda_0 & 3\lambda_1 & 2\lambda_2 & \lambda_3 & 0 \\ 0 & 0 & 4\lambda_0 & 3\lambda_1 & 2\lambda_2 & \lambda_3 \end{pmatrix} =$$

$= 16(-\lambda_1^2\lambda_2^2\lambda_3^2 - 256\lambda_4^3\lambda_0^3 + 4\lambda_1^2\lambda_2^3\lambda_4 + 27\lambda_0^2\lambda_3^4 + 4\lambda_1^3\lambda_3^3 + 27\lambda_1^4\lambda_4^2 + 6\lambda_1^2\lambda_3^2\lambda_4\lambda_0 + 192\lambda_1\lambda_3\lambda_4^2\lambda_0^2 - 18\lambda_1\lambda_3^3\lambda_0\lambda_2 - 18\lambda_1^3\lambda_3\lambda_4\lambda_2 + 80\lambda_1\lambda_3\lambda_4\lambda_0\lambda_2^2 - 144\lambda_4\lambda_0^2\lambda_3^2\lambda_2 - 144\lambda_1^2\lambda_4^2\lambda_0\lambda_2 + 128\lambda_4^2\lambda_0^2\lambda_2^2 + 4\lambda_2^3\lambda_3^2\lambda_0 - 166\lambda_2^4\lambda_4\lambda_0)$ for $n = 4$.

**2.3.2. Example.** In another extremal case $i = n - 2$ for a given $p = \sum \lambda_\nu t_0^\nu t_1^{n-\nu}$ we have to construct $n$ bilinear forms

$$F^\alpha(\psi, t) = \sum_{j=0}^{n-2} \psi_j (c_{j0}^\alpha t_0 + c_{j1}^\alpha t_1) , \quad \alpha = 0, 1, \ldots, n-1 .$$

Coefficient in parenthesis is equal to $\left(\frac{\partial}{\partial t_0}\right)^\alpha \left(\frac{\partial}{\partial t_1}\right)^{n-1-\alpha} t_0^j \left(\frac{\partial}{\partial t_1}\right)^j p(t_0, t_1)$ and the constants $c_{**}^*$ are given by

$$c_{j0}^\alpha = \begin{cases} (\alpha+1)!(n+j-\alpha-1)!\lambda_{\alpha+1-j}, & \text{for } -1 \le \alpha - j \le n-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$c^{\nu}_{i1} = \begin{cases} (\alpha)!(n+i-\alpha)!\lambda_{\alpha-i}, & \text{for } 0 \le \alpha - i \le n \\ 0, & \text{otherwise} \end{cases} .$$

The operator of multiplication by these forms

$$\partial_p : \underbrace{S(1,0) \oplus \cdots \oplus S(1,0)}_{n} \to S(2,1)$$

is represented by a square matrix of the size $n(n-1)$ and acts on the vectors of the above basis by the rule

$$\partial_p(\psi^{(\alpha)}_k) = \sum_{i=0}^{n-2} \psi_k \psi_j (c^{\alpha}_{j0} t_0 + c^{\alpha}_{j1} t_1) = \sum_{j,l} \xi^{\alpha}_{jkl} \psi_j \psi_k t_l \ ,$$

where

$$\xi^{\alpha}_{jkl} = \begin{cases} (\alpha+1)!(n+j-\alpha-1)!\lambda_{\alpha-j+1} & \text{for } l=0;\ -1 \le \alpha - j \le n-1 \\ (\alpha)!(n+j-\alpha)!\lambda_{\alpha-j} & \text{for } l=1;\ 0 \le \alpha - j \le n \\ 0, & \text{otherwise} \end{cases}$$

and $\nu = 0, \ldots, n-1;\ j,k = 0,1,\ldots n-2;\ l = 0,1.$

If we order $\psi^{(\alpha)}_k$ as

$$\psi^0_0, \ldots, \psi^0_{n-1}\ ,\ \psi^1_0, \ldots, \psi^1_{n-2}, \ldots, \ldots, \psi^{n-1}_0\ ,\ \ldots, \psi^{n-1}_{n-2}\ ,$$

take $\psi_j \psi_k t_l$ only with $j \le k$, and order them lexicographically, then, for example, for $n=3$ we get

$$\partial_p = \begin{pmatrix} 2!\lambda_1 & 3!\lambda_0 & 0 & 3!\lambda_0 & 0 & 0 \\ 0 & 2\lambda_1 & 3!\lambda_0 & 0 & 3!\lambda_0 & 0 \\ 2\lambda_2 & 4\lambda_1 & 0 & 2\lambda_1 & 3!\lambda_0 & 0 \\ 0 & 2\lambda_2 & 4\lambda_1 & 0 & 2\lambda_1 & 3!\lambda_0 \\ 3!\lambda_3 & 3!\lambda_2 & 0 & 2\lambda_2 & 4\lambda_1 & 0 \\ 0 & 3!\lambda_3 & 3!\lambda_2 & 0 & 2\lambda_2 & 4\lambda_1 \end{pmatrix},$$

$$\det \partial_p = 512(72\lambda_1^4\lambda_0\lambda_2 - 189\lambda_0^2\lambda_2^2\lambda_1^2 - 108\lambda_0^2\lambda_1^3\lambda_3 - 729\lambda_0^4\lambda_3^2 + 108\lambda_0^3\lambda_2^3 - 8\lambda_1^6 + 486\lambda_0^3\lambda_2\lambda_1\lambda_3)\ .$$

For $n=4$ the operator $\partial_p$ has the matrix

$$\partial_p = \begin{pmatrix} 6\lambda_1 & 24\lambda_0 & 0 & 0 & 0 & 0 & 24\lambda_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6\lambda_1 & 0 & 24\lambda_0 & 0 & 0 & 0 & 24\lambda_0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6\lambda_1 & 0 & 24\lambda_0 & 0 & 0 & 0 & 24\lambda_0 & 0 & 0 & 0 \\ 4\lambda_2 & 12\lambda_1 & 48\lambda_0 & 0 & 0 & 0 & 6\lambda_1 & 24\lambda_0 & 0 & 0 & 0 & 0 \\ 0 & 4\lambda_2 & 0 & 12\lambda_1 & 48\lambda_0 & 0 & 0 & 6\lambda_1 & 0 & 24\lambda_0 & 0 & 0 \\ 0 & 0 & 4\lambda_2 & 0 & 12\lambda_1 & 24\lambda_0 & 0 & 0 & 6\lambda_1 & 0 & 24\lambda_0 & 0 \\ 6\lambda_3 & 12\lambda_2 & 36\lambda_1 & 0 & 0 & 0 & 4\lambda_2 & 12\lambda_1 & 48\lambda_0 & 0 & 0 & 0 \\ 0 & 6\lambda_3 & 0 & 12\lambda_2 & 36\lambda_1 & 0 & 0 & 4\lambda_2 & 0 & 12\lambda_1 & 48\lambda_0 & 0 \\ 0 & 0 & 6\lambda_3 & 0 & 12\lambda_2 & 36\lambda_1 & 0 & 0 & 4\lambda_2 & 0 & 12\lambda_1 & 48\lambda_0 \\ 24\lambda_4 & 24\lambda_3 & 48\lambda_2 & 0 & 0 & 0 & 6\lambda_3 & 12\lambda_2 & 36\lambda_1 & 0 & 0 & 0 \\ 0 & 24\lambda_4 & 0 & 24\lambda_3 & 48\lambda_2 & 0 & 0 & 6\lambda_3 & 0 & 12\lambda_2 & 36\lambda_1 & 0 \\ 0 & 0 & 24\lambda_4 & 0 & 24\lambda_3 & 48\lambda_2 & 0 & 0 & 6\lambda_3 & 0 & 12\lambda_2 & 36\lambda_1 \end{pmatrix}$$

and zero set of its determinant consists of all polynomials

$$p = \lambda_0 t_1^4 + \lambda_1 t_1^3 t_0 + \lambda_2 t_1^2 t_0^2 + \lambda_3 t_1 t_0^3 + \lambda_4 t_0^4$$

such that there exists a nontrivial linear combination $\alpha_0 p + \alpha_1 p' + \alpha_2 p''$, which has a zero of multiplicity 4.

**2.3.3. Example.** In the simplest nonextremal case $\Sigma_1 \subset \mathcal{P}_4$ we get the following explicit expression for $\partial_p$ (where the coordinates of $\partial_p(\psi_\beta^{(\alpha)} t_0^\gamma t_1^{1-\gamma})$ will be written in rows to save space):

$$\partial_p = \begin{pmatrix}
12\lambda_0 & 6\lambda_1 & 2\lambda_2 & 0 & 0 & 24\lambda_0 & 6\lambda_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 12\lambda_0 & 6\lambda_1 & 2\lambda_2 & 0 & 0 & 24\lambda_0 & 6\lambda_1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 12\lambda_0 & 6\lambda_1 & 2\lambda_2 & 0 & 0 & 24\lambda_0 & 6\lambda_1 & 0 \\
0 & 0 & 0 & 0 & 0 & 12\lambda_0 & 6\lambda_1 & 2\lambda_2 & 0 & 0 & 24\lambda_0 & 6\lambda_1 \\
3\lambda_1 & 4\lambda_2 & 3\lambda_3 & 0 & 12\lambda_0 & 12\lambda_1 & 6\lambda_2 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\lambda_1 & 4\lambda_2 & 3\lambda_3 & 0 & 12\lambda_0 & 12\lambda_1 & 6\lambda_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3\lambda_1 & 4\lambda_2 & 3\lambda_3 & 0 & 12\lambda_0 & 12\lambda_1 & 6\lambda_2 & 0 \\
0 & 0 & 0 & 0 & 0 & 3\lambda_1 & 4\lambda_2 & 3\lambda_3 & 0 & 12\lambda_0 & 12\lambda_1 & 6\lambda_2 \\
2\lambda_2 & 6\lambda_3 & 12\lambda_4 & 0 & 6\lambda_1 & 12\lambda_2 & 12\lambda_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 2\lambda_2 & 6\lambda_3 & 12\lambda_4 & 0 & 6\lambda_1 & 12\lambda_2 & 12\lambda_3 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\lambda_2 & 6\lambda_3 & 12\lambda_4 & 0 & 6\lambda_1 & 12\lambda_2 & 12\lambda_3 & 0 \\
0 & 0 & 0 & 0 & 0 & 2\lambda_2 & 6\lambda_3 & 12\lambda_4 & 0 & 6\lambda_1 & 12\lambda_2 & 12\lambda_3
\end{pmatrix}$$

The zero set of the determinant $\det \partial_p$ consists of all polynomials

$$p = \lambda_0 t_1^4 + \lambda_1 t_1^3 t_0 + \lambda_2 t_1^2 t_0^2 + \lambda_3 t_1 t_0^3 + \lambda_4 t_0^4$$

such that there exists a nontrivial linear combination $\alpha_0 p + \alpha_1 p'$, which has a zero of multiplicity $\geq 3$.

## 3. Affine geometric and topological properties of $\Sigma_i$

**3.1. Differential operators.** Our studying of the affine geometry of the hypersurface $\Sigma_i \subset \mathcal{P}_n$ is based on the (non-canonical) affine isomorphism between the spaces of monic polynomials and invertible linear differential operators with constant coefficients.

**3.1.1. Notation.** Consider the vector space $\mathcal{V}_n \subset \mathbb{C}[x]$ of all polynomials of degree $\leq n$ and denote by $\mathcal{A}_n \subset End_{\mathbb{C}}(\mathcal{V}_n)$ the subalgebra of all linear endomorphisms commuting with all translations $p(x) \mapsto p(x-a)$. It is well known (see [3]) that $\mathcal{A}_n$ coincides with the truncated polynomial ring $\mathbb{C}[D]/D^{n+1}$, where $D = d/dx$. As in the previous section, we denote by $\mathcal{D}_i \subset \mathcal{A}_n$ the subspace formed by differential operators of order less or equal $i$ and by $\mathbb{P}(\mathcal{D}_i)$ the projectivization of $\mathcal{D}_i$.

Let $\mathcal{U}_n \subset \mathcal{A}_n$ be the multiplicative subgroup of all unipotent operators, i.e.

$$\mathcal{U}_n = \{1 + \lambda_1 D + \cdots + \lambda_n D^n \bmod(D^{n+1}) \mid \lambda_k \in \mathbb{C}\} .$$

Note that all unipotent operators are invertible. For a subset $M \subset \mathcal{U}_n$ we will denote by $M^{-1} = \{\, m^{-1} \mid m \in M \,\}$ the set of all inverse operators.

**3.1.2. Affine identification of $\mathcal{P}_n$ with $\mathcal{U}_n$.** Let us define *the affine isomorphism*

$$\iota :\ \mathcal{U}_n \longrightarrow \mathcal{P}_n$$

by taking a polynomial $p \in \mathcal{P}_n$ to the unique differential operator $\Psi_p \in \mathcal{U}_n$ such that $p = \Psi_p(x^n)$. In the rest of the paper we will usually identify both spaces by this isomorphism.

Almost all our geometric constructions will be done in the space $\mathcal{U}_n$. First of all, in the proposition 3.2. below we describe the $i$-th associated discriminant $\Sigma_i$ in terms of $\mathcal{U}_n$. In what follows we will denote this hypersurface in $\mathcal{U}_n$ by $\Sigma_i$ as well and will freely use both versions $\Sigma_i \subset \mathcal{P}_n$ and $\Sigma_i \subset \mathcal{U}_n$ without special indication.

**3.2. Proposition.** *The above isomorphism $\iota$ between $\mathcal{U}_n$ and $\mathcal{P}_n$ identifies $\Sigma_i$ with the hypersurface in $\mathcal{U}_n$ consisting of all operators $\Psi$ for which there exists a triple*

$$(\Xi, \Theta, a) \in \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-i-2}) \times \mathbb{C}$$

*such that $\Xi\Psi = \Theta \exp(-aD)$.*

*Proof.* By definition, $\Sigma_i$ consists of polynomials $p \in \mathcal{P}_n$ for which there exists a nontrivial $i$-tuple $\alpha_0, \alpha_1, \dots, \alpha_i$ such that $\alpha_0 p + \alpha_1 p' + \dots + \alpha_i p^{(i)}$ has a zero of multiplicity at least $i+2$ at some point $a \in \mathbb{C}$, i. e. $\alpha_0 p + \alpha_1 p' + \dots + \alpha_i p^{(i)} = (x-a)^{i+2} q(x)$, where $\deg(q) \le n-i-2$. In terms of differential operators this means that there exists a pair of operators $\Xi \in \mathbb{P}(\mathcal{D}_i)$, $\Theta \in \mathbb{P}(\mathcal{D}_{n-i-2})$ and a point $a \in \mathbb{C}$ such that $\Xi(p) = \Theta(x-a)^n$.

Furthermore, the 1-parameter family $(x-a)^n$ can be presented as the orbit of the polynomial $x^n$ under the action of the 1-parameter subgroup $\exp(-aD) \subset \mathcal{U}_n$, where $\exp(aD) = \sum (aD)^j/j! \in \mathcal{U}_n$. Therefore the condition $p \in \Sigma_i$ is equivalent to the existence of a triple

$$(\Xi, \Theta, a) \in \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-i-2}) \times \mathbb{C}$$

such that $p$ satisfies $\Xi(p) = \Theta \exp(-aD)(x^n)$. The isomorphism $\iota : \mathcal{U}_n \to \mathcal{P}_n$ identifies $p$ with the unique $\Psi_p$ such that $p = \Psi_p(x^n)$. Thus $p \in \Sigma_i$ if and only if $\Xi\Psi_p = \Theta \exp(-aD)$. $\square$

**3.2.1. Corollary.** *The involution* inv $:\ \Psi \to \Psi^{-1}$ *of $\mathcal{U}_n$ induces the biregular isomorphism between the affine algebraic hypersurfaces $\Sigma_i$ and $\Sigma_{n-2-i}$ for all $i = 0, \dots, n-2$.*

*Proof.* We have to show that the involution *inv* $: \mathcal{U}_n \to \mathcal{U}_n$ which sends each operator $\Psi \in \mathcal{U}_n$ onto its inverse $\Psi^{-1}$ maps $\Sigma_i$ isomorphically onto $\Sigma_{n-i-2}$. Indeed, $\Psi$ belongs to $\Sigma_i$ if and only if there exists at least one triple

$$\Xi \in \mathbb{P}(\mathcal{D}_i)\,,\ \Theta \in \mathbb{P}(\mathcal{D}_{n-i-2})\,,\ a \in \mathbb{C} \quad \text{such that} \quad \Xi\Psi = \Theta \exp(-aD)\,.$$

The last identity is equivalent to $\Xi \exp(aD) = \Theta\Psi^{-1}$, which exactly means that $\Psi^{-1}$ belongs to $\Sigma_{n-i-2}$. $\square$

**3.2.2. Remark.** The involution *inv* acts on the 1-parameter subgroup $\exp(aD) \subset \mathcal{U}_n$ by changing the sign of $a$. So, under the above isomorphism between $\Sigma_i$ and $\Sigma_{n-2-i}$ polynomials with a multiple generalized zero at a point $a \in \mathbb{C}$ are transformed into polynomials with a multiple generalized zero at the opposite point $-a$.

**3.2.3. Remark.** Since the projective hypersurfaces $\widehat{\Sigma}_i$ and $\widehat{\Sigma}_{n-i-2}$ (considered in the previous paragraph) have different degrees, the isomorphisms between $\Sigma_i$ and $\Sigma_{n-i-2}$ *can not* be extended to the *regular* isomorphisms of their projective closures and give only a birational equivalence.

**3.3. Subdivision of $\Sigma_i$ by $\Sigma_{i,j}$.** One can easily extend the above results to more general loci $\Sigma_{i,j} \subset \mathcal{P}_n$.

By the definition, $\Sigma_{i,j} \subset \mathcal{P}_n$ is the set of all polynomials $p \in \mathcal{P}_n$ such that some nontrivial linear combination $\alpha_0 p + \alpha_1 p' + \cdots + \alpha_i p^{(i)}$ has a zero of multiplicity greater or equal $j+2$. Fixing $a \in \mathbb{C}$ one gets $\Sigma_{i,j}(a) \subset \Sigma_{i,j}$ consisting of all $p \in \Sigma_{i,j}$ with a multiple $i$-generalized zero at $a$. We will call $\Sigma_{i,j}(a)$ *the fiber of* $\Sigma_{i,j}$ *at the point* $a \in \mathbb{C}$.

The locus $\Sigma_{i,i}$ coincides with our original discriminant hypersurface $\Sigma_i$. In the general case, the codimension of $\Sigma_{i,j}$ and $\Sigma_{i,j}(a)$ in $\mathcal{P}_n$ is equal to $j-i+1$ and $j-i+2$ respectively. We have a natural stratification

$$\Sigma_i = \Sigma_{i,i} \supset \Sigma_{i,i+1} \supset \cdots \supset \Sigma_{i,n-3} \supset \Sigma_{i,n-2} \supset \emptyset.$$

Obviously, $\Sigma_{ij} \subset \bigcap_{\nu=i}^{j} \Sigma_\nu = \Sigma_i \cap \Sigma_j$.

Exactly the same way as in Proposition 3.2. we get

**3.3.1. Proposition.** *The stratum* $\Sigma_{i,j}$ *is isomorphic to the set of all operators* $\Psi$ *for which there exists a triple*

$$(\Xi, \Theta, a) \in \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-j-2}) \times \mathbb{C}$$

*such that* $\Xi\Psi = \Theta \exp(-aD)$. □

**3.3.2. Corollary.** *The involution* inv : $\Psi \to \Psi^{-1}$ *of the affine space* $\mathcal{U}_n$ *induces the biregular algebraic isomorphisms between* $\Sigma_{i,j}$ *and* $\Sigma_{n-2-j,n-2-i}$ *for all* $0 \le i \le j \le n-2$ *and between* $\Sigma_{i,j}(a)$ *and* $\Sigma_{n-j-2,n-i-2}(-a)$ *for all* $i+j+1 \ge n$. □

**3.4. Desingularization of $\Sigma_{i,j}$.** Now we describe a natural resolution of singularities of $\Sigma_{i,j}$, which follows from prop. 3.3.1. Let

$$\mathcal{H}_{i,j} \subset \mathbb{P}(\mathcal{D}_i) \times \mathcal{U}_n \times \mathbb{C}$$

be the subvariety of triples $(\Xi, \Psi, a)$ such that $\Xi\Psi = \Theta \exp(-aD)$ for at least one $\Theta \in \mathbb{P}(\mathcal{D}_{n-i-2})$.

We denote by $\mathcal{H}_{i,j}(a)$ the fiber of $\mathcal{H}_{i,j}$ over the point $-a \in \mathbb{C}$ (i. e. the set of all the triples $(\Xi, \Psi, -a)$ with the same $a$). Note that any $\mathcal{H}_{i,j}(a)$ can be naturally algebraically identified with $\mathcal{H}_{i,j}(0)$:

$$(\Xi, \Psi, -a) \in \mathcal{H}_{i,j}(a) \Longleftrightarrow (\Xi, \Psi \exp(aD), 0) \in \mathcal{H}_{i,j}(0)\ .$$

Hence, $\mathcal{H}_{i,j} = \mathbb{C} \times \mathcal{H}_{i,j}(0)$.

We fix the notation

$$\pi : \ \mathcal{H}_{i,j} \to \Sigma_{ij} \in \mathcal{U}_n$$

for the natural projection of $\mathcal{H}_{i,j}$ onto the second factor in $\mathbb{P}(\mathcal{D}_i) \times \mathcal{U}_n \times \mathbb{C}$.

**3.4.1. Proposition.** *$\mathcal{H}_{i,j}(0)$ is a smooth complete intersection of quadrics and thus the restriction*

$$\pi|_{\mathcal{H}_{i,j}(0)} : \ \mathcal{H}_{i,j}(0) \to \Sigma_{ij}(0)$$

*gives a resolution of singularities of $\Sigma_{i,j}(a)$. Moreover, this resolution is semismall, i. e. it satisfies the condition:*

$$codim\,\{p \in \Sigma_{i,j}(a) : \ \dim \pi_i^{-1}(p) \geq l\} = 2l \text{ for all } l\ .$$

*Proof.* A pair $(\Xi, \Psi) = \Big(\alpha_0 + \alpha_1 D + ... + \alpha_i D, \ 1 + \psi_1 D + ... + \psi_n D^n\Big)$ presents a point $(\Xi, \Psi) \in \mathcal{H}_{i,j}(0)$ if and only if the coefficients of degrees $n - j - 1, ..., n$ in the expansion of

$$(\alpha_0 + \alpha_1 D + ... + \alpha_i D)(1 + \psi_1 D + ... + \psi_n D^n) \in \mathcal{A}_n$$

vanish. Thus $\mathcal{H}_{i,j}(0)$ is defined in $\mathbb{P}(\mathcal{D}_i) \times \mathcal{U}_n$ by the system of $j + 2$ quadratic equations in variables $(\alpha_0, \ldots, \alpha_i, \psi_1, \ldots \psi_n)$. In the case $i + j + 1 < n$ this system has the form

$$\begin{cases} \alpha_0\psi_n + \alpha_1\psi_{n-1} + \cdots + \alpha_i\psi_{n-i} = 0 \\ \alpha_0\psi_{n-1} + \alpha_1\psi_{n-2} + \cdots + \alpha_i\psi_{n-i-1} = 0 \\ \vdots \\ \alpha_0\psi_{n-j-1} + \alpha_1\psi_{n-j-2} + \cdots + \alpha_i\psi_{n-i-j-1} = 0 \end{cases}.$$

The Jacobi matrix $J$ of this system is the $(j + 2) \times (2i + j + 3)$ matrix of the form

$$\begin{pmatrix} \psi_n & \psi_{n-1} & \ldots & \psi_{n-i} & 0 & \ldots & 0 & \alpha_i & \alpha_{i-1} & \ldots & \alpha_0 \\ \psi_{n-1} & \psi_{n-2} & \ldots & \psi_{n-i-1} & \vdots & \iddots & \alpha_i & \alpha_{i-1} & \ldots & \alpha_0 & 0 \\ \vdots & \vdots & \vdots & \vdots & 0 & \iddots & \iddots & \iddots & \iddots & \iddots & \vdots \\ \psi_{n-j-1} & \ldots & \ldots & \psi_{n-i-j-1} & \alpha_i & \alpha_{i-1} & \ldots & \alpha_0 & 0 & \ldots & 0 \end{pmatrix}.$$

Since $(\alpha_0, \alpha_1, \ldots, \alpha_i) \neq (0, 0, \ldots, 0)$ the rank of $J$ is always equal to $j + 2$. This means that $\mathcal{H}_{i,j}(0)$ is a smooth complete intersection. The case $i + j + 1 \geq n$ is analogous. The Jacobi matrix contains a unit submatrix and a complementary submatrix of maximal rank.

Let us now consider the map $\pi : \mathcal{H}_{i,j}(0) \to \Sigma_{i,j}(0)$. The inverse image $\pi^{-1}(p)$ of any polynomial $p \in \Sigma_{i,j}(0)$ is a projective subspace. One has that $\dim(\pi^{-1}(p))$ equals to the corank of the above linear system reduced by 1. One can easily show that for a generic $\Psi \in \Sigma_{i,j}(0)$ the corank of the system equals 1 and one gets that $\mathcal{H}_{i,j}(0)$ is a resolution of singularities of $\Sigma_{i,j}(0)$. Moreover, more detailed consideration of the linear system (or of the space of Hankel matrices, see Section 4) shows that the set $\{p \in \Sigma_{i,j}(a) : \ \dim \pi^{-1}(p) \geq l\}$ has codimension $2l$ in $\Sigma_{i,j}(a)$ and $a$. $\square$

**3.4.2. Corollary.** *$\pi: \mathcal{H}_{i,j} \to \Sigma_{ij}$ is a resolution of singularities of $\Sigma_{i,j}$.*

**3.4.3. Remark.** In the case $i+j+1 < n$ the set $\mathcal{H}_{i,j}(0)$ has a nonsingular projective closure

$$\hat{\mathcal{H}}_{i,j}(0) \subset \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}\mathcal{V}_n \ .$$

$\hat{\mathcal{H}}_{i,j}(0)$ also is a smooth complete intersection of quadrics and one can calculate its cohomology as well as that of $\mathcal{H}_{i,j}(0)$.

The existence of a semismall resolution also gives a hope that one can calculate the intersection homology of $\Sigma_{i,j}(0)$. In the case $i+j+1 \geq n$ the variety $\hat{\mathcal{H}}_{i,j}(0)$ is singular at infinity with a conic type of singularities.

**3.5. Topology of the fiber $\boldsymbol{\Sigma_{i,j}(0)}$.** Here we are going to clarify the topology of the zero fiber $\Sigma_{i,j}(0)$. Recall that in terms of the space $\mathcal{U}_n$ of unipotent differential operators the space $\Sigma_{i,j}(0)$ consists of all operators $\Psi \in \mathcal{U}_n$ such that

$$\Xi\Psi = \Theta \qquad \text{for some } \Xi \in \mathbb{P}(\mathcal{D}_i)\,,\ \Theta \in \mathbb{P}(\mathcal{D}_{n-j-2}).$$

Consider the decreasing filtration

$$\Sigma_{i,j}(0) = \Sigma^0_{i,j}(0) \supset \Sigma^1_{i,j}(0) \supset \cdots \supset \Sigma^i_{i,j}(0)\ ,$$

where $\Sigma^l_{i,j}(0)$ consists of all $\Psi \in \Sigma_{i,j}(0)$ such that any $\Xi \in \mathcal{D}_i \setminus 0$ with the property that $\Xi\Psi \in \mathcal{D}_{n-j-2}$ is divisible by $D^l$.

In terms of the Proposition 3.4.1, the closed subset $\Sigma^l_{i,j}(0) \subset \Sigma_{i,j}(0)$ consists of all $\Psi$ such that the projection of the fiber $\pi^{-1}(\Psi) \subset \mathcal{H}_{i,j} \subset \mathbb{P}(\mathcal{D}_i) \times \mathcal{U}_n$ onto the first factor $\mathbb{P}(\mathcal{D}_i)$ is contained in the subspace $\mathbb{P}(D^l \cdot \mathcal{D}_i) \subset \mathbb{P}(\mathcal{D}_i)$.

**3.5.1. Lemma.** *The difference $\Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)$ is isomorphic to the space $CoPr(i, n-i-2)$ of all pairs of coprime polynomials*

$$\Big(1 + \alpha_1 D + \cdots + \alpha_r D^r\ , \quad 1 + \nu_1 D + \cdots + \nu_s D^s\Big)$$

*such that $r \leq i$, $s \leq (n-j-2)$.*

*Proof.* We are going to verify that the complement

$$\Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0) \subset \Sigma_{i,j}(0)$$

consists of all $\Psi(D) = A^{-1}(D)B(D)$, where $A \in \mathcal{U}_n \cap \mathcal{D}_i$, $B \in \mathcal{U}_n \cap \mathcal{D}_{n-j-2}$ are any two coprime (in $\mathbb{C}[D]$) polynomials, and that $\Psi$ and $(A, B)$ are uniquely determined by each other.

If $\Psi \in \Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)$ then there exists at least one pair

$$A' = 1 + \alpha_1 D + \cdots + \alpha_i D^i\ , \quad B' = 1 + \nu_1 D + \cdots + \nu_{n-j-2} D^{n-j-2}$$

such that $\Psi A' = B'$. If $A'$ and $B'$ are not coprime then $A' = HA$, $B' = HB$, where $H = 1 + h_1 D + \cdots + h_m D^m$ is their maximal common factor. Since $H$ is invertible in $\mathcal{U}_n$, one gets $\Psi A = B$ and $\Psi = A^{-1}B$ is represented in the required form.

On the other side, if $A \in \mathcal{U}_n \cap \mathcal{D}_i$ and $B \in \mathcal{U}_n \cap \mathcal{D}_{n-j-2}$ are coprime then the quotient $\Psi = A^{-1}B$, obviously, belongs to $\Sigma^0_{i,j}(0)\Sigma^1_{i,j}(0)$ and it remains to show that this quotient in $\mathcal{U}_n$ determines $A \in \mathcal{U}_n \cap \mathcal{D}_i$ and $B \in \mathcal{U}_n \cap \mathcal{D}_{n-j-2}$ uniquely. Let us assume that for some $A_1 \in \mathcal{U}_n \cap \mathcal{D}_i$, $B_1 \in \mathcal{U}_n \cap \mathcal{D}_{n-j-2}$. We have $A_1^{-1}B_1 = A^{-1}B$ in $\mathcal{U}_n$ and after multiplying by $A_1A$ in $\mathcal{U}_n$ we get the identity $A_1B = AB_1$, which holds in the usual polynomial ring $\mathbb{C}[D]$. Since $A$ and $B$ are coprime, therefore $A_1$ is divisible by $A$. Thus $A_1 = A$ and hence $B_1 = B$. $\square$

**3.5.2. Lemma.** *The homomorphism of reduction modulo $D^{n-k+1}$:*

$$res_{n,k} : \mathcal{U}_n \to \mathcal{U}_{n-k} ,$$

*which sends $A(D) \mod (D^{n+1}) \mapsto A(D) \mod (D^{n-k+1})$, maps the $k$-th complement*

$$\Sigma^k_{i,j}(0) \setminus \Sigma^{k+1}_{i,j}(0) \qquad \text{in} \qquad \Sigma_{i,j}(0) \subset \mathcal{U}_n$$

*epimorphically onto the 0-th complement*

$$\Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0) \qquad \text{in} \qquad \Sigma_{i-k,j}(0) \subset \mathcal{U}_{n-k}$$

*and the corresponding restriction*

$$\Sigma^k_{i,j}(0) \setminus \Sigma^{k+1}_{i,j}(0) \xrightarrow{res_{n,k}} \Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)$$

*is the trivial bundle with the fiber $\mathbb{C}^k$.*

*Proof.* If $\Psi \in \Sigma^k_{i,j}(0) \setminus \Sigma^{k+1}_{i,j}(0) \subset \Sigma_{i,j}(0) \subset \mathcal{U}_n$, then

$$D^kA\Psi = BD^k \qquad \text{in} \quad \mathcal{U}_n$$

for some $A \in \mathcal{U}_n \cap \mathcal{D}_{i-k}$, $B \in \mathcal{U}_n \cap \mathcal{D}_{n-j-2}$. Hence, in $\mathcal{U}_{n-k}$ we have $A\tilde{\Psi} = B$, where $\tilde{\Psi} = res_{n,k}\Psi$ and

$$res_{n,k}\left(\Sigma^k_{i,j}(0) \setminus \Sigma^{k+1}_{i,j}(0)\right) \subset \Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0).$$

To finish the proof, we describe the complete pullback $res^{-1}_{n,k}\left(\Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)\right)$. Let $\tilde{\Psi} \in \Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)$ inside $\Sigma_{i-k,j}(0) \subset \mathcal{U}_{n-k}$. As we have seen above, $A\tilde{\Psi} = B$ in $\mathcal{U}_{n-k}$ for some $A \in \mathcal{U}_{n-k} \cap \mathcal{D}_{i-k}$, $B \in \mathcal{U}_{n-k} \cap \mathcal{D}_{n-j-2}$. If we consider these $A, B$ as elements of $\mathcal{U}_n$, then for any $\Psi \in \mathcal{U}_n$ such that $res_{n,k}\Psi = \tilde{\Psi}$ we get $D^k\Psi A = D^kB$ in $\mathcal{U}_n$. Thus, $\Psi$ belongs to $\Sigma^k_{i,j}(0) \in \mathcal{U}_n$ and

$$res^{-1}_{n,k}\left(\Sigma^0_{i,j}(0) \setminus \Sigma^1_{i,j}(0)\right) \subset \Sigma^k_{i,j}(0) \subset \Sigma_i \subset \mathcal{U}_n.$$

It remains to prove that the left side locus lies inside $\Sigma^k_{i,j}(0) \setminus \Sigma^{k+1}_{i,j}(0)$. But if above $\Psi$ has the property $\Psi \in \Sigma^{k+1}_{i,j}(0)$ then we get the relation

$$D^{k+1}\Psi A_1 = D^{k+1}B \qquad \text{in} \quad \mathcal{U}_n.$$

After the reduction modulo $D^{n-k+1}$ it takes the form $\tilde{\Psi} D A_1 = D B_1$, which contradicts to the condition $\tilde{\Psi} \notin \Sigma^1_{i,j}(0)$. □

**3.5.3. Topology of $\boldsymbol{CoPr(r,s)}$.** The space $CoPr(r,s)$ can be also interpreted as the space of all pairs of monic polynomials of degree $r$ and $s$ with no common roots possibly except for the origin. (The possible multiplicity of a common zero at the origin can vary from 0 to $\min(r,s)$.) Note that the cohomology of the space of all pairs of coprime monic polynomials of some given degrees is known, see e.g. [9] but is not of any immediate help for the space $CoPr(r,s)$. It would be very interesting to calculate the cohomology of the space $CoPr(r,s)$ and to compare it with that of the usual space of rational functions and the braid space, see e.g. [10]. But, unfortunately, the complete information about the cohomology of $\Sigma_{i,j}(0)$ is unavailable at the present moment. This makes it impossible to calculate the cohomology even for the case of stable strata which have the simplest form among $\Sigma_{i,j}$, (see 3.7 below).

**Conjecture.** *The Leray spectral sequence of the above filtrations with* $\mathbf{Z}$*-coefficients collapses at the* $E_2$*-term.*

**3.6. Rationality of $\boldsymbol{\Sigma_{i,j}}$.** In this subsection we will prove the rationality of $\Sigma_{i,j}$ using the filtration of the fiber $\Sigma_{i,j}(0)$ considered in the previous section.

**3.6.1. Lemma.** $\Sigma_{i,j}(0)$ *is a rational variety.*

*Proof.* In fact, a birational equivalence between $\Sigma_{i,j}(0)$ and $\mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-j-2})$ is given in the proof of Lemma 3.5.1. Namely, $\Sigma_{i,j}(0)$ contains a Zariski open subset of all pairs of coprime polynomials

$$\left(1 + \alpha_1 D + \cdots + \alpha_i D^i\,,\ 1 + \mu_1 D + \cdots + \mu_{n-j-2} D^{n-j-2}\right),$$

which have the degrees exactly equal to $i$ and $n-j-2$. □

The following proposition implies the rationality of $\Sigma_{i,j}$.

**3.6.2. Proposition.** *There exists a Zariski open subset in the space* $\mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-j-2}) \times \mathbb{C}$ *isomorphic to a Zariski open subset in* $\Sigma_{i,j}$ *and therefore* $\Sigma_{i,j}$ *is rational.*

*Proof.* As above we have a family of codimension 1 subvarieties $\Sigma_{i,j}(a)$, $a \in \mathbb{C}$ in $\Sigma_{i,j}$, such that for all $a_1 \neq a_2$, $\Sigma_{i,j}(a_1)$ and $\Sigma_{i,j}(a_2)$ are isomorphic. By Propositon 3.6.1 every $\Sigma_{i,j}(a)$ is a rational variety. Since group of affine transformations $x \to cx + d$ acts bitransitively on $\mathbb{C}$, to prove rationality of $\Sigma_{i,j}$ it suffices to show that $\operatorname{codim}_{\mathbb{C}} (\Sigma_{i,j}(0) \cap \Sigma_{i,j}(1))$ in $\Sigma_{i,j}(0)$ is at least 2. This will imply that for a Zariski open subset in $\Sigma_{i,j}$ there exists exactly one triple $(\Xi, \Theta, a) \in \mathbb{P}(\mathcal{D}_i) \times \mathbb{P}(\mathcal{D}_{n-j-2}) \times \mathbb{C}$ such that $\Xi\Psi_p = \Theta e^{aD}$.

Indeed, consider $\Omega = \bigcup_{a_2 \in \mathbb{C} \setminus a_1} (\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2))$. If $\operatorname{codim}_{\mathbb{C}}(\Sigma_{i,j}(0)) \cap \Sigma_{i,j}(1))$ in $\Sigma_{i,j}(0)$ is at least 2 then $\Omega$ has codimension at least 1 in $\Sigma_{i,j}(a)$ and its complement is a Zariski open subset. Varying $a_1$ and taking the union of the complements we get a Zariski open subset in the whole $\Sigma_{i,j}$ for which there exists only one triple $(\Xi, \Theta, a)$ such that $\Xi\Psi_p = \Theta e^{aD}$. The following lemma accomplishes the proof. □

**3.6.3. Lemma.** *For any pair $(a_1, a_2)$, $(a_1 \neq a_2)$ the intersection $\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2)$ has codimension 2 in $\Sigma_{i,j}$.*

*Proof.* As was mentioned before $\mathrm{codim}(\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2))$ does not depend on particular choice of $(a_1, a_2)$. Now let $\hat{\Sigma}_{i,j}(a) \subset \hat{\mathcal{P}}_n$ be the projective closure of $\Sigma_{i,j}(a) \subset \mathcal{P}_n$ and let $\hat{\Sigma}_{i,j} \subset \hat{\mathcal{P}}_n$ be the projective closure of $\Sigma_{i,j}$. One can easily see that the fiber at infinity $\hat{\Sigma}_{i,j}(\infty) = \hat{\Sigma}_{i,j} \setminus \Sigma_{i,j}$ is a projective subspace of dimension $n - 2 - j + i$.

Take the correspondence $Cor$ in the space of $\Sigma_{i,j} \times \mathbb{C}$ consisting of all pairs $(\Psi, \alpha)$ where $\Psi \in \Sigma_{i,j}$ and $\alpha$ is a generalized zero of $\Psi$ and consider its closure $\widehat{Cor} \subset \hat{\Sigma}_{i,j} \times \mathbb{P}_1$. The inverse image of any point $a \in \mathbb{P}_1$ is $\hat{\Sigma}_{i,j}(a)$. It suffices to show that $\mathrm{codim}(\hat{\Sigma}_{i,j}(\infty) \cap \Sigma_{i,j}(a_2))$ in $\Sigma_{i,j}$ equals 2 since the codimension of intersection of any two fibers gives the upper bound for the codimension of intersection of two generic fibers.

Notice that $\hat{\Sigma}_{i,j}(a) \cap \hat{\mathcal{P}}^l \subset \hat{\mathcal{P}}^n$ is isomorphic to $\hat{\Sigma}_{i,j}(a) \subset \hat{\mathcal{P}}^l$ where $\hat{P}^l$ is the projectivized space of polynomials of degree at most $l$ in $t_0 : t_1$. Since the special fiber $\hat{\Sigma}_{i,j}(\infty)$ is isomorphic to $\hat{\mathcal{P}}^{n-j+i-2}$ one gets that $\hat{\Sigma}_{i,j}(\infty) \cap \hat{\Sigma}_{i,j}(a_2) \subset \hat{\mathcal{P}}^n$ is isomorphic to $\hat{\Sigma}_{i,j}(a_2) \subset \hat{\mathcal{P}}^{n-j+i-2}$. The latter space, obviously, has codimension 1 in $\hat{\Sigma}_{i,j}(\infty)$. Thus the former space has codimension 2 in $\hat{\Sigma}_{i,j}$. □

**3.6.4. Lemma.** *For a generic $k$-tuple $(a_1, \ldots, a_k)$ $(a_1 \neq a_2 \neq \cdots \neq a_k)$, $k \leq \frac{n}{j-i+2}$, the intersection $\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2) \cap \cdots \cap \Sigma_{i,j}(a_k)$ has codimension $k$ in $\Sigma_{i,j}$.*

*Proof.* We are going to iterate the above arguments using induction on the number of points and dimension. Namely, assume that we have proved that the codimension is the expected one for a Zariski open set in the space of $(k-1)$-tuples and for all $m < n$. For generic $a_1, a_2, \ldots, a_k$ one has

$$\mathrm{codim}\left(\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2) \cap \cdots \cap \Sigma_{i,j}(a_k)\right) \leq \mathrm{codim}\left(\hat{\Sigma}_{i,j}(\infty) \cap \Sigma_{i,j}(a_2) \cap \cdots \cap \Sigma_{i,j}(a_k)\right).$$

But the later intersection is isomorphic to the intersection of $(k-1)$-tuples in the space $\tilde{\mathcal{P}}^{n-j+i-2}$ and thus has the expected codimension by the inductive assumption. □

**3.6.5. Remark.** It is worth mentioning that not for all $a_1, a_2, \ldots a_k$, $k > 2$, the intersection $\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2) \cap \cdots \cap \Sigma_{i,j}(a_k)$ has the expected codimension. Namely, there exist polynomials $p \in \mathcal{P}_n$ which have more than $[n/2]$ $i$-generalized multiple zeros. This means that the intersection of more than the expected number of $\Sigma_i(a_l)$ is still nonempty, see Section 1. It would be very interesting to describe such 'Weierstrass' $k$-tuples.

**3.7. Some special cases.** A stratum $\Sigma_{i,j} \subset \mathcal{P}_n$ is called *stable* if $j - i + 2 > [\frac{n}{2}]$ and *unstable* otherwise. The following proposition reduces the study of the topology of any stable $\Sigma_{i,j}$ to that of its $\Sigma_{i,j}(0)$.

**3.7.1. Proposition.** *Any stable stratum $\Sigma_{i,j} \subset \mathcal{U}_n$ is isomorphic to $\Sigma_{i,j}(0) \times \mathbb{C}$.*

*Proof.* One has to show that if $j - i + 2 > [n/2]$ then each $p \in \Sigma_{i,j}$ belongs to the unique $\Sigma_{i,j}(a)$, i.e. that the intersection $\Sigma_{i,j}(a_1) \cap \Sigma_{i,j}(a_2)$, $a_1 \neq a_2$ is empty. This can be proved exactly along the same lines as those of the lemma in 3.6.3. □

**3.7.2. Remark.** The stable stratum $\hat{\Sigma}_{0,[\frac{n}{2}]-1} \subset \mathcal{P}_n$ is called the nullcone and is of a fundamental importance in the representation theory of $SL_2(\mathbb{C})$. The following problem was formulated to the authors by J.Weyman in April 1995.

**Question.** *Is it true that $\hat{\Sigma}_{0,[\frac{n}{2}]-1}$ is a set-theoretical complete intersection?*

One can even speculate that set-theoretically one has $\hat{\Sigma}_{0,[\frac{n}{2}]-1} = \bigcap_{i+0}^{[\frac{n}{2}]-1} \hat{\Sigma}_i$.

## 4. Example: cohomology of $\Sigma_1 \subset \mathcal{P}_4$

**4.1. Vassiliev's resolution.** A natural approach to the problem how to calculate the cohomology of $\Sigma_i \subset \mathcal{P}_n$ with compact supports is to try to generalize the simplicial resolution which was succesfully used by V.Vassiliev for the series of strata $\Sigma_0$, see [10] (recall that $\mathcal{P}_n \setminus \Sigma_0$ is the usual braid space). Such a generalized Vassiliev's resolving space for $\Sigma_i$ coincides with the set of all pairs $\{p, S_p\}$, where $p \in \Sigma_i$ and $S_p$ is the formal simplex spanned by all pairwise different multiple $i$-generalized roots of $p$. There exists a natural geometric realization of $\bar{\Sigma}_i$ in $\mathbb{C}^N$ for some sufficiently big $N$.

By construction, this resolving space $\bar{\Sigma}_i$ has a natural filtration

$$\bar{\Sigma}_i = F_m \supset F_{m-1} \supset \cdots \supset F_1 ,$$

where $F_i$ is the set of all pairs $\{p, S_p^i\}$, $p \in \Sigma_i$, $S_p^i$ denotes the $(i-1)$-skeleton of $S_p$, and $m$ is the maximal possible number of multiple pairwise different $i$-generalized zeros. (We use the convention that if $\dim S_p \le i$ then $S_p^i = S_p$.) Note that $F_1$ is homeomorphic to $\Sigma_i(0) \times \mathbb{C}$.

One can easily see that the obvious projection $\bar{\Sigma}_i \to \Sigma_i$ is a homotopy equivalence extendable to their 1-point compactifications. Therefore, $H_c^*(\Sigma_i) = H_c^*(\bar{\Sigma}_i)$.

We will apply this program to $\Sigma_1 \subset \mathcal{P}_4$. In this case we have only the 3-term filtration

$$\bar{\Sigma}_1 = F_3 \supset F_2 \supset F_1 ,$$

since the maximal number of multiple 1-generalized roots of a quartic polynomial equals 3, see 1.3. The main result of this section is the following

**4.2. Proposition.** *The cohomology with compact supports of $\Sigma_1 \subset \mathcal{P}_4$ is equal*

$$\Big(0, 0, 0, 0, 0, \mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}\Big) .$$

The proof of this proposition splits into a series of lemmas and will be finished in 4.6 below.

**4.3. $\Sigma_i(0)$ and Hankel matrices.** Recall that a matrix is called *Hankel* if it has the same entry on each anti-diagonal. Let us denote by $Hank(k, l)$ the set of all $(k \times l)$ Hankel matrices with complex coefficients and we denote by $Hank_{\deg}(k, l) \subset Hank(k, l)$, $k \le l$ the subset of matrices of rank $< k$.

**4.3.1. Lemma.** *If $i+j+1 < n$ then $\Sigma_{i,j}(a)$ is isomorphic to $Hank_{\deg}(i+1, j+2) \times \mathbb{C}^{n-i-j-2}$. If $i+j+1 \geq n$ then $\Sigma_{i,j}(a)$ is isomorphic to $\Sigma_{n-j-2,n-i-2}(-a)$.*

*Proof.* Since $\Sigma_{i,j}(a)$ is isomorphic to $\Sigma_{i,j}(0)$ we consider only the last case. For $\Sigma_{i,j}(0)$ the statement follows directly from the definition. □

Now we can describe the cohomology with compact supports of the first term of the filtration $F_1 = \Sigma_1(0) \times \mathbb{C}$.

**4.3.2. Lemma.** *The cohomology with compact supports of the variety $Hank_{\deg}(2, n)$ is equal to $(0, 0, \mathbb{Z}_n, 0, \mathbb{Z})$.*

*Proof.* The set $Hank_{\deg}(2, n)$ in question is the cone with the vertex at the origin 0 over the rational normal scroll in $\mathbb{C}^{n+1}$. The set $Hank_{\deg}(2, n) \setminus 0$ has the structure of a $\mathbb{C}^*$-bundle over this rational normal scroll $\mathbb{CP}^1$. The first Chern class $c_1$ of this bundle is equal to the degree of the scroll and therefore $c_1 = n$. Thus the usual cohomology of $Hank_{\deg}(2, n) \setminus 0$ is equal to $(\mathbb{Z}, 0, \mathbb{Z}_n, \mathbb{Z})$. Since it is a manifold its cohomology with compact supports is dual to the usual cohomology and is equal $(0, \mathbb{Z}, \mathbb{Z}_n, 0, \mathbb{Z})$. From the long exact sequence

$$H_c^*(Hank_{\deg}(2, n)) \to H_c^*(Hank_{\deg}(2, n)) \setminus 0 \to H_c^*(0)$$

one gets that $H_c^*(Hank_{\deg}(2, n)) = (0, 0, \mathbb{Z}_n, 0, \mathbb{Z})$. □

**4.3.3. Corollary.** $H_c^*(F_1) = (0, 0, 0, 0, \mathbb{Z}_3, 0, \mathbb{Z})$.

*Proof.* $F_1$ is homeomorphic to $Hank_{\deg}(2, 3) \times \mathbb{C}$. □

**4.4. Cohomology of $F_2 \setminus F_1$.** We are going to describe explicitly the set of all quartic polynomials, which have exactly two 1-generalized roots at given points $a$ and $b$ in $\mathbb{C}$. We use here the technique and notation from 3.4 and 3.5.

**4.4.1. Lemma.** *For any pair of distinct points $(a, b)$ there exist exactly 4 polynomials in $\Sigma_1 \subset \mathcal{P}_4$ such that their 1-generalized multiple zeros are $a$ and $b$.*

*Proof.* As above, it suffices to consider the case $a = 0$, $b = t$. Denote by $\Sigma_1^{\text{open}}(t)$ a Zariski open subset in $\Sigma_1(t)$ consisting of all $\Psi = A^{-1}(D)B(D)\exp(tD)$, where $A(D) = 1 + \alpha D$, $B = 1 + \beta D$, and $\alpha \neq \beta$. For $t = 0$ the set $\Sigma_1^{\text{open}}(t)$ is nothing more than the complement $\Sigma_1^0(0) \setminus \Sigma_1^1(0)$ considered in 3.5.

In the case $\Sigma_1 \subset \mathcal{A}_4$ the intersection $\Sigma_1(0) \cap \Sigma_1(t)$ is, obviously, contained inside $\Sigma_1^{\text{open}}(0) \cap \Sigma_1^{\text{open}}(t)$. Thus, $\Psi = A_1^{-1}(D)B_1(D)$ belongs to $\Sigma_1^{\text{open}}(0) \cap \Sigma_1^{\text{open}}(t)$ if and only if for some unipotent linear differential operators $A_2(D)$, $B_2(D)$ one gets

$$A_1^{-1}(D)B_1(D) = e^{tD}A_2^{-1}(D)B_2(D)$$

or, equivalently, $A_2(D)B_1(D) = e^{tD}A_1(D)B_2(D)$.

To determine $A_1, A_2, B_1.B_2$ we have to look at the intersection $\mathcal{D}_2 \cap \exp(tD)\mathcal{D}_2$. Direct computation shows that this intersection is a 1-dimensional subspace in $\mathcal{D}_2$ spanned by the polynomial

$$1 + tD/2 + t^2D^2/12 = (1 + \varrho tD)(1 + \bar{\varrho}tD) \ ,$$

where $-\varrho$, $-\bar{\varrho}$ are the complex conjugate roots of the quadratic polynomial $x^2 + x/2 + 1/12$, i. e. $\varrho = (3 + i\sqrt{3})/12$, and we have

$$(1 + \varrho tD)(1 + \bar{\varrho}tD) = \exp(tD) \cdot (1 - \varrho tD)(1 - \bar{\varrho}tD) \ .$$

Thus $\Sigma_1(0) \cap \Sigma_1(t)$ consists of the following four operators:

$$\begin{aligned}
(1 + \varrho tD)(1 - \varrho tD)^{-1} &= 1 + \frac{3 + i\sqrt{3}}{6}tD + \frac{1 + i\sqrt{3}}{12}t^2D^2 + \frac{i\sqrt{3}}{36}t^3D^3 + \frac{-1 + i\sqrt{3}}{144}t^4D^4 \\
(1 + \bar{\varrho}tD)(1 - \bar{\varrho}tD)^{-1} &= 1 + \frac{3 - i\sqrt{3}}{6}tD + \frac{1 - i\sqrt{3}}{12}t^2D^2 - \frac{i\sqrt{3}}{36}t^3D^3 + \frac{-1 - i\sqrt{3}}{144}t^4D^4 \\
(1 + \varrho tD)(1 - \bar{\varrho}tD)^{-1} &= 1 + \frac{1}{2}tD + \frac{3 - i\sqrt{3}}{24}t^2D^2 + \frac{1 - i\sqrt{3}}{48}t^3D^3 - \frac{i\sqrt{3}}{144}t^4D^4 \\
(1 + \bar{\varrho}tD)(1 - \varrho tD)^{-1} &= 1 + \frac{1}{2}tD + \frac{3 + i\sqrt{3}}{24}t^2D^2 + \frac{1 + i\sqrt{3}}{48}t^3D^3 - \frac{i\sqrt{3}}{144}t^4D^4
\end{aligned}$$

□

The previous formulas give also the exact description of how the fundamental group of the space of all pairs $(a, b)$, $a \neq b$, acts on the constructed quadruples of the polynomials. Namely, the interchange of $a$ and $b$ via the standard $\mathbb{Z}$-generator of this fundamental group leads, obviously, to the inversion of the above four differential operators and change of the sign of $t$. This procedure preserves the first and the second of the above operators and interchanges the last two of them. We get

**4.4.2. Corollary.** *The four polynomials from Lemma 4.4.1. are naturally organized in two pairs such that the replacement of $(a, b)$ by $(b, a)$ (via the standard generator of the fundamental group of the space of all pairs $(a, b)$ such that $a \neq b$) preserves both polynomials in one pair and interchanges polynomials in the other pair.*

Now we can easily calculate the cohomology of $F_2 \setminus F_1$.

**4.4.3. Lemma.** *The cohomology with compact supports of $F_2 \setminus F_1$ is*

$$(0, 0, 0, 0, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) \ .$$

*Proof.* Indeed, the set $F_2 \setminus F_1$ is a bundle over the set of all unordered distinct pairs of points with the fiber consisting of 4 segments. The action of the generator of the fundamental group of the base inverts the orientation of two of these segments and interchanges the two others with the inversion of orientation. Thus our fiber bundle consists of two copies of a 3-dimensional cylinder over a Möbius band and a 3-dimensional cylinder over an annulus.

Cohomology with compact supports of a Möbius band and an annulus is equal to $(0, 0, \mathbb{Z}_2)$ and $(0, \mathbb{Z}, \mathbb{Z})$ resp. Thus, $H_c^*(F_2 \setminus F_1)$ is equal to

$$(0, 0, 0, 0, \mathbb{Z}, \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2) . \qquad \square$$

**4.5. Cohomology of $F_3 \setminus F_2$** is also calculated via the exact description of all polynomials having three 1-generalized roots.

**Lemma.** *The intersection $\Sigma_1(a) \cap \Sigma_1(b) \cap \Sigma_1(c)$ for a triple of pairwise different points $(a, b, c)$ is nonempty if and only if there exist two constants $d$ and $e \neq 0$ such that $a - d, b - d, c - d$ are 3 different roots of the equation $x^3 = e$.*
*In this case $\Sigma_1(a) \cap \Sigma_1(b) \cap \Sigma_1(c)$ contains exactly one polynomial $(x + d)^4 - 2ex$.*

*Proof.* By Proposition 1.3.1 a polynomial $p \in \Sigma_1 \subset \mathcal{P}_4$ has at most three 1-generalized multiple zeros. Thus $p$ belongs to at most some triple intersection $\Sigma_1(a) \cap \Sigma_1(b) \cap \Sigma_1(c)$. Now let us determine the set of all $p$ which have exactly 3 pairwise different multiple 1-generalized zeros. This is equivalent to finding such $p$ that $\Delta_1(p)$ has 3 different double zeros which are not multiple zeros of $p$ itself. Using the action on the affine group we can assume that $p = x^4 + \lambda_1 x^2 + \lambda_2 x + \lambda_3$ which gives

$$\Delta_1(p) = -4\left[x^6 + \frac{\lambda_1}{2}x^4 - \lambda_2 x^3 + \left(\frac{\lambda_1^2}{2} - 3\lambda_3\right)x^2 + \frac{\lambda_1\lambda_2}{2}x - \frac{\lambda_1\lambda_3}{2} + \frac{\lambda_2^2}{4}\right].$$

$\Delta_1(p)$ has 3 zeros if and only if $\Delta_1(p) = [(x+\alpha)(x+\beta)(x-\alpha-\beta)]^2$ for some $\alpha \neq \beta \neq -\alpha-\beta$. (Omitted) consideration of the last condition implies that this is possible if and only if $p = x^4 + \lambda_2 x$. In this case $\Delta_1(p) = -4(x^3 - \frac{\lambda_2}{2})^2$ with 3 pairwise different double zeros. This implies that $\Sigma_1(a) \cap \Sigma_1(b) \cap \Sigma_1(c)$ is nonempty (and consists of exactly 1 polynomial $(x + d)^4 - 2\epsilon$) iff for some $d$ the numbers $a - d, b - d, c - d$ are roots of $x^3 = \epsilon$, $\epsilon \neq 0$. $\square$

**4.5.1. Corollary.** *The cohomology with compact supports of $F_3 \setminus F_2$ is equal*

$$(0, 0, 0, 0, 0, \mathbb{Z}, \mathbb{Z}) .$$

*Proof.* Indeed, $F_3 \setminus F_2$ is a bundle over the set of all equilateral triangles on $\mathbb{C}$ with the fiber consisting of the simplex formally spanned by triangle's vertices. The base is homeomorphic to $\mathbb{C} \times \mathbb{R}^+ \times S^1$. The generator of $S^1$ of the base acts trivially on the orientation of the fiber since it induces the cyclic shift of vertices. Thus $F_3 \setminus F_2$ is homeomorphic to $\mathbb{R}^5 \times S^1$. $\square$

**4.6. End of the proof of Proposition 4.2.** We accomplish the proof of Proposition 4.2 by using the Leray spectral sequence for the cohomology with compact supports for the whole $\bar{\Sigma}_1$. Its $E_1$- and $E_2$- pages are given on Fig. 1. (Potentially nontrivial differentials are shown by arrows.)

**4.6.1. Lemma.** a) *The differential* $\mathbb{Z} \to \mathbb{Z}$ *in the 3rd row of* $E_1$ *is multiplication by 3;*
b) *the differential* $\mathbb{Z} \to \mathbb{Z}$ *in the 4th row of* $E_1$ *is an isomorphism,*
c) *the differential* $\mathbb{Z}_3 \to \mathbb{Z}_3$ *on* $E_2$ *is an isomorphism.*

*Proof.* We consider the boundary map for the corresponding homology cycles. By Lemma 4.4 the part $\tilde{F}_2$ of $F_2 \setminus F_1$ to which $F_3 \setminus F_2$ is glued is a bundle over the braid space $Br(2)$ with the fiber consisting of 2 segments which can be interpreted as formal simplices spanned by the ordered pairs of points. Under the action of $\pi_1$ they change places with the inversion of their orientations. The 4-dimensional relative homology cycle $\Theta_1$ pairing with the 4-dimensional cohomology class in $F_2 \setminus F_1$ is obtained by taking fibers over all horizontal pairs of points in $Br(2)$. Analogously, the 5-dimensional relative homology class $\Theta_2$ in $F_3 \setminus F_2$ pairing with the 5-dimensional generator in cohomology consists of all formal 2-simplices over all equilateral triangles with one horizontal side. One easily gets $\partial(\Theta_2) = 3\Theta_1$ and a) follows. At the same time the boundary of the fundamental cycle in $F_3 \setminus F_2$ equals $\tilde{F}_2$ and b) follows. More detailed consideration of the boundary maps implies c). □

| | 1 | 2 | 3 |
|---|---|---|---|
| 6 | $\mathbb{Z}$ | | |
| 5 | | | |
| 4 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z} \longrightarrow$ | $\mathbb{Z}$ |
| 3 | | $\mathbb{Z} \longrightarrow$ | $\mathbb{Z}$ |

$E_1$

| | 1 | 2 | 3 |
|---|---|---|---|
| 6 | $\mathbb{Z}$ | | |
| 5 | | | |
| 4 | $\mathbb{Z}/3\mathbb{Z}$ | $\mathbb{Z}/2\mathbb{Z}+\mathbb{Z}/2\mathbb{Z}$ | |
| 3 | | | $\mathbb{Z}/3\mathbb{Z}$ |

$E_2$

Figure 1: The first two pages in the calculation of cohomology of $\Sigma_1$

## 5. Final remarks

The above partial results on the algebraic and topological properties of the associated discriminants show that their structure is substantially more complicated than that of the classical discriminants.

Below we formulate a few (of many more) questions which seem natural in the context of the associated discriminants.

1) Calculate the number of connected components to the union of the associated discriminants in the space $\mathcal{P}_n$ with real coefficients and, more generally, for the space of solutions to some linear homogeneous ordinary differential equation with constant coefficients. (This question is the most interesting from the point of view of its application to Toda lattices.)

2) Calculate the usual and intersection (co)homology of $\Sigma_i(0)$ or equivalently of the space $Hank_{\deg}(i,i)$. (For the usual cohomology one can use the stratification described in 3.5. For the intersection homology one can use the semismall desingularization. Many analogous spaces of matrices were sucessfully studied before. Recently Prof. R. MacPherson has suggested to the second author a recursive procedure applicable to determination of intersection homology of the space of degenerate Hankel matrices.)

3) Calculate the maximal number of pairwise different multiple $i$-generalized zeros for polynomials of degree $n$.

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