Beiträge zur Algebra und Geometrie
Contributions to Algebra and Geometry
Volume 39 (1998), No. 1, 135-153.

Plane Models of Semi-biplanes

B. Polster[1] **A. E. Schroth**

Department of Pure Mathematics, University of Adelaide
Adelaide, South Australia 5005, Australia

Institut für Topologie, TU Braunschweig
Pockelsstr. 14, D-38106 Braunschweig, Germany

Abstract. The connection between semi-biplanes and antiregular generalized quadrangles is employed to construct different plane models of semi-biplanes. The connection between antiregular generalized quadrangles and circle planes is used to generalize these models. The plane models of the classical semi-biplane are described explicitly. They can be visualized easily. The treatment shows how the different models are connected.

MSC 1991: 51A50, 51E12, 51B05, 51H15
Keywords: Semi-biplanes, generalized quadrangles, antiregular quadrangles, circle planes.

1. Introduction

The connection between semi-biplanes and generalized antiregular quadrangles established in [9] can be used to obtain nice models of semi-biplanes that can be easily visualized. Although these models can be best understood in the context of generalized quadrangles, they can be described without the notion of generalized quadrangles. The description is elementary.

Semi-biplanes are rather weak geometric structures consisting of points and blocks. The basic requirement is that any two points can be joined by exactly two blocks or by no block. So 'bi' stands for the two joining blocks and 'semi' indicates that joining blocks need not exist. There are several methods to construct semi-biplanes, but most of these constructions

[1]This research was supported by the Australian Research Council.

cannot be visualized easily. We will construct a model of the classical semi-biplane that lives in real three-space. Actually, it lives on two cones. Points of the semi-biplane are points on one cone while blocks are points on the other cone. Incidence is given by orthogonality with respect to a certain quadratic form.

This model can then be projected in different fashions to the real plane. These projections yield models where points and blocks are circles, or more generally ellipses, of the plane. Incidence is given by touching. There is one particularly striking model where both point set and block set correspond to the same set of circles.

All the models discussed in Section 3 are models of the same semi-biplane. They are described explicitly in terms of coordinates. Some of these models have already appeared in [7]. However, the treatment presented here provides a new interpretation of these models. Moreover, it becomes apparent how these seemingly different models are connected in a very geometric fashion.

Antiregular generalized quadrangles are also connected to circle planes. This connection enables us to view the models of semi-biplanes as substructures of circle planes. We extend this view to non-classical semi-biplanes. This yields models of non-classical semi-biplanes as subgeometries of circle planes.

The paper splits into two parts. The first part is a very down to earth discussion of the classical case. You need not be familiar with generalized quadrangles to understand the models. Anyhow, we give an explicit description of the real orthogonal quadrangle and the different models of the associated semi-biplane. So the first part serves several purposes: It presents concrete and easy to visualize models for semi-biplanes, it provides an introduction to generalized quadrangles and it illustrates the construction introduced in [9].

The second part generalizes the ideas discussed in the first part. This generalisation employs the connection between semi-biplanes and antiregular quadrangles and the connection between antiregular quadrangles and circle planes. This connection is especially strong when the geometries involved are topological geometries. We present several ways to construct semi-biplanes as substructures of circle planes. For topological circle planes where the point space has dimension two, that is, for locally compact circle planes with parameter 1, this yields models which generalize the models discussed in the first part.

This paper is part of a series of papers on semi-biplanes and generalized quadrangles. In [9] we introduce a construction of semi-biplanes from antiregular generalized quadrangles. This construction is used and illustrated in the present paper. In [8] we will characterize the semi-biplanes obtained by this construction. In [10] we introduce another construction of semi-biplanes from antiregular generalized quadrangles. The two constructions yield different types of semi-biplanes.

2. Preliminaries

An *incidence structure* $\mathfrak{I} = (\mathcal{V}_1, \mathcal{V}_2, I)$ consists of two sets $\mathcal{V}_1$, $\mathcal{V}_2$ and an incidence relation I between elements of $\mathcal{V}_1$ and elements of $\mathcal{V}_2$. Often elements of $\mathcal{V}_2$ are considered as subsets of $\mathcal{V}_1$ and incidence is given by inclusion in which case incidence is not mentioned explicitly.

Elements of $\mathcal{V}_1$ are usually called *points*, those of $\mathcal{V}_2$ *blocks*. Instead of 'blocks' we say *lines* if any two points are incident with at most one block and *circles* if any three points are incident with at most one block.

A pair $(v, w) \in \mathcal{V}_1 \times \mathcal{V}_2$ is called a *flag* if v is incident with w, that is, if $v\ I\ w$. The set of flags is denoted $\mathcal{F}$. An element of $\mathcal{V}_1 \times \mathcal{V}_2 \setminus \mathcal{F}$ is called an *anti-flag*.

2.1. Semi-biplanes

A *semi-biplane* $\mathfrak{S} = (\mathcal{P}, \mathcal{B})$ is an incidence structure with point set $\mathcal{P}$ and block set $\mathcal{B}$ satisfying the following axioms:

S1. Any two distinct points are incident with 0 or 2 blocks.

S2. Any two distinct blocks are incident with 0 or 2 points.

S3. The incidence graph of $\mathfrak{S}$ is connected.

S4. Every point is incident with at least three blocks and every block is incident with at least three points.

Two points p and q of a semi-biplane are called *parallel* if $p = q$ or if there is no block incident with p and q; dually for blocks. A semi-biplane is called *divisible* if the parallel relation on the point set and the parallel relation on the set of blocks are equivalence relations.

An incidence structure that satisfies S1, S2 and S4 but not S3 is called a *weak* semi-biplane.

A *topological divisible semi-biplane* is a semi-biplane where the set of points and the set of blocks carry Hausdorff topologies such that the natural geometric operations are continuous.

For a more detailed discussion of semi-biplanes the reader is referred to [13] or [14]. For an introduction to topological semi-biplanes and further constructions of semi-biplanes the reader is referred to [9] and [10]. In Section 3 we will give an explicit description of a particular semi-biplane which we call the classical semi-biplane.

2.2. Antiregular Quadrangles

A *generalized quadrangle* is an incidence structure $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ with point set $\mathcal{P}$ and line set $\mathcal{L}$ satisfying the following axioms:

Q1. Any two distinct points have at most one joining line.

Q2. For every anti-flag $(p, k) \in \mathcal{P} \times \mathcal{L} \setminus \mathcal{F}$ there exists exactly one flag $(q, l) \in \mathcal{F}$ such that $(p, l) \in \mathcal{F}$ and $(q, k) \in \mathcal{F}$.

Q3. Every point is incident with at least three lines, and every line is incident with at least three points.

An incidence structure that satisfies Axiom Q1 and Axiom Q2 and where every line contains at least 3 points and every point is incident with precisely two lines is called a *point thin* generalized quadrangle.

Axiom Q2 yields two mappings $\pi : \mathcal{P} \times \mathcal{L} \setminus \mathcal{F} \to \mathcal{P}$ and $\lambda : \mathcal{P} \times \mathcal{L} \setminus \mathcal{F} \to \mathcal{L}$ with $\pi(p, k) = q$ and $\lambda(p, k) = l$.

Two points p, q are said to be *collinear*, denoted $p \sim q$, if they can be joined by a line. The set of points collinear with a given point p is denoted $p^\perp$.

A generalized quadrangle is called *antiregular* if for any three pairwise non-collinear points a, b and c the *centre* $a^\perp \cap b^\perp \cap c^\perp$ is either empty or contains precisely two points.

A *topological quadrangle* is a generalized quadrangle where $\mathcal{P}$ and $\mathcal{L}$ carry Hausdorff topologies such that the maps π and λ are continuous. For an introduction to topological quadrangles we refer to [2] and [4]. A topological quadrangle is called a *compact* quadrangle if $\mathcal{P}$ and $\mathcal{L}$ carry compact topologies. If $\mathfrak{GQ}$ is a compact quadrangle whose point rows and line pencils are topological manifolds of positive dimensions, then there are parameters $m, n \in \mathbb{N}$ such that all point rows are homeomorphic to the n-dimensional sphere $\mathbb{S}_n$ and all line pencils to the m-dimensional sphere $\mathbb{S}_m$ (cf. [5]). In this case we speak of a compact quadrangle $\mathfrak{GQ}$ with *topological parameters* (m, n). If $m = n$, we say the quadrangle $\mathfrak{GQ}$ has parameter m rather than parameters (m, m). Compact quadrangles with parameter 1 are also known as three-dimensional quadrangles (cf. [2]). A more detailed discussion of compact quadrangles can be found in [6] and [11].

Suppose p, q are two non-collinear points and suppose g is a line of a quadrangle such that no point of the *trace* $p^\perp \cap q^\perp$ is incident with g. In particular, this implies that neither p nor q is incident with g. Then the restriction

$$\mu_{p,q,g} : p^\perp \cap q^\perp \to g : x \mapsto \pi(x, g)$$

is called the *square projection* from $p^\perp \cap q^\perp$ to g. Especially in antiregular quadrangles square projections play an essential rôle.

In Section 3 we will give an explicit description of the classical antiregular quadrangle.

2.3. Construction of semi-biplanes from antiregular quadrangles

In [9, 5.1] the following construction of semi-biplanes from antiregular generalized quadrangles is introduced:

Theorem 2.1. *For any two non-collinear points p and q of an antiregular quadrangle $\mathfrak{GQ}$, the incidence structure*

$$\mathfrak{S}(p, q) := (p^\perp \setminus (\{p\} \cup q^\perp), q^\perp \setminus (\{q\} \cup p^\perp), \sim)$$

is a weak semi-biplane.

In general $\mathfrak{S}(p, q)$ consists of more than one semi-biplane. If $\mathfrak{GQ}$ is a compact antiregular quadrangle with parameter 1, then $\mathfrak{S}(p, q)$ consists of two divisible semi-biplanes. To extract one of the semi-biplanes consider the following construction:

Suppose $\mathfrak{S}(p, q) = (p^\perp \setminus (\{p\} \cup q^\perp), q^\perp \setminus (\{q\} \cup p^\perp), \sim)$ is constructed from a compact antiregular quadrangle as above. For $s \in p^\perp \setminus (\{p\} \cup q^\perp)$ let $\mathcal{P}_\mathfrak{S}$ denote the connected component of $p^\perp \setminus (\{p\} \cup q^\perp)$ containing s. Let $\mathcal{B}_\mathfrak{S}$ denote the connected component of $q^\perp \setminus (\{q\} \cup p^\perp)$ containing points collinear with s. Then

$$\mathfrak{S}(p, q, s) := (\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$$

is a topological divisible semi-biplane. This has been proved in [9, 5.2]. Parallel classes of points are the lines through p restricted to $\mathcal{P}_\mathfrak{S}$, parallel classes of blocks are the lines through q restricted to $\mathcal{B}_\mathfrak{S}$.

3. Models for the classical case

The orthogonal quadrangle $Q(4,\mathbb{R})$ over $\mathbb{R}$ is the classical antiregular generalized quadrangle. It can be described as follows: Consider the orthogonality on the 4-dimensional projective space $PG(4,\mathbb{R})$ induced by a symmetric bilinear form of Witt index 2. Let $\mathcal{P}$ denote the set of isotropic points. Let $\mathcal{L}$ denote the set of totally isotropic lines, that is, the set of lines of $PG(4,\mathbb{R})$ that consist entirely of isotropic points. Then $(\mathcal{P},\mathcal{L})$ is the orthogonal quadrangle over $\mathbb{R}$. The construction implies that points of the quadrangle are collinear if and only if they are orthogonal. This construction works for every field. However, the generalized quadrangle is antiregular if and only if the field is not of characteristic 2.

We use homogeneous coordinates for $PG(4,\mathbb{R})$. The orthogonal quadrangle can be described explicitly as follows: Two points $p_i = \langle a_i, b_i, c_i, d_i, e_i\rangle \in PG(4,\mathbb{R})$, where $i = 1, 2$, are orthogonal if and only if $a_1a_2 + b_1b_2 - c_1c_2 + d_1e_2 + e_1d_2 = 0$. This implies

$$\mathcal{P} = \{\, \langle a, b, c, d, e\rangle \in PG(4,\mathbb{R}) : a^2 + b^2 - c^2 + 2de = 0 \,\}.$$

Let $p_\infty := \langle 0, 0, 0, 0, 1\rangle \in \mathcal{P}$. Then $\langle a, b, c, d, e\rangle \in p_\infty^\perp$ if and only if $d = 0$. Consequently, $\langle a, b, c, d, e\rangle \in \mathcal{P} \setminus p_\infty^\perp$ if and only if $d \neq 0$ and $a^2 + b^2 - c^2 + 2de = 0$. Since $d \neq 0$ we may assume $d = 1$. Then $e = -{}^1\!/_2(a^2 + b^2 - c^2)$. Therefore, the space $\mathcal{P} \setminus p_\infty^\perp$ can be identified with $\mathbb{R}^3$ via the correspondence

$$(x, y, z) \longleftrightarrow \langle x, y, z, 1, \tfrac{1}{2}(z^2 - x^2 - y^2)\rangle.$$

The next step is to determine within $\mathbb{R}^3$ the set of points collinear with a given point in $\mathbb{R}^3$. We have:

$$\begin{aligned}(x, y, z) \sim (a, b, c) \quad &\iff \quad \langle x, y, z, 1, \tfrac{1}{2}(z^2 - x^2 - y^2)\rangle \perp \langle a, b, c, 1, \tfrac{1}{2}(c^2 - a^2 - b^2)\rangle \\ &\iff \quad (x - a)^2 + (y - b)^2 = (z - c)^2.\end{aligned}$$

So, $(a, b, c)^\perp \cap \mathbb{R}^3 = \{\, (x, y, z) : (x - a)^2 + (y - b)^2 = (z - c)^2 \,\}$. This is a double cone with vertex (a, b, c) (see Figure 1).

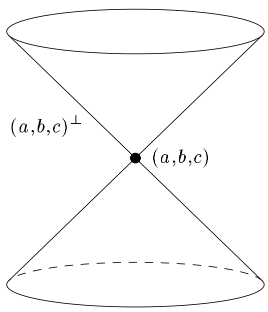

Figure 1

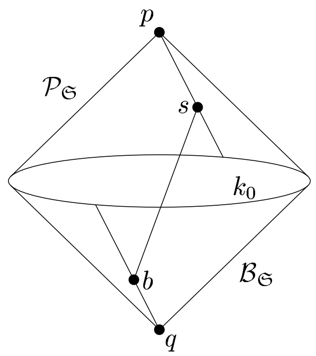

Figure 2

Figure 1 reminds of the usual pictures of light cones in Minkowski space. This is no coincidence. The orthogonal quadrangle $Q(4,\mathbb{R})$ is the geometry behind a "Minkowski space" with only two space dimensions and one time dimension. Lines of the quadrangle are light lines of the Minkowski space. The full Minkowski space corresponds to the generalized quadrangle $Q(5,\mathbb{R})$.

Our next aim is to exhibit the semi-biplanes obtained from the orthogonal quadrangle. The trick is to choose points for the derivation in such a fashion that the semi-biplane is embedded in $\mathbb{R}^3$. Let $p=(0,0,1)$ and $q=(0,0,-1)$. Then

$$\begin{aligned} p^\perp \cap q^\perp &= \{\,(x,y,z) : x^2+y^2=(z-1)^2=(z+1)^2\,\} \\ &= \{\,(x,y,0) : x^2+y^2=1\,\} \\ &=: k_0. \end{aligned}$$

Let

$$\begin{aligned} \mathcal{P}_{\mathfrak{S}} &:= \{\,(x,y,z) : x^2+y^2=(z-1)^2 \text{ and } 0<z<1\,\} \\ &= \{\,(x,y,1-|(x,y)|) : 0<|(x,y)|<1\,\}, \\ \mathcal{B}_{\mathfrak{S}} &:= \{\,(x,y,z) : x^2+y^2=(z+1)^2 \text{ and } 0>z>-1\,\} \\ &= \{\,(x,y,|(x,y)|-1) : 0<|(x,y)|<1\,\} \end{aligned}$$

denote the set of points and the set of blocks, respectively. A point $a=(x_1,y_1,z_1)\in\mathcal{P}_{\mathfrak{S}}$ is incident with a block $b=(x_2,y_2,z_2)\in\mathcal{B}_{\mathfrak{S}}$ if and only if $a\sim b$, that is, if and only if $(x_1-x_2)^2+(y_1-y_2)^2=(z_1-z_2)^2$. Note that the point $s:=(1/2,0,1/2)$ is incident with the block $b:=(-1/2,0,-1/2)$. Consider $\mathcal{P}_{\mathfrak{S}}$ and $\mathcal{B}_{\mathfrak{S}}$ as subsets of $\mathcal{P}$ rather than $\mathbb{R}^3$. Then $\mathcal{P}_{\mathfrak{S}}$ is the connected component of $p^\perp\setminus(\{p\}\cup q^\perp)$ containing s and $\mathcal{B}_{\mathfrak{S}}$ is the connected component of $q^\perp\setminus(\{q\}\cup p^\perp)$ containing b. By [9, 5.2], it follows that $(\mathcal{P}_{\mathfrak{S}},\mathcal{B}_{\mathfrak{S}})=\mathfrak{S}(p,q,s)$ is indeed a semi-biplane. We call this the classical semi-biplane.

Already, this model is visible. Parallel classes are lines through p or q. A point $s\in\mathcal{P}_{\mathfrak{S}}$ is incident with a block $b\in\mathcal{B}_{\mathfrak{S}}$ if and only if the line joining s to b in $\mathbb{R}^3$ is parallel to a line on the double cone $\{\,(x,y,z) : x^2+y^2=z^2\,\}$. Figure 2 illustrates this model. It depicts a parallel class of points, a parallel class of blocks and an incident point-block pair. However, this is not a plane model. The general idea to obtain plane models from this model is to take a surface $S\subset\mathbb{R}^3$ and make the following identifications:

$$\begin{aligned} \mathcal{P}_{\mathfrak{S}} &\cong S\cap\mathcal{P}_{\mathfrak{S}}\cup\{\,S\cap s^\perp : s\in\mathcal{P}_{\mathfrak{S}}\setminus S\,\}, \\ \mathcal{B}_{\mathfrak{S}} &\cong S\cap\mathcal{B}_{\mathfrak{S}}\cup\{\,S\cap s^\perp : s\in\mathcal{B}_{\mathfrak{S}}\setminus S\,\}. \end{aligned}$$

A point is incident with a block if the point is contained in the block, if the block is contained in the point or if the point touches the block. A model obtained from a surface S is denoted $\mathfrak{M}_S$.

We start with $S=\mathcal{P}_{\mathfrak{S}}=\{\,(x,y,z) : x^2+y^2=(z-1)^2 \text{ and } 0<z<1\,\}$. Then no block is in S and for every block $b\in\mathcal{B}_{\mathfrak{S}}$ the intersection $b^\perp\cap S$ is an ellipse that touches the circle $k_0=\{\,(x,y,0) : |(x,y)|=1\,\}$. The touching point is not a point of the geometry. Two points are parallel if and only if they lie on a line through the point p. Two blocks are parallel if and only if the corresponding ellipses touch the circle k_0 in the same point.

To obtain a plane model we project the whole situation onto the xy-plane. Then the point set becomes the set $\{ (x,y) : 0 < |(x,y)| < 1 \}$. Blocks become ellipses touching the unit circle. Calculations reveal that one obtains precisely those ellipses whose long diameter starts at a point on the unit circle and runs through $(0,0)$. Moreover, if 2α is the length of the short diameter, then $0 < \alpha < 1$ and $1+\alpha$ is the length of the long diameter. Parallel classes of points are rays emanating from the origin. Parallel classes of blocks are touching pencils where the common touching point lies on the unit circle. This model appeared already in [7].

Figure 3 indicates the unflattened and Figure 4 the flattened version of this model. Both times some blocks of a parallel class are drawn.

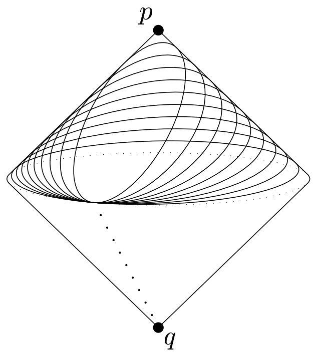

Figure 3

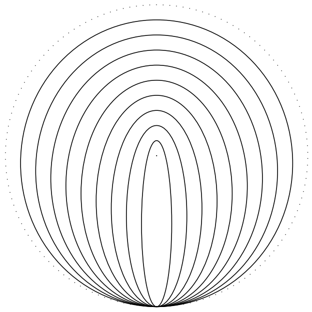

Figure 4

Instead of $S = p^\perp \cap \mathbb{R}^3$ one can take any double cone $S = S_v := v^\perp \cap \mathbb{R}^3$, where $v \in \mathbb{R}^3$. For $v = q$ the dual of the above model is obtained. Interesting models arise for $v \in p^\perp \cap q^\perp$ and for $v \in \mathbb{R}^3 \setminus (p^\perp \cup q^\perp)$. We leave it to the reader to work out these models in detail. Some of these models are related to models constructed below.

Another interesting class of models is obtained if S is a plane orthogonal to the z-axis. So $S = S_h := \{ (x,y,z) : z = h \}$. In all these cases points become circles touching the circle $S \cap p^\perp$ and blocks become circles touching the circle $S \cap q^\perp$. If a point or a block is contained in S then the corresponding circle has radius 0. We say that a circle s of radius 0, that is, a point of S, touches a proper circle c if and only if s lies on c. In these models two points or two blocks are parallel if and only if the corresponding circles touch in a point of $S \cap p^\perp$ or $S \cap q^\perp$, respectively. A point is incident with a block if and only if the corresponding circles touch in a point neither incident with $S \cap p^\perp$ nor with $S \cap q^\perp$. Figure 5 sketches the models for S_0, S_1 and S_2. Depicted are the circles corresponding to a point s and the circles corresponding to a block b collinear with s.

Particularly nice models arise with S_h if we choose $h = 0, 1$ or $h > 1$. In the first case $S \cap p^\perp = S \cap q^\perp$ and in the second case $S \cap p^\perp$ is a point. We examine these models in more detail.

Let $S = S_0$. Then every point $(a,b,c) = (a,b,1-|(a,b)|)$ gets identified with the circle $\{ (x,y) : |(x-a, y-b)| = 1 - |(a,b)| \}$. This is the circle with centre (a,b) that touches the unit circle. Every block $(a,b,c) = (a,b,|(a,b)|-1)$ gets identified with the circle $\{ (x,y) : |(x-a,y-b)| = 1 - |(a,b)| \}$. Again, this is the circle with centre (a,b) that touches the unit circle. So, $\mathcal{P}_\mathfrak{S}$ as well as $\mathcal{B}_\mathfrak{S}$ correspond to the set of circles in the unit disc that

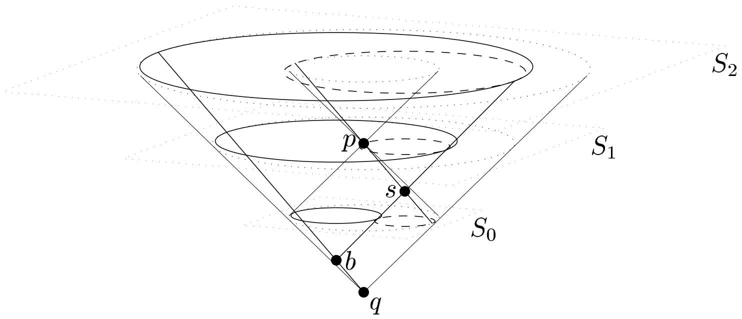

Figure 5

touch the unit circle. Figure 6 illustrates this model. It depicts some points of a parallel class and some blocks of a parallel class. Points are drawn with dashed lines while blocks are drawn with full lines. This model already appeared in [7]. The present paper makes it obvious that this model is closely related to the model depicted in Figure 4.

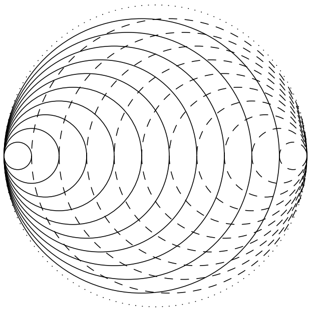

Figure 6

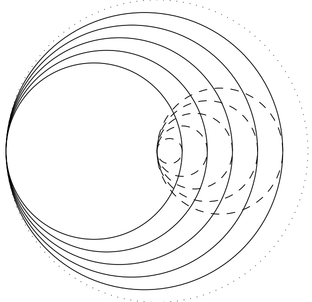

Figure 7

Now let $S = S_1$. Then every point $(a,b,c) = (a,b,1-|(a,b)|)$ gets identified with the circle $\{ (x,y) : |(x-a,y-b)| = |(a,b)| \}$. This is the circle with centre (a,b) that is incident with the origin. Every block $(a,b,c) = (a,b,|(a,b)|-1)$ gets identified with the circle $\{ (x,y) : |(x-a,y-b)| = 1+|(a,b)| \}$. This is the circle with centre (a,b) that touches the circle $\{ (x,y) : |(x,y)| = 2 \}$. After rescaling we obtain the following model: Points are all circles through the origin that are contained entirely in the open unit disc. Blocks are all those circles that touch the unit circle, that are contained in the unit disc and that contain the origin in their interior. Parallel classes of blocks are touching pencils of circles. Two points are parallel if and only if the corresponding circles touch and if the two circles are not separated by the line through the origin tangent to the two circles. Obviously, the two circles touch in the origin. Figure 7 illustrates this model. Again, some points of a parallel class and some blocks of a parallel class are drawn.

Let $S := S_h$ where $h > 1$. A point $(a,b,c) = (a,b,1-|(a,b)|)$ gets identified with the circle $\{ (x,y) : |(x-a,y-b)| = h-1+|(a,b)| \}$. This is the circle with centre (a,b) that touches the circle $\{ (x,y) : |(x,y)| = h-1 \}$. It is contained entirely in the half open annulus $\{ (x,y) : h-1 \leq |(x,y)| < h+1 \}$. It is not contractible within this annulus. A block $(a,b,c) = (a,b,|(a,b)|-1)$ gets identified with the circle $\{ (x,y) : |(x-a,y-b)| = h+1-|(a,b)| \}$.

This is the circle with centre (a, b) that touches the circle $\{ (x, y) : |(x, y)| = h + 1 \}$. It is contained entirely in the half open annulus $\{ (x, y) : h - 1 < |(x, y)| \le h + 1 \}$. It is not contractible within this annulus. Figure 8 illustrates this model. This model as well as the

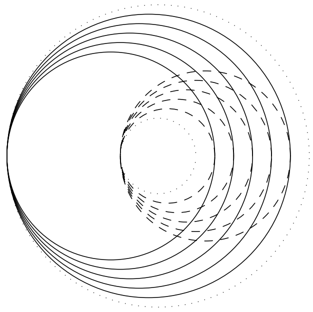

Figure 8

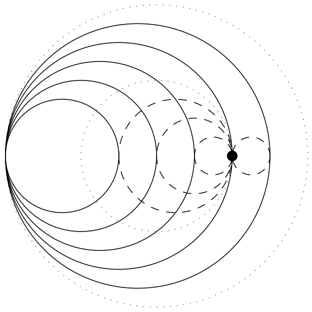

Figure 9

model illustrated in Figure 4 can also be viewed as geometries on the cylinder. Essentially, the models on the cylinder are obtained by projection using as centre a suitable point. Figure 10 and Figure 11 illustrate these models.

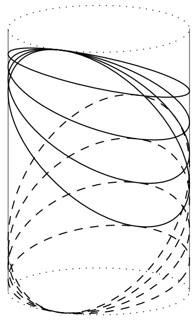

Figure 10

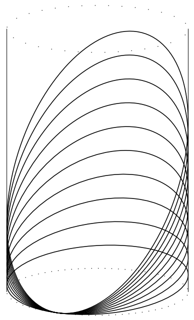

Figure 11

For $0 < h < 1$ the model obtained using S_h is not quite as nice as the the preceding models. Therefore, we will not describe it explicitly but only refer to Figure 9. For $h < 0$ the model obtained using S_h is the dual of the model obtained using S_{-h}.

4. The general case

The close connection between antiregular quadrangles and circle planes provides a way to generalize the models discussed in the previous section. The general idea is to view semi-biplanes as subgeometries of circle planes.

Throughout this section $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ denotes an antiregular generalized quadrangle.

4.1. Circle planes

A *circle plane* $\mathfrak{Cp} = (\mathcal{P}, \mathcal{C}, \mathcal{E})$ is an incidence structure with point set $\mathcal{P}$, set of circles $\mathcal{C}$ and a collection $\mathcal{E}$ of equivalence relations on $\mathcal{P}$ such that the following axioms hold, where two points are said to be non-parallel if they are not in relation for any of the equivalence relations in $\mathcal{E}$ and where two circles touch if they coincide or if they have precisely one point in common:

C1. Any three pairwise non-parallel points are incident with precisely one circle.

C2. For any two non-parallel points s and t and any circle k incident with s there is precisely one circle incident with s and t that touches k.

C3. For every point s, every circle k and every relation in $\mathcal{E}$ different from the identity relation there is precisely one point on k in relation to s.

C4. For any points s and t and any two distinct equivalence relations $\|_i$, $\|_j \in \mathcal{E}$ different from the identity relation there is precisely one point u such that $s \|_i u \|_j t$.

C5. Every circle is incident with at least three points and not all points are incident with the same circle.

A *Möbius plane* is a circle plane where $\mathcal{E}$ contains just the identity relation. A *Laguerre plane* is a circle plane where $\mathcal{E}$ contains just one relation different from the identity relation. A *Minkowski plane* is a circle plane where $\mathcal{E}$ contains precisely two relations different from the identity relation.

The classical models for Möbius, Laguerre and Minkowski planes are obtained as the geometries of plane sections of an elliptic quadric, an elliptic cone with its vertex removed or a ruled quadric, respectively, in projective three-space.

For an introduction to circle planes we refer to [11] and [12].

A *locally compact connected* circle plane is a circle plane where point set and circle set are locally compact Hausdorff spaces such that all geometric operations are continuous. A basic result, proved in [1], states that if $\mathfrak{Cp}$ is a locally compact connected circle plane such that point space and circle space have finite topological dimension, then there exists $i \in \{1, 2\}$ such that all circles are homeomorphic to $\mathbb{S}_i$. We call such a circle plane a locally compact circle plane with parameter i.

4.2. Circle planes and generalized quadrangles

An involutory collineation τ on $\mathfrak{GQ}$ with fixed point set Σ is called a *Möbius involution* if the following conditions are satisfied:

1. No line is fixed by τ.
2. Every line contains precisely one fixed point.
3. Either $a^\perp \cap b^\perp \cap c^\perp \neq \emptyset$ for any three points $a, b, c \in \Sigma$ or there is an involution σ on Σ such that $a^\perp \cap b^\perp \cap c^\perp = \emptyset \iff \sigma(a)^\perp \cap \sigma(b)^\perp \cap \sigma(c)^\perp \neq \emptyset$ for any three points $a, b, c \in \Sigma$.

If there is such an involution σ we call it the involution associated to τ.

An involutory collineation τ on $\mathfrak{GQ}$ with fixed point set Σ and fixed line set Λ is called a *Minkowski involution* if the following conditions are satisfied:

1. (Σ, Λ) is a point thin quadrangle.
2. A line not fixed by τ contains precisely one fixed point.

3. Either $a^\perp \cap b^\perp \cap c^\perp \neq \emptyset$ for any three non-collinear points $a, b, c \in \Sigma$ or there is an involutory collineation σ on (Σ, Λ) such that
$$a^\perp \cap b^\perp \cap c^\perp = \emptyset \iff \sigma(a)^\perp \cap \sigma(b)^\perp \cap \sigma(c)^\perp \neq \emptyset$$
for any three non-collinear points $a, b, c \in \Sigma$.

A Möbius or Minkowski involution is called *centric* if the centre of any three non-collinear points in Σ is not empty. If a Möbius or Minkowski involution is not centric, the involution σ need not be unique. The existence of a centric Möbius or Minkowski involution does not imply that the generalized quadrangle itself is centric. The real orthogonal quadrangle $Q(4, \mathbb{R})$ is not centric but there are centric Möbius involutions.

In the real orthogonal quadrangle $Q(4, \mathbb{R})$ Möbius and Minkowski involutions are rather natural maps.

Let $Q(4, \mathbb{R})$ be as described in Section 3. For $q = \langle a, b, c, d, e\rangle \in PG(4, \mathbb{R})$ let $r_q := a^2 + b^2 - c^2 + 2de$ and $H_q := \{\, \langle s, t, u, v, w\rangle : as + bt - cu + dw + ev = 0 \,\}$. Reflection at H_q is a Möbius involution if $r_q < 0$ and a Minkowski involution if $r_q > 0$.

As in Section 3, let $p_\infty := \langle 0, 0, 0, 0, 1\rangle$ and identify $\mathcal{P} \setminus p_\infty^\perp$ with $\mathbb{R}^3$. Then every reflection at a plane perpendicular to the z-axis corresponds to a Möbius involution fixing p_∞. Every reflection at a plane parallel to the z-axis corresponds to a Minkowski involution fixing p_∞.

The connection between antiregular quadrangles and circle planes is as follows: Suppose $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ is an antiregular generalized quadrangle. Then

1. For every $p \in \mathcal{P}$ the derived structure
$$\mathfrak{Lp}(\mathfrak{GQ}, p) := (p^\perp \setminus \{p\}, \{\, p^\perp \cap q^\perp : q \in \mathcal{P} \setminus p^\perp \,\})$$
is a Laguerre plane.
2. For every centric Möbius or Minkowski involution τ with fixed point set Σ the derived structure
$$\begin{aligned}\mathfrak{Cp}(\mathfrak{GQ}, \tau) &:= (\Sigma, \{\, \Sigma \cap q^\perp : q \in \mathcal{P} \setminus \Sigma \,\}) \\ &= (\Sigma, \{\, q^\perp \cap \tau(q)^\perp : q \in \mathcal{P} \setminus \Sigma \,\})\end{aligned}$$
is a Möbius or Minkowski plane, respectively.
3. For every non-centric Möbius or Minkowski involution τ with fixed point set Σ and associated involution σ the derived structure
$$\begin{aligned}\mathfrak{Cp}(\mathfrak{GQ}, \tau, \sigma) &:= (\Sigma, \{\, \Sigma \cap q^\perp : q \in \mathcal{P} \setminus \Sigma \,\} \cup \{\, \sigma(\Sigma \cap q^\perp) : q \in \mathcal{P} \setminus \Sigma \,\}) \\ &= (\Sigma, \{\, q^\perp \cap \tau(q)^\perp : q \in \mathcal{P} \setminus \Sigma \,\} \cup \{\, \sigma(q^\perp \cap \tau(q)^\perp) : q \in \mathcal{P} \setminus \Sigma \,\})\end{aligned}$$
is a Möbius or Minkowski plane, respectively.

Proofs for these statements can be found in [11]. For finite Laguerre planes of odd order and for locally compact finite-dimensional connected circle planes the above construction can also be reversed (cf. [11]). Most likely, the above construction can also be reversed for Möbius or Minkowski planes of odd order. We will call a circle plane *good* if it is related to a generalized quadrangle $\mathfrak{GQ}$ in the above way.

4.3. Semi-biplanes in Laguerre planes

If p and q are two non-collinear points of an antiregular quadrangle, then for any point r the weak semi-biplane $\mathfrak{S}(p,q)$ can be viewed as a subgeometry of the Laguerre plane $\mathfrak{Lp}(\mathfrak{GQ},r)$. This generalizes the models $\mathfrak{M}_S$ where S is the double cone $v^\perp$ for some point v. How the weak semi-biplane $\mathfrak{S}(p,q)$ is embedded in the Laguerre plane $\mathfrak{Lp}(\mathfrak{GQ},r)$ depends on the position of r relative to p and q.

If $\mathfrak{Lp}$ is a good Laguerre plane, in particular, if $\mathfrak{Lp}$ is a finite Laguerre plane of odd order or a locally compact Laguerre plane with parameter $m \in \{1,2\}$, then any such subgeometry is a weak semi-biplane. If the Laguerre plane is locally compact with parameter 2, then every such subgeometry is a divisible semi-biplane. This follows from [9, 5.3], since the corresponding quadrangle has also parameter 2. If the Laguerre plane is locally compact with parameter 1, then every such subgeometry is a weak semi-biplane consisting of two divisible semi-biplanes. This follows from [9, 5.3], since the corresponding quadrangle has also parameter 1. For a locally compact Laguerre plane with parameter 1 the two semi-biplanes can be distinguished by connectedness properties.

We now discuss the different possibilities for r relative to p and q. We say two circles of a Laguerre plane touch if and only if they have precisely one point in common.

If $r = p$, then q corresponds to a circle k of the Laguerre plane $\mathfrak{Lp}(\mathfrak{GQ},r)$. The points of $\mathfrak{S}(p,q)$ correspond to the points of the Laguerre plane not incident with k. The blocks of $\mathfrak{S}(p,q)$ correspond to the circles of the Laguerre plane that touch k. A point of the semi-biplane is incident with a block if and only if the point is incident with the corresponding circle.

If $r = q$, one obtains the dual of the above model.

If $r \in p^\perp \cap q^\perp$, then p and q correspond to non-parallel points s_p and s_q of the Laguerre plane. The points and blocks of $\mathfrak{S}(p,q)$ are:

points:	points parallel to but different from s_p,
	circles through s_p not incident with s_q,
blocks:	points parallel to but different from s_q,
	circles through s_q not incident with s_p.

A point is incident with a block if the corresponding objects form a flag or are two touching circles.

If $r \in \mathcal{P} \setminus (p^\perp \cup q^\perp)$, then p and q correspond to non-touching circles k_p and k_q of the Laguerre plane. The points and blocks of $\mathfrak{S}(p,q)$ are:

points:	points incident with k_p but not with k_q,
	circles that touch k_p but not k_q,
blocks:	points incident with k_q but not with k_p,
	circles that touch k_q but not k_p.

A point is incident with a block if the corresponding objects form a flag or are two touching circles.

If $r \in p^\perp \setminus (\{p\} \cup q^\perp)$, then p corresponds to a point s and q corresponds to a circle k of the Laguerre plane. The point s is not incident with k. The points and blocks of $\mathfrak{S}(p,q)$ are:

points: points parallel to but different from s that are not incident with k,
circles incident with s that do not touch k,
one extra point,
blocks: points incident with k but not parallel to s,
circles not incident with s that touch k.

A point is incident with a block if the corresponding objects form a flag or are two touching circles. The extra point is incident with those blocks that correspond to points on k.

If $r \in q^\perp \setminus (\{q\} \cup p^\perp)$ one obtains the dual of the above model.

As noted before, any such subgeometry in any good Laguerre plane forms a weak semi-biplane. We now restrict our attention to locally compact Laguerre planes with parameter 1. Then two of these subgeometries yield nice descriptions of semi-biplanes.

Let $\mathfrak{Lp}$ denote a locally compact Laguerre plane with parameter 1. Then the point space of $\mathfrak{Lp}$ is homeomorphic to the cylinder $\mathbb{S}_1 \times \mathbb{R}$, every circle is homeomorphic to $\mathbb{S}_1$ and separates the point space into two connected components (cf. [3, 3.5]).

Lemma 4.1. *Suppose k is a circle of a locally compact Laguerre plane $\mathfrak{Lp} = (\mathcal{P}_{\mathfrak{Lp}}, \mathcal{C})$ with parameter 1. Let $\mathcal{P}_\mathfrak{S}$ be a connected component of $\mathcal{P}_{\mathfrak{Lp}} \setminus k$. Let $\mathcal{B}_\mathfrak{S}$ denote the set of circles that touch k in one point and that are contained in $\mathcal{P}_\mathfrak{S} \cup k$. Then $(\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$ is a topological divisible semi-biplane.*

Proof. Let $s \in \mathcal{P}_\mathfrak{S}$. Let $\mathfrak{GQ}$ denote the quadrangle associated with the Laguerre plane $\mathfrak{Lp}$. This is a compact antiregular quadrangle with parameter 1 and a point p distinguished by the construction. The circle k corresponds to a point q not collinear with p and the point s can be viewed as a point collinear with p but not with q. Then $(\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$ is isomorphic to $\mathfrak{S}(p,q,s)$. This is a topological divisible semi-biplane by [9, 5.3]. □

Remark. These semi-biplanes are generalisations of the model illustrated in Figure 11 and Figure 4.

Lemma 4.2. *Suppose k_1 and k_2 are two non-intersecting circles of a locally compact Laguerre plane $\mathfrak{Lp} = (\mathcal{P}_{\mathfrak{Lp}}, \mathcal{C})$ with parameter 1. Let C denote the relatively compact connected component of $\mathcal{P}_{\mathfrak{Lp}} \setminus (k_1 \cup k_2)$. Let $\mathcal{P}_\mathfrak{S}$ denote the set of circles that touch k_1 and that are contained in $C \cup k_1$. Let $\mathcal{B}_\mathfrak{S}$ denote the set of circles that touch k_2 and that are contained in $C \cup k_2$. Then $(\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$ is a topological divisible semi-biplane where incidence is given by touching.*

Proof. This follows from [9, 5.3] with arguments as above. □

Remark. These semi-biplanes are generalisations of the model illustrated in Figure 10 and Figure 8.

4.4. Semi-biplanes in Möbius planes

Suppose $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ is an antiregular quadrangle with a Möbius involution τ. Let Σ denote the fixed point set of τ. Let $\mathfrak{Mo}$ denote the Möbius plane $\mathfrak{Cp}(\mathfrak{GQ}, \tau)$ if τ is centric and the Möbius plane $\mathfrak{Cp}(\mathfrak{GQ}, \tau, \sigma)$ if τ is not centric and σ an involution on Σ associated with τ.

Let $p \in \mathcal{P}$. If $p \notin \Sigma$, then $p^\perp \cap \Sigma = p^\perp \cap \tau(p)^\perp$ is a circle of the Möbius plane $\mathfrak{Mo}$. If $p \in \Sigma$, then $p^\perp \cap \Sigma = \{p\}$ is a point of $\mathfrak{Mo}$. We will regard points of the Möbius plane as degenerate circles, that is, as circles of radius 0. Non-degenerate circles are also called proper circles. We define touching of circles as follows: Two non-degenerate circles touch if and only if they have precisely one point in common. A degenerate circle, that is, a point, touches a proper circle if and only if the point is incident with the circle. Two degenerate circles never touch.

For circles k_p and k_q corresponding to two non-collinear points p and q in $\mathfrak{GQ}$ there are essentially three possibilities:

1. $k_p \neq k_q$ and k_p does not touch k_q,
2. k_p and k_q are proper touching circles,
3. $k_p = k_q$ is a proper circle.

If k_p and k_q are two different non-touching circles, then points of the derived weak semi-biplane $\mathfrak{S}(p, q)$ are all circles that touch k_p but not k_q. Blocks of $\mathfrak{S}(p, q)$ are all circles that touch k_q but not k_p. Note that by our definition of touching this includes points incident with one but not the other circle. A point is incident with a circle if and only if the corresponding circles touch. If at least one of the two circles is degenerate and if the Möbius involution is not centric, then $\mathfrak{S}(p, q)$ is in fact just a subgeometry of the structure described above.

If k_p touches k_q, then, since p and q are not collinear, neither circle is degenerate. Let s denote the point incident with k_p and k_q. Then points of $\mathfrak{S}(p, q)$ are all circles touching k_p but not k_q and all circles touching k_p in s. Blocks of $\mathfrak{S}(p, q)$ are all circles touching k_q but not k_p and all circles touching k_q in s. A point is incident with a block if and only the corresponding circles touch in a point different from s. Note that the circles that touch k_p and k_q in s correspond to points as well as blocks.

If $k_p = k_q$, then points as well as blocks of $\mathfrak{S}(p, q)$ are all non-degenerate circles that touch k_p. A point is incident with a block if and only if the corresponding circles touch in a point not on k_p.

If $\mathfrak{Mo}$ is a locally compact Möbius plane with parameter 1, then one can describe the semi-biplanes that form the weak semi-biplane $\mathfrak{S}(p, q)$. Again, we will only consider the nice cases.

Let $\mathfrak{Mo}$ denote a locally compact Möbius plane with parameter 1. Then the point space of $\mathfrak{Mo}$ is homeomorphic to $\mathbb{S}_2$, every circle is homeomorphic to $\mathbb{S}_1$ and separates the point space into two connected components (cf. [1, 3.2]).

Lemma 4.3. *Suppose k_1 and k_2 are two non-intersecting proper circles of a locally compact Möbius plane $\mathfrak{Mo} = (\mathcal{P}_{\mathfrak{Mo}}, \mathcal{C})$ with parameter 1. Let C_1 denote the connected component of $\mathcal{P}_{\mathfrak{Mo}} \setminus k_1$ that does not contain k_2. Let C_2 denote the connected component of $\mathcal{P}_{\mathfrak{Mo}} \setminus (k_1 \cup k_2)$*

bounded by k_1 and k_2. Let C_3 denote the connected component of $\mathcal{P}_{\mathfrak{Mo}} \setminus k_2$ that does not contain k_1.

Let $\mathcal{P}_{\mathfrak{S}}$ denote the set of circles that touch k_2, that are contained in $C_2 \cup k_2$ and that are not contractible in $C_2 \cup k_2$. Let $\mathcal{B}_{\mathfrak{S}}$ denote the set of circles that touch k_1, that are contained in $C_2 \cup k_1$ and that are not contractible in $C_2 \cup k_1$. Let $\mathcal{P}'_{\mathfrak{S}}$ denote points on k_2 and the set of circles that touch k_2, that are contained in $C_2 \cup C_3 \cup k_2$ and that are contractible in $C_2 \cup k_2$ if they are contained in $C_2 \cup k_2$. Let $\mathcal{B}'_{\mathfrak{S}}$ denote the set of circles that touch k_1, that are contained in $C_2 \cup C_3 \cup k_1 \cup k_2$ and that are not contractible in $C_2 \cup k_1 \cup k_2$ if they are contained in $C_2 \cup k_1 \cup k_2$.

Then $(\mathcal{P}_{\mathfrak{S}}, \mathcal{B}_{\mathfrak{S}})$ and $(\mathcal{P}'_{\mathfrak{S}}, \mathcal{B}'_{\mathfrak{S}})$ are topological divisible semi-biplanes where incidence is given by touching.

Proof. Let $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ denote the generalized quadrangle associated with $\mathfrak{Mo}$. Then $\mathfrak{GQ}$ is a compact antiregular quadrangle with parameter 1. There is a centric Möbius involution τ such that $\mathfrak{Mo} = \mathfrak{Cp}(\mathfrak{GQ}, \tau)$ (cf. [11, 4.15]). Let Σ denote the fixed point set of τ. Let $\{p, q\} \subset \mathcal{P} \setminus \Sigma$ such that $p^\perp \cap \Sigma = k_2$ and $q^\perp \cap \Sigma = k_1$.

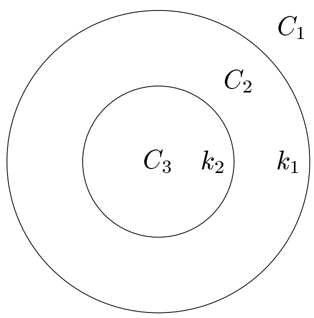

Figure 12

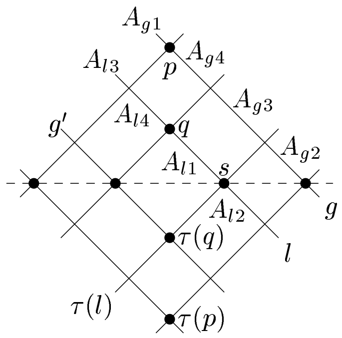

Figure 13

For $x \in \mathcal{P}$ we denote the line pencil through x by $\mathcal{L}_x$. Let $l \in \mathcal{L}_q$ and $s := l \cap \Sigma = \pi(\tau(q), l)$. We claim that $\pi(p, l)$ and $\pi(\tau(p), l)$ lie in the same connected component of $l \setminus \{s, q\}$. To prove this claim consider the square projection

$$\mu := \mu_{p,\tau(p),l} : p^\perp \cap \tau(p)^\perp = p^\perp \cap \Sigma = k_1 \to l.$$

The point $s = l \cap \Sigma$ is not in the image of μ since no two points in Σ are collinear. The point q is not in the image of μ since

$$q^\perp \cap p^\perp \cap \tau(p)^\perp = q^\perp \cap p^\perp \cap \Sigma = p^\perp \cap \Sigma \cap q^\perp \cap \Sigma = k_2 \cap k_1 = \emptyset.$$

Thus the image of μ is a connected set containing $\pi(p, l)$ and $\pi(\tau(p), l)$ but neither q nor s. This proves the claim. In fact, we even obtain that the image of μ is the closure of the connected component of $l \setminus \{p^\perp, \tau(p)^\perp\}$ not containing q.

After interchanging p and $\tau(p)$, if necessary, we obtain that the punctured line $l \setminus \{q, s, \pi(p, l), \pi(\tau(p), l)\}$ splits into four connected components A_{l1}, A_{l2}, A_{l3} and A_{l4} such that:

$$
\begin{aligned}
A_{l1} &= (q, s), \\
A_{l2} &= (s, \pi(\tau(p), l)), \\
A_{l3} &= (\pi(\tau(p), l), \pi(p, l)), \\
A_{l4} &= (\pi(p, l), q).
\end{aligned}
$$

The notation (a, b) is to be understood as follows. Let $x_1, \ldots, x_n \in l \in \mathcal{L}$, where $n \geq 3$. If x_i, x_j are two of these points such that one of the connected components of $l \setminus \{x_i, x_j\}$ contains none of the points x_k, where $1 \leq k \leq n$, then (x_i, x_j) denotes this connected component.

Since $q^\perp \cap p^\perp \cap \tau(p)^\perp$ is empty, the above situation holds for all lines l through q simultaneously. If l is a line through q and if x is a point on l, then $x^\perp \cap \Sigma$ determines the connected component A_{li} that contains x. One easily verifies:

If $x \in A_{l1}$, then $x^\perp \cap \Sigma$ is contained in $C_1 \cup k_1$.
If $x \in A_{l2}$, then $x^\perp \cap \Sigma$ is contained in $C_2 \cup k_1$ and contractible in $C_2 \cup k_1$.
If $x \in A_{l3}$, then $x^\perp \cap \Sigma$ intersects k_2 in two points.
If $x \in A_{l4}$, then $x^\perp \cap \Sigma$ is contained in $C_2 \cup k_1$ but not contractible in $C_2 \cup k_1$.

In all cases $x^\perp \cap \Sigma$ touches k_1.

Let $l \in \mathcal{L}_q$. Then $\lambda(\tau(q), l) = \tau(l)$ since $\tau(q) \in \tau(l)$ and $l \cap \tau(l) = \pi(\tau(q), l) \in \Sigma$ (see Figure 13). Let $g := \lambda(p, \tau(l))$ and $g' := \lambda(\tau(p), \tau(l))$. Then $\tau(g) = \lambda(\tau(p), l)$ and $\tau(g') = \lambda(p, l)$. The points $\tau(q)$, $\pi(\tau(q), g)$, $\pi(\tau(q), g')$ and $\pi(\tau(q), l)$ are all incident with the line $\tau(l)$. The points $\tau(p)$, $\pi(\tau(p), g)$, $\pi(\tau(p), l)$ are all incident with the line $\tau(g)$. The points p, $\pi(p, g')$, $\pi(p, l)$ are all incident with the line $\tau(g')$.

As above one verifies that $\pi(q, g)$ and $\pi(\tau(q), g)$ lie in the same connected component of $g \setminus (\{p\} \cup \Sigma) = g \setminus \{p, \pi(\tau(p), g)\}$. Also, $\pi(q, g')$ and $\pi(\tau(q), g')$ lie in the same connected component of $g' \setminus (\{\tau(p)\} \cup \Sigma) = g' \setminus \{\tau(p), \pi(p, g')\}$.

This implies that $g \setminus \{p, \pi(q, g), \pi(\tau(q), g), \pi(\tau(p), g)\}$ consists of four connected components A_{g1}, A_{g2}, A_{g3} and A_{g4} such that:

$$
\begin{aligned}
A_{g1} &= (p, \pi(\tau(p), g)), \\
A_{g2} &= (\pi(\tau(p), g), \pi(\tau(q), g)), \\
A_{g3} &= (\pi(\tau(q), g), \pi(q, g)), \\
A_{g4} &= (\pi(q, g), p).
\end{aligned}
$$

Also, $g' \setminus \{\tau(p), \pi(q, g'), \pi(\tau(q), g'), \pi(p, g')\}$ consists of four connected components $A_{g'1}$, $A_{g'2}$, $A_{g'3}$ and $A_{g'4}$ such that:

$$
\begin{aligned}
A_{g'1} &= (\tau(p), \pi(p, g')), \\
A_{g'2} &= (\pi(p, g'), \pi(q, g')), \\
A_{g'3} &= (\pi(q, g'), \pi(\tau(q), g')), \\
A_{g'4} &= (\pi(\tau(q), g'), \tau(p)).
\end{aligned}
$$

As above one easily verifies that for $h \in \{g, g'\}$ the following holds:
If $x \in A_{h1}$, then $x^\perp \cap \Sigma$ is contained in $C_3 \cup k_2$.
If $x \in A_{h2}$, then $x^\perp \cap \Sigma$ is contained in $C_2 \cup k_2$ and contractible in $C_2 \cup k_2$.
If $x \in A_{h3}$, then $x^\perp \cap \Sigma$ intersects k_1 in two points.
If $x \in A_{h4}$, then $x^\perp \cap \Sigma$ is contained in $C_2 \cup k_2$ but not contractible in $C_2 \cup k_2$.

Let $l \in \mathcal{L}_q$, $g \in \mathcal{L}_p$ and $g' \in \mathcal{L}_{\tau(p)}$ as above. Then l intersects neither g nor g'. Choose $b \in A_{l4}$ and let $s := \pi(b, g) \in A_{g4}$ and $s' := \pi(b, g') \in A_{g',2}$. By [9], the derived structures $\mathfrak{S}(p, q, s) = (\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$ and $\mathfrak{S}(\tau(p), q, s') = (\mathcal{P}'_\mathfrak{S}, \mathcal{B}'_\mathfrak{S})$ are semi-biplanes where:

$\mathcal{P}_\mathfrak{S}$ is the connected component of $p^\perp \setminus (\{p\} \cup q^\perp)$ containing s,
$\mathcal{B}_\mathfrak{S}$ is the connected component of $q^\perp \setminus (\{q\} \cup p^\perp)$ containing b,
$\mathcal{P}'_\mathfrak{S}$ is the connected component of $p^\perp \setminus (\{p\} \cup \tau(q)^\perp)$ containing s',
$\mathcal{B}'_\mathfrak{S}$ is the connected component of $\tau(q)^\perp \setminus (\{\tau(q)\} \cup p^\perp)$ containing b.

One easily verifies:

$$\begin{aligned}
\mathcal{P}_\mathfrak{S} &= \bigcup_{g \in \mathcal{L}_p} A_{g4}, \\
\mathcal{B}_\mathfrak{S} &= \bigcup_{h \in \mathcal{L}_q} A_{h4}, \\
\mathcal{P}'_\mathfrak{S} &= \bigcup_{g \in \mathcal{L}_{\tau(p)}} (A_{g1} \cup A_{g2}) \cup p^\perp \cap \tau(p)^\perp, \\
\mathcal{B}'_\mathfrak{S} &= \bigcup_{h \in \mathcal{L}_q} (A_{h4} \cup A_{h3}) \cup p^\perp \cap q^\perp.
\end{aligned}$$

Thus $\mathcal{P}_\mathfrak{S}$ corresponds to the set of circles that touch k_1 and that are contained in $C_2 \cup k_1$ but not contractible in $C_2 \cup k_1$. The set $\mathcal{B}_\mathfrak{S}$ corresponds to the set of circles that touch k_2 and that are contained in $C_2 \cup k_2$ but not contractible in $C_2 \cup k_2$. The set $\mathcal{P}'_\mathfrak{S}$ corresponds to the set of possibly degenerate circles that touch k_2, that are contained in $C_2 \cup C_3 \cup k_2$ and that are contractible in $C_2 \cup k_2$ if they are contained in $C_2 \cup k_2$. Also, $\mathcal{B}'_\mathfrak{S}$ corresponds to the set of circles that touch k_1, that are contained in $C_2 \cup C_3 \cup k_1 \cup k_2$ and that are not contractible in $C_2 \cup k_1 \cup k_2$ if they are contained in $C_2 \cup k_1 \cup k_2$. □

Remark. These semi-biplanes are generalisations of the model illustrated in Figure 8 and Figure 9.

Lemma 4.4. *Suppose s is a point and k a proper circle not incident with s of a locally compact Möbius plane $\mathfrak{Mo} = (\mathcal{P}_{\mathfrak{Mo}}, \mathcal{C})$ with parameter 1. Let C denote the connected component of $\mathcal{P}_{\mathfrak{Mo}} \setminus k$ that contains s. Let $\mathcal{P}_\mathfrak{S}$ denote the set of circles incident with s that are contained in $C \cup \{s\}$. Let $\mathcal{B}_\mathfrak{S}$ denote the set of circles that touch k and that are contained but not contractible in $C \cup k$. Then $(\mathcal{P}_\mathfrak{S}, \mathcal{B}_\mathfrak{S})$ is a topological divisible semi-biplane where incidence is given by touching in a point of C.*

Proof. The proof of this result runs along the same lines as the proof of Lemma 4.3 and can be safely left to the reader. Alternatively, one can obtain the result directly from Lemma 4.3 as follows. The circle k corresponds to k_1 and the proper circle k_2 shrinks to the point s, that is, the point q is moved to Σ. Then C_3 is empty and $C = C_2$. □

Remark. These semi-biplanes are generalisations of the model illustrated in Figure 7.

Lemma 4.5. *Suppose k is a proper circle of a locally compact Möbius plane $\mathfrak{Mo} = (\mathcal{P}_{\mathfrak{Mo}}, \mathcal{C})$ with parameter 1. Let C denote a connected component of $\mathcal{P}_{\mathfrak{Mo}} \setminus k$. Let $\mathcal{P}_{\mathfrak{S}}$ and $\mathcal{B}_{\mathfrak{S}}$ denote the set of circles that touch k and that are contained in $C \cup k$. Then $(\mathcal{P}_{\mathfrak{S}}, \mathcal{B}_{\mathfrak{S}})$ is a topological divisible semi-biplane where incidence is given by touching in a point of C.*

Proof. One can view this configuration as a limit case of Lemma 4.3 where $k_1 = k_2$, that is, where $q = \tau(p)$. Then C corresponds to C_3, the connected component C_2 is empty and the model $(\mathcal{P}'_{\mathfrak{S}}, \mathcal{B}'_{\mathfrak{S}})$ becomes the above semi-biplane. However, the proof of 4.3 does not work here. One has to introduce a further line of reference neither incident with p nor with q.

A direct proof of the results runs as follows: Let $\mathfrak{GQ} = (\mathcal{P}, \mathcal{L})$ denote the generalized quadrangle associated with $\mathfrak{Mo}$. Let τ denote the Möbius involution such that $\mathfrak{Mo} = \mathfrak{Cp}(\mathfrak{GQ}, \tau)$. Let Σ denote the fixed point set of τ and let $p \in \mathcal{P}$ be such that $p^\perp \cap \Sigma = k$. The set $p^\perp \setminus (\{p\} \cup \tau(p)^\perp)$ corresponds to the set of circles that touch k. Now, the set $p^\perp \setminus (\{p\} \cup \tau(p)^\perp)$ consists of two connected components. One of these connected components corresponds to the set of circles that touch k and that are contained in $C \cup k$. The other connected component corresponds to the set of circles that touch k and that do not intersect C. The same is true for $\tau(p)^\perp \setminus (\{\tau(p)\} \cup p^\perp)$ since $p^\perp \cap \Sigma = \tau(p)^\perp \cap \Sigma = p^\perp \cap \tau(p)^\perp$. Let $\mathcal{P}_{\mathfrak{S}}$ and $\mathcal{B}_{\mathfrak{S}}$ denote the set of circles that touch k and that are contained in $C \cup k$. Choose $l \in \mathcal{P}_{\mathfrak{S}}$. Let m denote a circle touching k that touches l in a point t not on k. Then $t \in C$, hence $m \in \mathcal{B}_{\mathfrak{S}}$. Thus $(\mathcal{P}_{\mathfrak{S}}, \mathcal{B}_{\mathfrak{S}}) = \mathfrak{S}(p, \tau(p), l)$ is a topological divisible semi-biplane. □

Remark. These semi-biplanes are generalisations of the model illustrated in Figure 6.

4.5. Semi-biplanes in Minkowski planes

For Minkowski planes the general treatment of the connection between semi-biplanes and Minkowski planes runs as for Möbius planes. The only difference is that now degenerate circles, that is points, can touch. We say that two degenerate circles touch if they are parallel.

The point space of the classical Minkowski plane is embedded in $PG(3, \mathbb{R})$ rather than $\mathbb{R}^3$. One consequence of this is that most plane models of the semi-biplane embedded in the Minkowski plane are not easy to describe. Those models that can be described easily are not new. They appeared already in the discussion of semi-biplanes in Laguerre planes. Therefore, we will not discuss these models here.

References

[1] Buchanan, T.; Hähl, H.; Löwen, R.: *Topologische Ovale.* Geom. Dedicata **9** (1980), 401–424.

[2] Forst, M.: *Topologische 4-Gone.* Mitt. d. Math. Sem. Gießen **147** (1981), 65–129.

[3] Groh, H.: *Topologische Laguerre-Ebenen II.* Abh. Math. Sem. Univ. Hamburg **34** (1969), 11–21.

[4] Grundhöfer, Th.; Knarr, N.: *Topology in generalized quadrangles.* Topology Appl. **34** (1990), 139–152.

[5] Knarr, N.: *The nonexistence of certain topological polygons.* Forum Math. **2** (1990), 603–612.

[6] Kramer, L.: *Compact Polygons.* PhD thesis, Tübingen, 1994.

[7] Polster, B.: *Semi-biplanes on the cylinder.* Geom. Dedicata **58** (1995), 145–160.

[8] Polster, B.; Schroth, A. E.: *Homology semi-biplanes in antiregular generalized quadrangles.* In preparation.

[9] Polster, B.; Schroth, A. E.: *Semi-biplanes and antiregular generalized quadrangles.* To appear in Geom. Dedicata.

[10] Polster, B.; Schroth, A. E.: *Split semi-biplanes in antiregular generalized quadrangles.* To appear in Bull. Belg. Math. Soc.

[11] Schroth, A. E.: *Topological Circle Planes and Topological Quadrangles.* Pitman Research Notes in Mathematics Series **337**, Longman, Harlow 1995.

[12] Steinke, G. F.: *Topological circle geometries.* In: F. Buekenhout, (ed.), Handbook of Incidence Geometry: Buildings and Foundations. North-Holland, Amsterdam 1995, 1325–1354.

[13] Wild, P.: *On semi-biplanes.* PhD thesis, Westfield College, University of London, 1980.

[14] Wild, P.: *Some families of semibiplanes.* Discrete Math. **138** (1995), 397–403.

Received April 26, 1997