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Contact Homogeneity and Envelopes of Riemannian Metrics

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Abstract. We study contact of higher order of Riemannian manifolds (or metrics). The main results concern "contact homogeneous Riemannian manifolds" (of arbitrary order) and envelopes of Riemannian metrics (of first and second order).

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1. The contact properties of Riemannian manifolds

We start with the basic definitions.

Definition 1. *Let (M, g), $(\overline{M}, \overline{g})$ be two Riemannian manifolds (of the same dimension n and of class C^∞) and let $p \in M$, $\overline{p} \in \overline{M}$ be two points. We say that these manifolds have contact of order s (≥ 0) at the pair of points $(p, \overline{p})$ if there are two systems of local coordinates $(u^1, \dots, u^n)$, $(\overline{u}^1, \dots, \overline{u}^n)$ in the neighborhoods of the points p and $\overline{p}$ respectively such that*

$$u^i(p) = \overline{u}^i(\overline{p}) = 0 \qquad \text{for} \quad i = 1, \dots, n \tag{1}$$

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and

$$(\partial^r g_{ij}/\partial u^{i_1}\cdots\partial u^{i_r})(p) = (\partial^r \overline{g}_{ij}/\partial \overline{u}^{i_1}\cdots\partial \overline{u}^{i_r})(\overline{p}) \tag{2}$$

hold for all $r \in \{0,1,\dots,s\}$ *and all* $i,j,i_1,\dots,i_r \in \{1,\dots,n\}$.

We see that the above definition is independent of the choice of the local coordinates in the following restricted sense: if we take new systems of local coordinates $(v^1,\dots,v^n)$, $(\overline{v}^1,\dots,\overline{v}^n)$ which come from $(u^1,\dots,u^n)$ and $(\overline{u}^1,\dots,\overline{u}^n)$ respectively by *the same transformation rule,* then (2) is still satisfied for the new systems of local coordinates.

We also see that the condition (1) is not essential. In fact, if the local coordinate systems (u^i), $(\overline{u}^i)$ do not satisfy (1) but only (2), then we can replace them by the coordinate systems $v^i = u^i - u^i(p)$, $\overline{v}^i = \overline{u}^i - \overline{u}^i(\overline{p})$ for $i = 1,\dots,n$ which satisfy both (1) and (2).

Let us remark that V. V. Slavskii [29] is treating a special case of Definition 1 where $\overline{M} = M$ and $\overline{p} = p$. He is using the notions “tangent”, “osculating” and “superosculating” for the contact of order 0, 1 and 2, respectively. A reference is given to E. Cartan [4].

Let now ∇, $\overline{\nabla}$, R, $\overline{R}$ denote the corresponding Levi-Civita connections and the corresponding Riemannian curvature tensors, respectively. For the sake of simplicity, the components of the tensors $\nabla^r R$ and $\overline{\nabla}^r\overline{R}$ respectively with respect to some vector basis will be marked only by the corresponding indices.

Definition 2. *We say that two Riemannian manifolds* (M,g) *and* $(\overline{M},\overline{g})$*, have the same curvature up to order* s *at the pair of points* $(p,\overline{p})$ *if there is an orthonormal basis* $(e_1,\dots,e_n)$ *for* g *in* T_pM *and an orthonormal basis* $(\overline{e}_1,\dots,\overline{e}_n)$ *for* $\overline{g}$ *in* $T_{\overline{p}}\overline{M}$ *such that*

$$\left(\nabla^r_{i_1,\dots,i_r}R\right)_{ijkl} = \left(\overline{\nabla}^r_{i_1,\dots,i_r}\overline{R}\right)_{ijkl} \tag{3}$$

hold for all $r \in \{0,1,\dots,s\}$ *and all* $i,j,k,l,i_1,\dots,i_r \in \{1,\dots,n\}$.

Next, we shall prove that the definitions 1 and 2 are equivalent in some sense.

Proposition 1. *Two Riemannian manifolds* (M,g), $(\overline{M},\overline{g})$ *have contact of order* $s+2$ *at the pair of points* $(p,\overline{p})$ *if and only if they have the same curvature up to order* s *at* $(p,\overline{p})$.

Proof. The “only if” part: suppose that (1) and (2) hold for the corresponding local coordinate systems (u^i), $(\overline{u}^i)$ up to the order $s+2$ $(s \geq 0)$. The classical local formulas imply that all the *coordinate* components of the tensors R_p, $(\nabla R)_p,\dots,(\nabla^s R)_p$ can be expressed as rational functions of the metric components and their derivatives up to order $s+2$ and hence they are equal to the corresponding coordinate components of $\overline{R}_{\overline{p}}$, $(\overline{\nabla}\overline{R})_{\overline{p}},\dots,(\overline{\nabla}^s\overline{R})_{\overline{p}}$. Now, for some matrix (a_i^j), the basis $\{f_i = \sum_j a_i^j(\partial/\partial u^j)_p,\ i = 1,\dots,n\}$ is orthonormal with respect to g and, because $g_{ij}(p) = \overline{g}_{ij}(\overline{p})$, the basis $\{\overline{f}_i = \sum_j a_i^j(\partial/\partial\overline{u}^j)_{\overline{p}},\ i = 1,\dots,n\}$ is orthonormal with respect to the second metric $\overline{g}$. Hence the corresponding components in (3) are still equal when expressed through the new vector bases (f_i) and $(\overline{f}_i)$, q.e.d.

The "if" part: for a system of *normal* coordinates $(x^1, \dots, x^n)$ centered at p we have a (formal) Taylor series of the form

$$g_{ij} = \delta_{ij} - \tfrac{1}{3} \sum_{k,l} (R_{ikjl})_p x^k x^l + (\text{higher order terms}). \tag{4}$$

Moreover, the coefficients of the monomials of degree $r+2$ in the higher order terms are certain (universal) polynomials in the corresponding components of R_p, $(\nabla R)_p, \dots, (\nabla^r R)_p$ (see [8], [10], [11], [14]). In other words, all derivatives $(\partial^r g_{ij} / \partial x^{i_1} \cdots \partial x^{i_r})(p)$ up to order $s+2$ can be expressed, in a unique way, through these curvature components up to order s.

Let (e_i), $(\overline{e}_i)$ be orthonormal bases as in Definition 2 and let (x^i), $(\overline{x}^i)$ be the corresponding normal coordinate systems, i.e.

$$e_i = (\partial / \partial x^i)_p, \quad \overline{e}_i = (\partial / \partial \overline{x}^i)_p, \qquad \text{for} \quad i = 1, \dots, n. \tag{5}$$

Then the equalities (3) imply the equalities (2) for all $r \leq s+2$ with respect to the normal coordinate systems (x^i), $(\overline{x}^i)$. (The latter also satisfy (1).) This concludes the proof of the proposition. □

We shall close this section with the following result concerning a special case of Definition 1.

Proposition 2. *Let g and $\overline{g}$ be two Riemannian metrics defined on the same manifold M and let $p \in M$. If g and $\overline{g}$ have contact of order s at the common point p, and if $(v^1, \dots, v^n)$ is a fixed local coordinate system around p, then there is a metric $\widetilde{g}$ defined in some neighborhood $U_p \subset M$ such that*

a) *$(U_p, \widetilde{g})$ is isometric to $(V_p, \overline{g})$, where V_p is another neighborhood of p,*

b) $(\partial^r g_{ij} / \partial v^{i_1} \cdots \partial v^{i_r})(p) = (\partial^r \widetilde{g}_{ij} / \partial v^{i_1} \cdots \partial v^{i_r})(p)$

for all $r = 0, 1, \dots, s$ and all $i, j, i_1, \dots, i_r \in \{1, \dots, n\}$.

Proof. Consider local coordinate systems (u^i), $(\overline{u}^i)$ around $p \in M$ for the metrics g and $\overline{g}$ respectively such that the conditions (1) and (2) from Definition 1 are satisfied (for the special case $\overline{M} = M$, $\overline{p} = p$). In the local coordinates we can write

$$g = \sum g_{ij}(u^1, \dots, u^n) \mathrm{d}u^i \mathrm{d}u^j, \quad \overline{g} = \sum \overline{g}_{ij}(\overline{u}^1, \dots, \overline{u}^n) \mathrm{d}\overline{u}^i \mathrm{d}\overline{u}^j. \tag{6}$$

Consider now a new metric $\widetilde{g}$ on the neighborhood of p whose local expression has the form $\sum \overline{g}_{ij}(u^1, \dots, u^n) \mathrm{d}u^i \mathrm{d}u^j$. Then the property a) is obviously satisfied and the conditions (2) are satisfied for the metrics g and $\widetilde{g}$ when expressed in the common coordinate system (u^i). But this must be also true for the given local coordinate system (v^i), which is our condition b). □

Notice that we *do not* require here the conditions $v^i(p) = 0$ for $i = 1, \dots, n$.

2. Contact homogeneous Riemannian manifolds

We start with

Definition 3. *A Riemannian manifold (M, g) is said to be contact homogeneous of order s if, for every two points $p, q \in M$, the manifold (M, g) has contact of order s with itself at the pair of points (p, q).*

The following definition is essentially due to I. M. Singer (see [28], [27] and [3]).

Definition 4. *A Riemannian manifold (M, g) is said to be curvature homogeneous up to order s if, for every two points $p, q \in M$, there is a linear isometry $\varphi: T_pM \to T_qM$ such that*

$$\varphi^* R_q = R_p, \dots, \qquad \varphi^*(\nabla^s R)_q = (\nabla^s R)_p. \tag{7}$$

As a direct consequence of Proposition 1, we have now the following

Proposition 3. *A Riemannian manifold (M, g) is curvature homogeneous up to order s if and only if it is contact homogeneous of order $s + 2$.*

Obviously, a locally homogeneous Riemannian manifold (M, g) is curvature homogeneous, and also contact homogeneous, in each finite order. We also see that every Riemannian manifold is contact homogeneous of order 1. (One can just use normal coordinate systems at two different points and the formula (4).) The contact homogeneous spaces of order 2 are the same as curvature homogeneous spaces in the sense of I. M. Singer. One of the basic problems in [28] was the question if there are any curvature homogeneous spaces which are not locally homogeneous. The first example was constructed in 1973 by S. Sekigawa ([24], see also [25]). Presently, the theory of curvature homogeneous spaces is well-developed: see [17], [18], [19], [2] and especially the survey article in [3]. A complete classification is known in dimension $n = 3$ (see [13], [16], [20] and the next section).

The main result of [28] can be now presented in the following way: let (M, g) be a connected Riemannian manifold of dimension n (and class C^∞). For each $p \in M$ and $s = 0, 1, 2, \dots$ define the Lie algebra

$$\mathfrak{g}(p; s) = \{A \in \mathfrak{gl}(T_pM) \,|\, A \cdot g_p = A \cdot R_p = \dots = A \cdot (\nabla^s R)_p = 0\}, \tag{8}$$

where A acts as a derivation on the tensor algebra of T_pM. Now, denote by $k(M, p)$ the least integer for which the sequence

$$\mathfrak{g}(p; 0) \supseteq \mathfrak{g}(p; 1) \supseteq \dots \supseteq \mathfrak{g}(p; s) \supseteq \dots \tag{9}$$

stabilizes. Obviously $k(M, p) < \frac{n(n-1)}{2}$. Let us denote by k_M the maximum of all numbers $k(M, p)$, $p \in M$, and by k_n the maximum of all numbers k_M for all manifolds (M, g) of dimension n. We usually call k_n *the Singer number for the dimension* n. (An estimate $k_n + 1 < 3n/2$ is given without proof as exercise in [9], p. 165.)

By translating the main theorem from [28] (see also [23]) in our language we obtain

Theorem 1. *Let* (M, g) *be a Riemannian manifold of dimension* n *which is contact homogeneous up to order* $k_n + 3$. *Then* (M, g) *is locally homogeneous.*

In [26] and [27] the following results have been (in fact) proved:

Theorem 2. *Let* (M, g) *be a Riemannian manifold of dimension* 3 *or* 4. *If* (M, g) *is contact homogeneous of order* 3, *then it is locally homogeneous.*

The existence of contact homogeneous Riemannian manifolds of order 3 which are not locally homogeneous (for the dimensions $n \geq 5$) remains an interesting and rather difficult open problem.

3. Envelopes of Riemannian metrics

Let M be an n-dimensional manifold and let $\{g^p | \, p \in M\}$ be a family of Riemannian metrics given in the neighborhoods U_p of the corresponding points $p \in M$.

Definition 5. *A Riemannian metric* g *on* M *is called a* k*-th order envelope of the family* $\{g^p | \, p \in M\}$ *if, for each point* $p \in M$, *the manifolds* (M, g) *and* (U_p, g^p) *have a contact of order* k *at the pair of points* (p, p).

We shall treat the first order envelopes and the second order envelopes in this section. (For the higher order envelopes we are still missing interesting examples or results.)

A) We start with the following observation, which is well-known: any Riemannian metric g on a manifold M can be approximated, at each point $p \in M$, by a Euclidean metric which has contact of first order with it. In other words, *each Riemannian metric on a manifold* M *is a first order envelope of a family of (local) Euclidean metrics.* This assertion seems to be trivial but it becomes less trivial if we look for an explicit expression of this fact in a unique coordinate system. If M is an open domain of a Cartesian space $R^n(x^1, \ldots, x^n)$, then such an explicit expression must exist due to Proposition 2. In fact, we shall prove

Proposition 4. *Each Riemannian metric* $g = \sum g_{ij} \mathrm{d}x^i \mathrm{d}x^j$ *on an open subset* $U \subset R^n(x^1, \ldots, x^n)$ *is the first order envelope of a system of Euclidean metrics* $\{h^p | \; p \in U\}$. *These metrics can be expressed, in the standard coordinates, through the polynomials of the metric components* g_{ij}, *the square roots of such polynomials and through the linear functions of the derivatives* $(\partial g_{ij}/\partial x^k)$.

Proof. The existence part of the Proposition is trivial. We are going to prove (and clear up) the second statement.

Let $p = (x_o^1, \dots, x_o^n) \in U$ be fixed. We shall describe a Euclidean metric h^p on U which has the first order contact with g at the point p and has the property stated in the proposition. We shall look for $h = h^p$ in the form

$$h = \sum_{k=1}^{n} (\mathrm{d}X^k)^2 \tag{10}$$

where X^k are quadratic polynomials of $x^1, \dots, x^n$, namely

$$X^k = \sum_{k=1}^{n} a_i^k (x^i - x_o^i) + \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}^k (x^i - x_o^i)(x^j - x_o^j) \quad (k = 1, \dots, n). \tag{11}$$

Here a_i^k and a_{ij}^k are coefficients to be determined such that

$$a_{ij}^k = a_{ji}^k. \tag{12}$$

Substituing from (11) into (10), we obtain easily

$$h = \sum_{i,j=1}^{n} \left\{ \sum_{k=1}^{n} \Big(a_i^k + \sum_{u=1}^{n} a_{iu}^k (x^u - x_o^u) \Big) \Big(a_j^k + \sum_{v=1}^{n} a_{jv}^k (x^v - x_o^v) \Big) \right\} \mathrm{d}x^i \mathrm{d}x^j. \tag{13}$$

If we write (13) in the form $h = \sum_{i,j=1}^{n} h_{ij} \mathrm{d}x^i \mathrm{d}x^j$, the conditions for the first order contact say

$$h_{ij}(p) = \sum_{k=1}^{n} a_i^k a_j^k = g_{ij}(p) \qquad (i, j = 1, \dots, n), \tag{14}$$

$$(\partial h_{ij}/\partial x^l)(p) = \sum_{k=1}^{n} \left(a_i^k a_{jl}^k + a_j^k a_{il}^k \right) = (\partial g_{ij}/\partial x^l)(p) \qquad (i, j, l = 1, \dots, n). \tag{15}$$

We first construct a matrix $A = (a_i^j)$ satisfying (14). Here we just use the Gram-Schmidt orthogonalization process for constructing an orthonormal basis $(e_1, \dots, e_n)$ in the tangent space $(T_p R^n,\, g_p)$ from the coordinate basis $\big((\partial/\partial x^1), \dots, (\partial/\partial x^n)\big)_p$. Then expressing the second one through the first one in the form

$$(\partial/\partial x^i)_p = \sum_{k=1}^{n} a_i^k e_k, \qquad i = 1, \dots, n, \tag{16}$$

we obtain immediately the wanted matrix. As well-known, the coefficients a_i^k are uniquely determined functions of the components $g_{ij}(p)$ which involve only polynomials of g_{ij} and the square roots of such polynomials. The matrix A is triangular.

Now, let us look for the coefficients a^k_{ij}. Consider the inverse matrix $A^{-1} = (A^j_i)$ of the matrix $A = (a^j_i)$. An easy check shows that the formula

$$a^k_{ij} = \tfrac{1}{2} \sum_{u=1}^{n} \Big(\frac{\partial g_{ui}}{\partial x^j} + \frac{\partial g_{uj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^u} \Big)(p) A^u_k \tag{17}$$

gives coefficients satisfying (12) and (15). This completes the proof of the Proposition. □

B) This part dealing with the second order envelopes was inspired by the work of Ü. Lumiste (see, e.g., [21], [22]). Lumiste studied second order envelopes of (locally) symmetric submanifolds of R^n (i.e., of those with parallel second fundamental form, cf. Ferus [7]). Such envelopes (if properly constructed) are the same as so-called *semi-parallel submanifolds* (see [5], [6]). These submanifolds satisfy automatically , as Riemannian manifolds, the curvature identity

$$R(X,Y)\cdot R = 0 \qquad (X, Y \in T_pM,\ p \in M). \tag{18}$$

(Here the curvature transformations $R(X,Y)$ act as derivations on the corresponding tensor algebra.)

On the other hand, Riemannian manifolds satisfying (18) are called *semi-symmetric spaces.* They have been classified by Z. Szabó ([30], [31], [32]); for the more explicit classification see [12], [1] and [3]. It is an open problem which semi-symmetric spaces can be isometrically immersed as semi-parallel submanifolds of Euclidean spaces, but the class of all semi-symmetric spaces seems to be much broader. The locally symmetric spaces and all Riemannian manifolds of the form $M_2 \times \mathsf{R}^k$ satisfy (18) but they form a negligible subclass of the whole class of semi-symmetric spaces. (We call them "trivial" ones.)

Lumiste proved that, in some situations, one can obtain semi-parallel submanifolds which are neither locally symmetric nor locally decomposable as Riemannian manifolds. These examples are so-called "Szabó cones" (cf. [31]), which are always incomplete as Riemannian manifolds. They occur as second order envelopes of some families of generalized rotational cylinders [22].

On the other hand, it is not known if there exist second order envelopes of (locally) symmetric submanifolds of Euclidean spaces which produce nontrivial and complete semi-symmetric spaces.

In the intrinsic geometry, the situation is different. In fact, we can easily prove the following

Proposition 5. *Each semi-symmetric metric g on a manifold M is a second order envelope of a family $\{h^p \mid p \in M\}$ of locally symmetric metrics.*

Proof. Choose a point $p \in M$. Because $R_p(X,Y)\cdot R_p = 0$ at p, a well-known result by E. Cartan says that there exists a symmetric space (N,h) with the "typical" curvature tensor R_p. More precisely, for each point $q \in N$, the spaces (M,g) and (N,h) have the same curvature of order 0 (which means contact of order 2) at the pair of points (p,q). Using two systems of local coordinates as in Definition 1, one can identify corresponding neighborhoods of p and q in M and N respectively. This gives a locally symmetric metric h^p in a neighborhood of p on M which has the second order contact with g at p. Hence the result follows. □

To show a good explicit illustration of Proposition 5, it is natural to start with the 3-dimensional example by K. Sekigawa [25] and its generalizations. We recall here that the original Sekigawa's examples are complete Riemannian manifolds.

In [12] and [18] the following family of metrics was investigated on an open domain $U \subset \mathsf{R}^3(x^1, x^2, x^3)$ (we only change the notation):

$$g = (f(x^1, x^2)\mathrm{d}x^1)^2 + (\mathrm{d}x^2 + x^3\mathrm{d}x^1)^2 + (\mathrm{d}x^3 - x^2\mathrm{d}x^1)^2. \tag{19}$$

Here $f(x^1, x^2)$ is an arbitrary smooth function such that $f''_{22} \neq 0$. Each metric g of the form (19) is semi-symmetric, locally irreducible and not locally homogeneous. If $U = \mathsf{R}^3$ and $0 < \alpha(x^2) \leq f(x^1, x^2) \leq \beta(x^2) < +\infty$ for some smooth functions α, β, then the metric g is complete.

The scalar curvature of (19) is given by the formula

$$\tau = -2f^{-1}f''_{22}. \tag{20}$$

It was proved in [12] that, in dimension three, the family (19) describes "semisymmetric foliated spaces of parabolic type" in the sense of Z. Szabó [30]. If $\tau = -2r^2$ is a negative constant, i.e., if

$$f = a(x^1)\exp(rx^2) + b(x^1)\exp(-rx^2), \tag{21}$$

and if, moreover, $a(x^1)$ and $b(x^1)$ are positive constants, we obtain (in different coordinates) the original example by Sekigawa. Each metric (21) such that $a(x^1) > a > 0$, $b(x^1) > b > 0$ on R is complete and curvature homogeneous with the same curvature as the symmetric space $H^2(-r^2) \times \mathsf{R}$. If $\tau = 2r^2$ is a positive constant, then we get a family of metrics in which

$$f = a(x^1)\cos(rx^2) + b(x^1)\sin(rx^2). \tag{22}$$

These spaces are again curvature homogeneous, with the same curvature as the symmetric space $S^2(r^2) \times \mathsf{R}$. Yet, the metrics satisfying (22) are never defined on the whole of R^3 and they are never complete.

In the general case (19), if the domain U is connected, then the scalar curvature is either everywhere positive (i.e., $\tau = 2r^2$) or everywhere negative (i.e., $\tau = -2r^2$) but it can now change from point to point. Yet, at each point $p \in U$, the space (U, g) has the same curvature as the symmetric space $S^2(r^2(p)) \times \mathsf{R}$, or $H^2(-r^2(p)) \times \mathsf{R}$, respectively. Hence we have the following specific result:

Proposition 6. *Every Riemannian metric g on U of the form (19) with $\tau > 0$ (or $\tau < 0$, respectively) is a second order envelope of the cylindrical metrics which are, at each point p, those of the space $S^2(r^2(p)) \times \mathsf{R}$ (or $H^2(-r^2(p)) \times \mathsf{R}$, respectively).*

It remains to give explicit formulas in the standard coordinates showing how the metric g is produced as a second order envelope of locally symmetric metrics of the above type. Our computational method will be based on Proposition 2.

First, let us notice that the metric of $S^2(r^2) \times \mathsf{R}$ has a local form

$$\overline{h} = (\mathrm{d}X^2)^2 + \cos^2(rX^2)(\mathrm{d}X^1)^2 + (\mathrm{d}X^3)^2 \tag{23}$$

and the metric of $H^2(-r^2) \times \mathsf{R}$ has a local form

$$\overline{h} = (\mathrm{d}X^2)^2 + \cosh^2(rX^2)(\mathrm{d}X^1)^2 + (\mathrm{d}X^3)^2. \tag{24}$$

(Here $X^1,\ X^2,\ X^3$ are "cylindrical" local coordinates.)

We see that, at the origin $X^i = 0$, the metric (23), or (24), has contact of order 2 with the metric

$$h = (\mathrm{d}X^2)^2 + (1 - \varepsilon(rX^2)^2)(\mathrm{d}X^1)^2 + (\mathrm{d}X^3)^2 \tag{25}$$

where $\varepsilon = 1$, or $\varepsilon = -1$, respectively. Hence it is sufficient to express the metric g from (19) as a second order envelope of the metrics which are locally isometric to (25). Here, for each point $p \in U$, the coordinates $X^1,\ X^2,\ X^3$ must be functions of the standard coordinates $x^1,\ x^2,\ x^3$ such that $X^i(p) = 0$ for $i = 1, 2, 3$. We can see easily that these functions can be chosen as third degree polynomials of the standard coordinates because we are interested in the second order contact only.

Let $o \in U$ be any fixed point, $o = (x_o^1, x_o^2, x_o^3)$. (Here we use the symbol o instead of p, which is more convenient for the notation.) Let us put, for $k = 1, 2, 3$,

$$X^k = \sum_{i=1}^{n} a_i^k (x^i - x_o^i) + \tfrac{1}{2} \sum_{i,j=1}^{n} a_{ij}^k (x^i - x_o^i)(x^j - x_o^j) + \tfrac{1}{6} \sum_{i,j,l=1}^{n} a_{ijl}^k (x^i - x_o^i)(x^j - x_o^j)(x^l - x_o^l) \tag{26}$$

where a_{ij}^k and a_{ijl}^k are all symmetric with respect to the lower indices.

Expressing the conditions for the second order contact of g and h at o, we obtain, after a routine calculation, the following identities:

$$h_{ij}(o) = \sum_{k=1}^{n} a_i^k a_j^k = g_{ij}(o), \tag{27}$$

$$\frac{\partial h_{ij}}{\partial x^u}(o) = \sum_{k=1}^{n} (a_i^k a_{ju}^k + a_j^k a_{iu}^k) = \frac{\partial g_{ij}}{\partial x^u}(o), \tag{28}$$

$$\begin{aligned} \frac{\partial^2 h_{ij}}{\partial x^u \partial x^v}(o) =& \sum_{k=1}^{n} (a_i^k a_{juv}^k + a_j^k a_{iuv}^k + a_{iu}^k a_{jv}^k + a_{iv}^k a_{ju}^k) - 2\varepsilon r^2 a_i^1 a_j^1 a_u^2 a_v^2 \\ =& \frac{\partial^2 g_{ij}}{\partial x^u \partial x^v}(o). \end{aligned} \tag{29}$$

This represents $6+18+36=60$ algebraic equations.

From the geometrical condition of the second order contact one obtains additional equations to (27), namely

$$a_3^1 = a_3^2 = 0. \tag{30}$$

(These equations express the fact that the nullity directions of the curvature tensors coincide for h and g at the given point.)

Recall that, for the metric (19),

$$(g_{ij}) = \begin{bmatrix} f^2+(x^2)^2+(x^3)^2 & -x^3 & x^2 \\ -x^3 & 1 & 0 \\ x^2 & 0 & 1 \end{bmatrix}. \tag{31}$$

Now, rather long but routine calculations give the following (simplest) solution of the system (27), (28), (29) and (30). (Here we mark by the *index o* the value of any function at the point o.) First we have

$$(a_i^j) = \begin{bmatrix} f_o & -x_o^3 & x_o^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{32}$$

Further, the only nonzero coefficients a_{ij}^k (satisfying symmetry in lower indices) are given by the formulas

$$\begin{cases} a_{11}^1 = f'_{1o} - x_o^3 f'_{2o}, \quad a_{12}^1 = f'_{2o}, \quad a_{11}^2 = -f_o f'_{2o} - x_o^2, \\ a_{13}^2 = -1, \qquad a_{11}^3 = -x_o^3, \qquad a_{12}^3 = 1. \end{cases} \tag{33}$$

Finally, the only nonzero coefficients a_{ijl}^k (satisfying symmetry in lower indices) are given by

$$\begin{cases} a_{111}^1 = f''_{11o} - x_o^3 f''_{12o} - 2(x_o^3)^2 f''_{22o} - f_o(f'_{2o})^2 - x_o^2 f'_{2o}, \\ a_{112}^1 = f''_{12o} + x_o^3 f''_{22o}, \qquad a_{113}^1 = -f'_{2o}, \\ a_{111}^2 = x_o^3(1+(f'_{2o})^2 - f_o f''_{22o}) - f_o f''_{12o} - 2f'_{1o}f'_{2o}, \\ a_{112}^2 = -(1+(f'_{2o})^2), \quad a_{111}^3 = -(x_o^2 + f_o f'_{2o}), \quad a_{113}^3 = -1. \end{cases} \tag{34}$$

Moreover, we get an extra condition

$$\varepsilon r^2 = -(f_o)^{-1} f''_{22o}, \tag{35}$$

which is the same as (20). (Let us mention that, once the coefficients a_i^j are fixed, as in (32), then the coefficients of higher order are uniquely determined.)

This gives, step by step, the explicit form of each metric $\overline{h}$ from (23) and (24).

In particular, we can represent in this way the Sekigawa example with the complete metric on R^3, which has (so far) no analog in the "extrinsic theory" by Lumiste. As we know, this is a curvature homogeneous space.

We shall conclude this paper with the following question: *Can every curvature homogeneous metric on a manifold M be represented as a second order envelope of locally homogeneous metrics?*

The answer is negative and the present authors solved this question completely in dimension 3. To see the point better, we shall give some basic information about curvature homogeneous spaces of dimension 3. (See [3] for more details.) First of all, a 3-dimensional Riemannian manifold is curvature homogeneous (or, what is the same, contact homogeneous of order 2) if and only if its principal Ricci curvatures $\varrho_1,\ \varrho_2,\ \varrho_3$ are constant. Two such Riemannian manifolds have the same curvature (in order 0) if and only if they have the same principal Ricci curvatures, up to a numeration. The first author [13] proved that all spaces with any prescribed constant $\varrho_1 = \varrho_2 \neq \varrho_3$ are locally parametrized by two arbitrary functions of one variable. The paper [20] solves definitely the case of any prescribed distinct constants $\varrho_1,\ \varrho_2,\ \varrho_3$: the corresponding spaces are locally parametrized by three arbitrary functions of two variables. Hence we see that there are a lot of curvature homogeneous spaces which are not locally homogeneous (the latter ones depend locally, in any given dimension, only on a finite number of parameters!).

Now, rewriting the main result from [15] via Proposition 1, we obtain the following

Theorem 3. *Let (M, g) be a curvature homogeneous space of dimension 3. Then the metric g is a second order envelope of a family of locally homogeneous metrics $\{h^p \,|\, p \in M\}$ if and only if the principal Ricci curvatures $\varrho_1 \geq \varrho_2 \geq \varrho_3$ of (M, g) satisfy the following conditions:*

(a) *The Ricci form does not have the signature $(+,+,-)$ or $(+,0,-)$,*
(b) *for the Ricci signature $(+,+,0)$ one has $\varrho_1 = \varrho_2$,*
(c) *for the Ricci signature $(-,-,-)$ or $(0,-,-)$ one has either $\varrho_1 = \varrho_2 = \varrho_3$ or* $\dfrac{(\varrho_1)^2 + (\varrho_3)^2}{\varrho_1 + \varrho_3} \leq \varrho_2 < \dfrac{\varrho_1 + \varrho_3}{2}$.

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