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An endpoint estimate for the Kunze-Stein phenomenon and related maximal operators

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Abstract

One of the purposes of this paper is to prove that if G is a noncompact connected semisimple Lie group of real rank one with finite center, then

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G).$$

Let K be a maximal compact subgroup of G and $X = G/K$ a symmetric space of real rank one. We will also prove that the noncentered maximal operator

$$\mathcal{M}_2 f(z) = \sup_{z\in B} \frac{1}{|B|}\int_B |f(z')|\, dz'$$

is bounded from $L^{2,1}(X)$ to $L^{2,\infty}(X)$ and from $L^p(X)$ to $L^p(X)$ in the sharp range of exponents $p \in (2,\infty]$. The supremum in the definition of $\mathcal{M}_2 f(z)$ is taken over all balls containing the point z.

1. Introduction

A central result in the theory of convolution operators on semisimple Lie groups is the Kunze-Stein phenomenon which, in its classical form, states that if G is a connected semisimple Lie group with finite center and $p \in [1,2)$, then

$$L^2(G) * L^p(G) \subseteq L^2(G). \tag{1.1}$$

The usual convention, which will be used throughout this paper, is that if $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ are Banach spaces of functions on G then the notation $\mathcal{U} * \mathcal{V} \subseteq \mathcal{W}$ indicates both the set inclusion and the associated norm inequality. The inclusion (1.1) was established by Kunze and Stein [10] in the case when the group G is $\mathrm{SL}(2,\mathbb{R})$ (and, later on, for a number of other particular groups) and by Cowling [3] in the general case stated above. For a more complete account of the development of ideas leading to (1.1) we refer the reader to [3] and [4].

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More recently, Cowling, Meda and Setti noticed that if the group G has real rank one then the inclusion (1.1) can be strengthened. Following earlier work of Lohoué and Rychener [9], the key ingredient in their approach is the use of Lorentz spaces $L^{p,q}(G)$; they prove in [4] that if G is a connected semisimple Lie group of real rank one with finite center, $p \in (1,2)$ and $(u,v,w) \in [1,\infty]^3$ has the property that $1 + 1/w \leq 1/u + 1/v$, then

$$L^{p,u}(G) * L^{p,v}(G) \subseteq L^{p,w}(G). \tag{1.2}$$

In particular, $L^{p,1}$ convolves L^p into L^p for any $p \in [1,2)$. Our first theorem is an endpoint estimate for (1.2) showing what happens when $p = 2$.

Theorem A. *If G is a noncompact connected semisimple Lie group of real rank one with finite center then*

$$L^{2,1}(G) * L^{2,1}(G) \subseteq L^{2,\infty}(G). \tag{1.3}$$

Notice that (1.2) follows from Theorem A and a bilinear interpolation theorem ([4, Theorem 1.2]). Unlike the classical proofs of the Kunze-Stein phenomenon, our proof of Theorem A will be based on real-variable techniques only: the inclusion (1.3) is equivalent to an inequality involving a triple integral on G and we use certain nonincreasing rearrangements to control this triple integral. Easy examples, involving only K-bi-invariant functions, show that the inclusion (1.3) is sharp in the sense that neither of the $L^{2,1}$ spaces nor the $L^{2,\infty}$ space can be replaced with some $L^{2,u}$ space for any $u \in (1,\infty)$.

Let K be a maximal compact subgroup of the group G and $X = G/K$ the associated symmetric space. Assume from now on that the group G satisfies the hypothesis stated in Theorem A and let d be the distance function on $X \times X$ induced by the Killing form on the Lie algebra of the group G. Let $B(x,r)$ denote the ball in X centered at the point x of radius r (with respect to the distance function d) and let $|A|$ denote the measure of the set $A \subset X$. For any locally integrable function f on X, let

$$\mathcal{M}_2 f(z) = \sup_{z \in B} \frac{1}{|B|} \int_B |f(z')|\, dz', \tag{1.4}$$

where the supremum in the definition of $\mathcal{M}_2 f(z)$ is taken over all balls B containing z. We will prove the following:

Theorem B. *The operator $\mathcal{M}_2$ is bounded from $L^{2,1}(X)$ to $L^{2,\infty}(X)$ and from $L^p(X)$ to $L^p(X)$ in the sharp range of exponents $p \in (2,\infty]$.*

We recall that the more standard centered maximal operator

$$\mathcal{M}_1 f(z) = \sup_{r>0} \frac{1}{|B(z,r)|} \int_{B(z,r)} |f(z')|\, dz'$$

is bounded from $L^1(X)$ to $L^{1,\infty}(X)$ and from $L^p(X)$ to $L^p(X)$ for any $p > 1$, as shown in [5] and [12] (without the assumption that G has real rank one). Notice however that, unlike in the case of Euclidean spaces, balls on symmetric spaces do not have the basic doubling property (i.e. $|B(z, 2r)|$ is not proportional to $|B(z, r)|$ if r is large), thus the maximal operators $\mathcal{M}_1$ and $\mathcal{M}_2$ are not comparable. Easy examples (see [7, Section 4]) show that Theorem B is sharp in the sense that the maximal operator $\mathcal{M}_2$ is not bounded from $L^{2,u}(X)$ to $L^{2,v}(X)$ unless $u = 1$ and $v = \infty$.

This paper is organized as follows: in the next section we recall most of the notation related to semisimple Lie groups and symmetric spaces and prove a proposition that explains the role of the Lorentz space $L^{2,1}(G//K)$ – the subspace of K-bi-invariant functions in $L^{2,1}(G)$. In Section 3 we prove Theorem B. As a consequence of Theorem B we obtain in Section 4 a covering lemma on noncompact symmetric spaces of real rank one. In Section 5 we give a complete proof of Theorem A, which is divided into four steps. The main estimate in the proof of Theorem A uses the technique of nonincreasing rearrangements; we return to this technique in the last section and prove a general rearrangement inequality.

We conclude this section with some remarks on semisimple Lie groups of higher real rank. If the group G has real rank different from 1, then (1.2) fails (the estimate in Lemma 6 and the discussion following Proposition 7 in [1] show that the appropriate spherical function Φ_p fails to belong to $L^{p',\infty}(G)$, where p' is the conjugate exponent of p); therefore Theorem A fails to hold. On the other hand, the author has recently proved by a different method in [7] that the L^p estimate in Theorem B holds on symmetric spaces of arbitrary real rank. In the general case it is not known however whether the maximal operator $\mathcal{M}_2$ is bounded from $L^{2,1}(X)$ to $L^{2,\infty}(X)$.

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2. Preliminaries

Let G be a noncompact connected semisimple Lie group with finite center, and let $\mathfrak{g}$ be its Lie algebra. Most of our notation related to semisimple Lie groups and symmetric spaces is standard and can be found for example in [6]. Fix a Cartan involution θ of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the associated Cartan decomposition. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$; we will assume from now on that the group G has real rank one, i.e., $\dim \mathfrak{a} = 1$. Let $\mathfrak{a}^*$ denote the real dual of $\mathfrak{a}$, let $\Sigma \subset \mathfrak{a}^*$ be the set of nonzero roots of the pair $(\mathfrak{g}, \mathfrak{a})$ and let W be the Weyl group associated to Σ. It is well-known that $W = \{1, -1\}$ and Σ is either of the form $\{-\alpha, \alpha\}$ or of the form $\{-2\alpha, -\alpha, \alpha, 2\alpha\}$. Let $m_1 = \dim \mathfrak{g}_{-\alpha}$, $m_2 = \dim \mathfrak{g}_{-2\alpha}$, $\rho = \frac{1}{2}(m_1 + 2m_2)\alpha$ and $\mathfrak{a}_+ = \{H \in \mathfrak{a} : \alpha(H) > 0\}$. Finally let $\overline{\mathfrak{n}} = \mathfrak{g}_{-\alpha} + \mathfrak{g}_{-2\alpha}$, $\overline{N} = \exp \overline{\mathfrak{n}}$, $K = \exp \mathfrak{k}$, $A = \exp \mathfrak{a}$ and $A_+ = \exp \mathfrak{a}_+$ and let $X = G/K$ be a symmetric space of real rank one.

The group G has an Iwasawa decomposition $G = \overline{N}AK$ and a Cartan decomposition $G = K\overline{A_+}K$. Our proofs are based on relating these two decompositions, and for real rank one groups one has the explicit formula in [6, Ch.2, Theorem 6.1]. A similar idea was used by Strömberg [12] for groups of arbitrary real rank. Let $H_0 \in \mathfrak{a}$ be the unique element of $\mathfrak{a}$ for which $\alpha(H_0) = 1$ and let $a(s) = \exp(sH_0)$ for $s \in \mathbb{R}$ be a parametrization of the subgroup A. By [6, Ch.2, Theorem 6.1] one can identify the group $\overline{N}$ with $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ using a diffeomorphism $\overline{n} : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \overline{N}$. This diffeomorphism has the property that if $t \geq 0$ then $\overline{n}(v, w)a(s) \in Ka(t)K$ if and only if

$$(\cosh t)^2 = \Big[\cosh s + e^s |v|^2\Big]^2 + e^{2s}|w|^2. \tag{2.1}$$

In addition,

$$a(s)\overline{n}(v, w)a(-s) = \overline{n}(e^{-s}v, e^{-2s}w). \tag{2.2}$$

Let $|\rho| = \rho(H_0) = \frac{1}{2}(m_1 + 2m_2)$ and let dg, $d\overline{n}$ and dk denote Haar measures on G, $\overline{N}$ and K, the last one normalized such that $\int_K 1\, dk = 1$. Then the following integral formulae hold for any continuous function f with compact support:

$$\int_G f(g)\, dg \;=\; C_1 \int_K \int_{\mathbb{R}_+} \int_K f(k_1 a(t) k_2)(\sinh t)^{m_1} (\sinh 2t)^{m_2}\, dk_2\, dt\, dk_1, \tag{2.3}$$

and

$$\begin{aligned} \int_G f(g)\, dg \;&=\; C_2 \int_K \int_{\mathbb{R}} \int_{\overline{N}} f(\overline{n}a(s)k) e^{2|\rho|s}\, d\overline{n}\, ds\, dk \\ &=\; C_2' \int_K \int_{\mathbb{R}} \int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} f(\overline{n}(v, w)a(s)k) e^{2|\rho|s}\, dv\, dw\, ds\, dk. \end{aligned} \tag{2.4}$$

The measures dv and dw are the usual Lebesgue measures on $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, and the constants C_1, C_2 and C_2' depend on the normalizations of the various Haar measures. We will need a new integration formula, which is the subject of the following lemma.

LEMMA 1. *Suppose that* $f : G \to \mathbb{C}$ *is a* K*-bi-invariant (i.e.,* $f(k_1 g k_2) = f(g)$ *for any* $k_1, k_2 \in K$*) continuous function with compact support and* $F(t) = f(a(t))$ *for any* $t \in [0, \infty)$*. Then for any* $s \in \mathbb{R}$

$$e^{|\rho|s} \int_{\overline{N}} f(\overline{n}a(s))\, d\overline{n} = \int_{|s|}^{\infty} F(t)\psi(t, s)\, dt,$$

where the kernel $\psi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ *has the property that* $\psi(t,s) = 0$ *if* $t < |s|$ *and*

$$\psi(t,s) \approx \sinh t (\cosh t)^{m_2/2} (\cosh t - \cosh s)^{(m_1+m_2-2)/2} \tag{2.5}$$

if $t \geq |s|$.

As usual, the notation $U \approx V$ means that there is a constant $C \geq 1$ depending only on the group G such that $C^{-1}U \leq V \leq CU$. This lemma is essentially proved in [8, Section 5]. For later reference we reproduce its proof.

Proof of Lemma 1. For any $t \geq |s|$, let

$$T_{t,s} = \{(v,w) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} : (\cosh t)^2 = \left[\cosh s + e^s |v|^2\right]^2 + e^{2s}|w|^2\} \tag{2.6}$$

be the set of points $P = P(v,w) \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ with the property that $\overline{n}(P)a(s) \in Ka(t)K$ (these surfaces will play a key role in the proof of Theorem A). Let $d\omega_{t,s}$ be the induced measure on $T_{t,s}$ such that

$$\int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} \phi(v,w)\, dv\, dw = \int_{t \geq |s|} \left[\int_{T_{t,s}} \phi(P) d\omega_{t,s}(P) \right] dt$$

for any continuous compactly supported function ϕ. Then, since the function f is K-bi-invariant,

$$\begin{aligned} e^{|\rho| s} \int_{\overline{N}} f(\overline{n}a(s))\, d\overline{n} &= C e^{|\rho| s} \int_{\mathbb{R}^{m_1} \times \mathbb{R}^{m_2}} f(\overline{n}(v,w)a(s))\, dv\, dw \\ &= C e^{|\rho| s} \int_{t \geq |s|} F(t) \left[\int_{T_{t,s}} 1\, d\omega_{t,s} \right] dt. \end{aligned}$$

Let $\psi(t,s) = e^{|\rho| s} \int_{T_{t,s}} 1\, d\omega_{t,s}$ and assume that $m_2 \geq 1$. We make the change of variables $v = \left[e^{-s}(u \cosh t - \cosh s)\right]^{1/2} \omega_1$ and $w = e^{-s} \cosh t (1-u^2)^{1/2} \omega_2$, where $\omega_1 \in S^{m_1-1}$ (the $m_1 - 1$ dimensional sphere in $\mathbb{R}^{m_1}$), $\omega_2 \in S^{m_2-1}$ and $u \in [\frac{\cosh s}{\cosh t}, 1]$. We have

$$\psi(t,s) = C \sinh t (\cosh t)^{m_2} \int_{\frac{\cosh s}{\cosh t}}^{1} (u \cosh t - \cosh s)^{(m_1-2)/2} (1-u^2)^{(m_2-2)/2}\, du,$$

which easily proves (2.5). The computation of the function ψ is slightly easier if $m_2 = 0$ and the result is also given by (2.5). □

Our next proposition explains the role of the Lorentz space $L^{2,1}(G//K)$ which, by definition, is the subspace of K-bi-invariant functions in $L^{2,1}(G)$:

PROPOSITION 2. *The Abel transform*

$$\mathcal{A}f(a) = e^{\rho(\log a)} \int_{\overline{N}} f(\overline{n}a)\, d\overline{n}$$

is bounded from $L^{2,1}(G//K)$ *to* $L^\infty(A/W)$*. In other words, if* f *is a locally integrable* K*-bi-invariant function on* G *and* $a \in A$ *then*:

$$e^{\rho(\log a)} \int_{\overline{N}} f(\overline{n}a)\, d\overline{n} \le C||f||_{L^{2,1}(G)}. \tag{2.7}$$

Proof of Proposition 2. The usual theory of Lorentz spaces (see, for example, [11, Chapter V]) shows that it suffices to prove the inequality (2.7) under the additional assumption that f is the characteristic function of an open K-bi-invariant set of finite measure. For any $t \ge 0$, let $F(t) = f(a(t))$, so

$$||f||_{L^{2,1}(G)} = C\left[\int_{\mathbb{R}_+} F(t)(\sinh t)^{m_1}(\sinh 2t)^{m_2}\, dt\right]^{1/2}. \tag{2.8}$$

In view of Lemma 1 and (2.8), it suffices to prove that for any $s \in \mathbb{R}$

$$\int_{t\ge|s|} F(t)\psi(t,s)\, dt \le C\left[\int_{\mathbb{R}_+} F(t)(\sinh t)^{m_1}(\sinh 2t)^{m_2}\, dt\right]^{1/2} \tag{2.9}$$

for any measurable function $F : \mathbb{R}_+ \to \{0,1\}$. Notice that if $t \ge 1 + |s|$ then $\psi(t,s) \approx e^{|\rho|t}$, $(\sinh t)^{m_1}(\sinh 2t)^{m_2} \approx e^{2|\rho|t}$ and it follows from Lemma 3 below that

$$\int_{t\ge|s|+1} F(t)\psi(t,s)\, dt \le C\left[\int_{t\ge|s|+1} F(t)(\sinh t)^{m_1}(\sinh 2t)^{m_2}\, dt\right]^{1/2}. \tag{2.10}$$

In order to deal with the integral in t over the interval $[|s|, |s|+1]$ we consider two cases: $|s| \ge 1$ and $|s| \le 1$. If $|s| \ge 1$ and $t \in [|s|, |s|+1]$, then $\psi(t,s) \approx e^{|\rho||s|}(t-|s|)^{(m_1+m_2-2)/2}$, $(\sinh t)^{m_1}(\sinh 2t)^{m_2} \approx e^{2|\rho||s|}$ and, since $(m_1+m_2-2)/2 \ge -1/2$, it follows that

$$\begin{aligned}\int_{|s|}^{|s|+1} F(t)\psi(t,s)\, dt &\le Ce^{|\rho||s|}\int_{|s|}^{|s|+1} F(t)(t-|s|)^{-1/2}\, dt\\ &= Ce^{|\rho||s|}\int_0^1 F(|s|+u^2)\, du \le C\left[e^{2|\rho||s|}\int_0^1 F(|s|+u^2)u\, du\right]^{1/2}\\ &\le C\left[\int_{|s|}^{|s|+1} F(t)(\sinh t)^{m_1}(\sinh 2t)^{m_2}\, dt\right]^{1/2}.\end{aligned}$$

One of the inequalities in the sequence above follows from the estimate (2.11) below. This, together with (2.10), completes the proof of the proposition in the case $|s| \ge 1$. The estimation of the integrals over the interval $[|s|, |s|+1]$ is similar in the case $|s| \le 1$. □

LEMMA 3. *If* $\delta \ne 0$ *and* $d\mu_1(t) = e^{\delta t}dt$, $d\mu_2(t) = e^{2\delta t}dt$ *are two measures on* $\mathbb{R}$ *then*

$$||f||_{L^1(\mathbb{R},d\mu_1)} \le C_\delta ||f||_{L^{2,1}(\mathbb{R},d\mu_2)}.$$

Proof of Lemma 3. One can assume that f is the characteristic function of a set. The change of variable $t = (\log s)/\delta$ and the substitution $g(s) = f((\log s)/\delta)$ show that it suffices to prove that

$$\frac{1}{|\delta|}\int_{\mathbb{R}_+} g(s)\,ds \le C_\delta\left[\frac{1}{|\delta|}\int_{\mathbb{R}_+} g(s)s\,ds\right]^{1/2} \tag{2.11}$$

for any measurable function $g : \mathbb{R}_+ \to \{0,1\}$, which follows by a rearrangement argument. □

3. Proof of the maximal theorem

For any locally integrable function $f : X \to \mathbb{C}$ let

$$\widetilde{\mathcal{M}_2} f(z) = \sup_{r\ge 1} \frac{1}{|B(z,r)|^{1/2}} \int_{B(z,r)} |f(z')|\,dz'. \tag{3.1}$$

Most of this section will be devoted to the proof of the following theorem:

Theorem 4. *The operator* $\widetilde{\mathcal{M}_2}$ *is bounded from* $L^{2,1}(X)$ *to* $L^{2,\infty}(X)$.

Notice that Theorem B is an easy consequence of Theorem 4: let

$$\begin{aligned} \mathcal{M}_2^0 f(z) &= \sup_{z\in B, r(B)\le 1} \frac{1}{|B|}\int_B f(z')dz', \\ \mathcal{M}_2^1 f(z) &= \sup_{z\in B, r(B)\ge 1} \frac{1}{|B|}\int_B f(z')dz', \end{aligned}$$

where $r(B)$ is the radius of the ball B. We can assume that the Killing form on the Lie algebra $\mathfrak{g}$ is normalized such that $|H_0| = 1$. Let $o = \{K\}$ be the origin of the symmetric space X. Then the ball $B(o,r)$ is equal to the set of points $\{ka(t)\cdot o : k \in K, t \in [0,r)\}$ and one clearly has $|B(o,r)| \approx r^{m_1+m_2+1}$ if $r \le 1$ and $|B(o,r)| \approx e^{2|\rho|r}$ if $r \ge 1$. The operator $\mathcal{M}_2^0$, the local part of $\mathcal{M}_2$, is clearly bounded on $L^p(X)$ for any $p > 1$. On the other hand, if z belongs to a ball B of radius $r \ge 1$, then $B(z,2r)$ contains the ball B and $|B(z,2r)| \approx e^{2|\rho|\cdot 2r} \approx |B|^2$. Therefore

$$\frac{1}{|B|}\int_B f(z')\,dz' \le \frac{C}{|B(z,2r)|^{1/2}}\int_{B(z,2r)} f(z')\,dz'$$

which shows that $\mathcal{M}_2^1 f(z) \le C\widetilde{\mathcal{M}_2} f(z)$, and the conclusion of Theorem B follows by interpolation with the trivial L^∞ estimate.

Proof of Theorem 4. Let χ_r be the characteristic function of the K-bi-invariant set $\{g \in G : d(g\cdot o, o) < r\}$. Since the measure of a ball of radius r in X is proportional to $e^{2|\rho|r}$ if $r \geq 1$, one has

$$\widetilde{\mathcal{M}_2}f(g\cdot o) \approx \sup_{r\geq 1}\left[e^{-|\rho|r}\int_G f(g'\cdot o)\chi_r({g'}^{-1}g)\,dg'\right].$$

The change of variables $g = \overline{n}a(t)k$, $g' = \overline{n}'a(t')k'$ and the integral formula (2.4) show that

$$\begin{aligned}(3.2)\quad &\widetilde{\mathcal{M}_2}f(\overline{n}a(t)\cdot o)\\ &\leq C\sup_{r\geq 1}\left[e^{-|\rho|r}\int_{\mathbb{R}}\left(\int_{\overline{N}} f(\overline{n}'a(t')\cdot o)\chi_r(a(-t'){\overline{n}'}^{-1}\overline{n}a(t))\,d\overline{n}'\right)e^{2|\rho|t'}\,dt'\right].\end{aligned}$$

We first deal with the integral over the group $\overline{N}$ and dominate the right-hand side of (3.2) using a standard maximal operator on the nilpotent group $\overline{N}$. For any $u > 0$ let $\mathcal{B}_u$ be the ball in $\overline{N}$ defined as the set $\{\overline{n}(v, w) : |v| \leq u$ and $|w| \leq u^2\}$. Clearly, $\int_{\mathcal{B}_u} 1 d\overline{n} = Cu^{2|\rho|}$. The group $\overline{N}$ is equipped with non-isotropic dilations $\delta_u(\overline{n}(v, w)) = \overline{n}(uv, u^2w)$, which are group automorphisms, therefore the maximal operator

$$\mathcal{N}g(\overline{n}) = \sup_{u>0}\left[\frac{1}{u^{2|\rho|}}\int_{\mathcal{B}_u}|g(\overline{n}\,\overline{m}^{-1})|\,d\overline{m}\right]$$

is bounded from $L^p(\overline{N})$ to $L^p(\overline{N})$ for any $p > 1$ ([13, Lemma 2.2]). For any locally integrable function $f : X \to \mathbb{R}_+$ and any $\overline{n} \in \overline{N}$ and $a \in A$ let

$$\mathcal{M}_3f(\overline{n}a\cdot o) = \sup_{u>0}\left[\frac{1}{u^{2|\rho|}}\int_{\mathcal{B}_u}|f(\overline{n}\,\overline{m}^{-1}a\cdot o)|\,d\overline{m}\right].$$

Since the maximal operator $\mathcal{N}$ is bounded on $L^p(\overline{N})$ one has $||\mathcal{M}_3f||_{L^p(X)} \leq C_p||f||_{L^p(X)}$ for any $p > 1$. We will now use the function $\mathcal{M}_3f$ to control the integral over $\overline{N}$ in (3.2). Notice that (2.1) and (2.2), together with the fact that $d(ka(t)\cdot o, o) = t$ for any $t \geq 0$ and $k \in K$, show that if $\chi_r(a(-t')\overline{m}a(t)) = 1$ for some $\overline{m} \in \overline{N}$ then $\overline{m}$ has to belong to the ball $\mathcal{B}_{e^{(r-t-t')/2}}$; therefore

$$\begin{aligned}\int_{\overline{N}} f(\overline{n}'a(t')\cdot o)\chi_r(a(-t'){\overline{n}'}^{-1}\overline{n}a(t))\,d\overline{n}' &\leq \int_{\mathcal{B}_{e^{(r-t-t')/2}}} f(\overline{n}\,\overline{m}^{-1}a(t')\cdot o)\,d\overline{m}\\ &\leq Ce^{|\rho|(r-t-t')}\mathcal{M}_3f(\overline{n}a(t')\cdot o).\end{aligned}$$

If we substitute this inequality into (3.2) we conclude that

$$(3.3)\qquad \widetilde{\mathcal{M}_2}f(\overline{n}a(t)\cdot o) \leq Ce^{-|\rho|t}\int_{\mathbb{R}}\mathcal{M}_3f(\overline{n}a(t')\cdot o)e^{|\rho|t'}\,dt'.$$

We can now estimate the $L^{2,\infty}$ norm of $\widetilde{\mathcal{M}_2}f$: for any $\lambda > 0$, the set $E_\lambda = \{z \in X : \widetilde{\mathcal{M}_2}f(z) > \lambda\}$ is included in the set

$$\{\overline{n}a(t)\cdot o : e^{-|\rho|t}\int_{\mathbb{R}}\mathcal{M}_3f(\overline{n}a(t')\cdot o)e^{|\rho|t'}\,dt' > \lambda/C\}.$$

The measure dz in X is proportional to the measure $e^{2|\rho|t}\,d\overline{n}\,dt$ in $\overline{N}\times\mathbb{R}$ under the identification $z=\overline{n}a(t)\cdot o$. Therefore the measure of this last set is less than or equal to

$$\frac{C\int_{\overline{N}}\left[\int_{\mathbb{R}}\mathcal{M}_3 f(\overline{n}a(t')\cdot o)e^{|\rho|t'}\,dt'\right]^2 d\overline{n}}{\lambda^2};$$

hence

$$||\widetilde{\mathcal{M}_2} f||^2_{L^{2,\infty}}\le C\int_{\overline{N}}\left[\int_{\mathbb{R}}\mathcal{M}_3 f(\overline{n}a(t')\cdot o)e^{|\rho|t'}\,dt'\right]^2 d\overline{n}. \tag{3.4}$$

One can now use the following simple lemma to dominate the right-hand side of (3.4):

Lemma 5. *If U and V are two measure spaces with measures du and dv respectively, and $H: U\times V\to\mathbb{R}_+$ is measurable then*

$$\left[\int_U ||H(u,.)||^2_{L^{2,1}(V,dv)}\,du\right]^{1/2}\le C||H||_{L^{2,1}(U\times V,\,du\,dv)}.$$

The proof of this lemma is straightforward. Combining Lemma 3 (at the end of the previous section) and Lemma 5, one has

(3.5)

$$\begin{aligned}\int_{\overline{N}}\left[\int_{\mathbb{R}}\mathcal{M}_3 f(\overline{n}a(t')\cdot o)e^{|\rho|t'}\,dt'\right]^2 d\overline{n} &\le C\int_{\overline{N}}||\mathcal{M}_3 f(\overline{n}a(.)\cdot o)||^2_{L^{2,1}(\mathbb{R},e^{2|\rho|t'}dt')}\,d\overline{n}\\ &\le C||\mathcal{M}_3 f(\overline{n}a(t')\cdot o)||^2_{L^{2,1}(\overline{N}\times\mathbb{R},e^{2|\rho|t'}\,d\overline{n}\,dt')}\\ &\le C||\mathcal{M}_3 f||^2_{L^{2,1}(X)}.\end{aligned}$$

Finally, since the maximal operator $\mathcal{M}_3$ is bounded on $L^p(X)$ for any $p>1$, it follows by the general version of Marcinkiewicz interpolation theorem that $||\mathcal{M}_3 f||_{L^{2,1}(X)}\le C||f||_{L^{2,1}(X)}$ and Theorem 4 follows from (3.4) and (3.5). □

4. A covering lemma

A simple connection between covering lemmas and boundedness of maximal operators is explained in [2]. In our setting we have:

Corollary 6. *If a collection of balls $B_i\subset X$, $i\in I$, has the property that $|\cup B_i|<\infty$ then one can select a finite subset $J\subset I$ such that*

$$\text{(i)}\quad \left|\bigcup_{i\in I} B_i\right|\le C\left|\bigcup_{j\in J} B_j\right|; \tag{4.1}$$

$$\text{(ii)}\quad \left|\left|\sum_{j\in J}\chi_{B_j}\right|\right|_{L^{2,\infty}(X)}\le C\left|\bigcup_{i\in I} B_i\right|^{1/2}.$$

It follows from (4.1) that

$$\Big|\Big|\sum_{j\in J}\chi_{B_j}\Big|\Big|_{L^q(X)}\leq C_q\Big|\bigcup_{i\in I}B_i\Big|^{1/q}$$

for any $q\in[1,2)$. Thus, in the terminology of [2], the family of natural balls on symmetric spaces of real rank one has the covering property V_q if and only if $q\in[1,2)$.

5. Proof of the convolution theorem

In this section we will prove Theorem A. In view of the general theory of Lorentz spaces, it suffices to prove that

$$\iint_{G\times G}f(z)g(z^{-1}z')h(z')\,dz'\,dz\leq C||f||_{L^{2,1}}||g||_{L^{2,1}}||h||_{L^{2,1}} \tag{5.1}$$

whenever $f,g,h:G\to\{0,1\}$ are characteristic functions of open sets of finite measure. We can also assume that g is supported away from the origin of the group, for example in the set $\bigcup_{t>1}Ka(t)K$. The main part of our argument is devoted to proving that the left-hand side of (5.1) is controlled by an integral involving suitable rearrangements of the functions f, g and h, as in (5.19). Let $z=\overline{n}a(t)k$, $z'=\overline{n}'a(t')k'$ and the left-hand side of (5.1) becomes

$$\int_K\int_K\int_{\mathbb{R}}\int_{\mathbb{R}}I(k,k',t,t')e^{2|\rho|(t+t')}\,dt'\,dt\,dk'\,dk, \tag{5.2}$$

where

(5.3)
$$I(k,k',t,t')=\iint_{\overline{N}\times\overline{N}}f(\overline{n}a(t)k)g(k^{-1}a(-t)\overline{n}^{-1}\overline{n}'a(t')k')h(\overline{n}'a(t')k')\,d\overline{n}'\,d\overline{n}$$

We will show how to dominate the expression in (5.2) in four steps.

Step 1. *Integration on the subgroup $\overline{N}$.* As in the proof of the maximal theorems, we start by integrating on $\overline{N}$. Define $F_1,H_1:K\times\mathbb{R}\to\mathbb{R}_+$ by

$$F_1(k,t)=\int_{\overline{N}}f(\overline{n}a(t)k)\,d\overline{n}$$

and

$$H_1(k',t')=\int_{\overline{N}}h(\overline{n}'a(t')k')\,d\overline{n}'.$$

Using the simple inequality

$$\iint_{\overline{N}\times\overline{N}}a(\overline{n})b(\overline{n}^{-1}\overline{n}')c(\overline{n}')\,d\overline{n}'\,d\overline{n}$$
$$\leq\left(\int_{\overline{N}}b(\overline{n})\,d\overline{n}\right)\left[\min\left(\left(\int_{\overline{N}}a(\overline{n})\,d\overline{n}\right),\left(\int_{\overline{N}}c(\overline{n})\,d\overline{n}\right)\right)\right],$$

which holds for any measurable functions $a, b, c : \overline{N} \to [0,1]$ with compact support, it follows that the integral $I(k,k',t,t')$ in (5.3) is dominated by

$$\min\left[F_1(k,t), H_1(k',t')\right]\left[\int_{\overline{N}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')\,d\overline{n}_1\right]. \tag{5.4}$$

By (2.2), the map $\overline{n}_1 \to a(-t)\overline{n}_1 a(t) = \overline{n}_2$ is a dilation of $\overline{N}$ with $d\overline{n}_1 = e^{-2|\rho|t}\,d\overline{n}_2$; therefore

$$\begin{aligned} \int_{\overline{N}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')\,d\overline{n}_1 &= e^{-2|\rho|t}\int_{\overline{N}} g(k^{-1}\overline{n}_2 a(t'-t)k')\,d\overline{n}_2 \\ &= Ce^{-2|\rho|t}\int_{\mathbb{R}^{m_1}\times\mathbb{R}^{m_2}} g(k^{-1}\overline{n}(v,w)a(t'-t)k')\,dv\,dw \\ &= Ce^{-2|\rho|t}\int_{u\ge|t'-t|}\int_{T_{u,t'-t}} g(k^{-1}\overline{n}(P)a(t'-t)k')\,d\omega_{u,t'-t}(P)\,du. \end{aligned} \tag{5.5}$$

The surfaces $T_{u,s}$ defined in (2.6) for $\{(u,s) \in \mathbb{R}_+ \times \mathbb{R} : u \ge |s|\}$ and the associated measures $d\omega_{u,s}$ have the same meaning as in the proof of Lemma 1. Let

$$G_1(k,k',u,s) = \left(\int_{T_{u,s}} 1\,d\omega_{u,s}\right)^{-1}\left[\int_{T_{u,s}} g(k^{-1}\overline{n}(P)a(s)k')\,d\omega_{u,s}(P)\right] \tag{5.6}$$

be the average of the function $P \to g(k^{-1}\overline{n}(P)a(s)k')$ on the surface $T_{u,s}$ (the domain of definition of G_1 is $\{(k,k',u,s) \in K \times K \times \mathbb{R}_+ \times \mathbb{R} : u \ge |s|\}$, and $G_1(k,k',u,s) \in [0,1]$). If we substitute this definition in (5.5), we conclude that

$$\begin{aligned} &\int_{\overline{N}} g(k^{-1}a(-t)\overline{n}_1 a(t')k')\,d\overline{n}_1 \\ &\qquad = Ce^{-|\rho|(t+t')}\int_{u\ge|t'-t|} G_1(k,k',u,t'-t)\psi(u,t'-t)\,du. \end{aligned}$$

The function $\psi(u,s)$ was computed in the proof of Lemma 1 and is given by (2.5). Finally, if we substitute this last formula in (5.4), we find that the integral $I(k,k',t,t')$ is dominated by

$$Ce^{-|\rho|(t+t')}\min\left[F_1(k,t), H_1(k',t')\right]\int_{u\ge|t'-t|} G_1(k,k',u,t'-t)\psi(u,t'-t)\,du,$$

which shows that the left-hand side of (5.1) is dominated by

$$\begin{aligned} C\int_K\int_K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{u\ge|t'-t|} &\min\left[F_1(k,t), H_1(k',t')\right] \\ &G_1(k,k',u,t'-t)\psi(u,t'-t)e^{|\rho|(t+t')}\,du\,dt'\,dt\,dk'\,dk. \end{aligned} \tag{5.7}$$

For later use, we record the following properties of the functions F_1 and H_1:

$$\begin{aligned} ||f||_{L^{2,1}(G)} &= \left[C_2\int_K\int_{\mathbb{R}} F_1(k,t)e^{2|\rho|t}\,dt\,dk\right]^{1/2}, \\ ||h||_{L^{2,1}(G)} &= \left[C_2\int_K\int_{\mathbb{R}} H_1(k',t')e^{2|\rho|t'}\,dt'\,dk'\right]^{1/2}. \end{aligned} \tag{5.8}$$

Step 2. *Integration on the subgroup A.* Let χ_1 and χ_2, be the characteristic functions of the sets $\{(k,k',t,t') : F_1(k,t) \le H_1(k',t')\}$ and $\{(k,k',t,t') : H_1(k',t') \le F_1(k,t)\}$ respectively. For any k,k',t,t' one has

$$\begin{cases} F_1(k,t)\chi_1(k,k',t,t') \le H_1(k',t'), \\ H_1(k',t')\chi_2(k,k',t,t') \le F_1(k,t). \end{cases} \tag{5.9}$$

Since $\chi_1+\chi_2 \ge 1$, the expression (5.7) is less than or equal to the sum of two similar expressions of the form

$$C\int_K\int_K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{u\ge|t'-t|} F_1(k,t)\chi_1(k,k',t,t') \\ G_1(k,k',u,t'-t)\psi(u,t'-t)e^{|\rho|(t+t')}\,du\,dt'\,dt\,dk'\,dk.$$

The change of variable $t'=t+s$ in the expression above shows that it is equal to

$$C\int_K\int_K\int_{\mathbb{R}}\int_{\mathbb{R}}\int_{u\ge|s|} F_1(k,t)\chi_1(k,k',t,t+s) \\ G_1(k,k',u,s)\psi(u,s)e^{2|\rho|t}e^{|\rho|s}\,du\,dt\,ds\,dk'\,dk, \tag{5.10}$$

and the first of the inequalities in (5.9) becomes

$$F_1(k,t)\chi_1(k,k',t,t+s)e^{2|\rho|t} \le H_1(k',t+s)e^{2|\rho|t}. \tag{5.11}$$

Let $F(k)=\left[\int_{\mathbb{R}} F_1(k,t)e^{2|\rho|t}\,dt\right]^{1/2}$, $H(k')=\left[\int_{\mathbb{R}} H_1(k',t')e^{2|\rho|t'}\,dt'\right]^{1/2}$ and

$$A(k,k',s)=\int_{\mathbb{R}} F_1(k,t)\chi_1(k,k',t,t+s)e^{2|\rho|t}\,dt.$$

The expression (5.10) becomes

$$C\int_K\int_K\int_{\mathbb{R}}\int_{u\ge|s|} A(k,k',s)G_1(k,k',u,s)\psi(u,s)e^{|\rho|s}\,du\,ds\,dk'\,dk. \tag{5.12}$$

Clearly, $A(k,k',s) \le F(k)^2$ (since $\chi_1 \le 1$) and $A(k,k',s) \le e^{-2|\rho|s}H(k')^2$ by (5.11); therefore

$$e^{|\rho|s}A(k,k',s) \le \begin{cases} e^{|\rho|s}F(k)^2 & \text{if } e^{|\rho|s} \le H(k')/F(k), \\ e^{-|\rho|s}H(k')^2 & \text{if } e^{|\rho|s} \ge H(k')/F(k). \end{cases}$$

If we substitute this inequality in (5.12) we find that the left-hand side of (5.1) is dominated by

$$C\int_K\int_K\int_{e^{|\rho|s}\le H(k')/F(k)}\int_{u\ge|s|} F(k)^2G_1(k,k',u,s)\psi(u,s)e^{|\rho|s}\,du\,ds\,dk'\,dk \\ + C\int_K\int_K\int_{e^{|\rho|s}\ge H(k')/F(k)}\int_{u\ge|s|} H(k')^2G_1(k,k',u,s)\psi(u,s)e^{-|\rho|s}\,du\,ds\,dk'\,dk. \tag{5.13}$$

We pause for a moment to note that our estimates so far, together with the proof of Lemma 1 in the second section, suffice to prove that $L^{2,1}(G) * L^{2,1}(G//K) \subseteq L^{2,\infty}(G)$: if g is a K-bi-invariant function, then $G_1(k,k',u,s)$ depends only on u, and (2.9) shows that

$$\int_{u\geq|s|} G_1(k,k',u,s)\psi(u,s)\,du \leq C||g||_{L^{2,1}}.$$

As a consequence, both terms in (5.13) are dominated by

$$C||g||_{L^{2,1}}\int_K\int_K F(k)H(k')\,dk'\,dk;$$

therefore

$$\begin{aligned}\iint_{G\times G} f(z)g(z^{-1}z')h(z')\,dz'\,dz &\leq C||g||_{L^{2,1}}\int_K\int_K F(k)H(k')\,dk'\,dk\\ &\leq C||f||_{L^{2,1}}||g||_{L^{2,1}}||h||_{L^{2,1}}.\end{aligned}$$

Here we used the fact that, as a consequence of (5.8),

$$\begin{aligned}||f||_{L^{2,1}(G)} &= \left[C_2\int_K F(k)^2\,dk\right]^{1/2},\\ ||h||_{L^{2,1}(G)} &= \left[C_2\int_K H(k')^2\,dk'\right]^{1/2}.\end{aligned} \tag{5.14}$$

Step 3. *A rearrangement inequality.* In the general case (if g is not assumed to be K-bi-invariant) we will show that both terms in (5.13) are dominated by some expression of the form

$$C\int_0^1\int_0^1\int_{\mathbb{R}_+} F^*(x)H^*(y)G^{**}(x,y,u)e^{|\rho|u}\,du\,dy\,dx$$

where $F^*, H^* : (0,1] \to \mathbb{R}_+$ are the usual nonincreasing rearrangements of the functions F and H (recall that the measure of K is equal to 1) and $G^{**} : (0,1]\times(0,1]\times\mathbb{R}_+ \to \{0,1\}$ is a suitable "double" rearrangement of g. The precise definitions are the following: if $a : K \to \mathbb{R}_+$ is a measurable function then the nonincreasing rearrangement $a^* : (0,1] \to \mathbb{R}_+$ is the right semicontinuous nonincreasing function with the property that

$$|\{k \in K : a(k) > \lambda\}| = |\{x \in (0,1] : a^*(x) > \lambda\}| \text{ for any } \lambda \in [0,\infty).$$

Assume now that $a : K\times K \to \mathbb{R}_+$ is a measurable function. For almost every $k \in K$ let $a^*(k,y)$, $y \in (0,1]$, be the nonincreasing rearrangement of the function $k' \to a(k,k')$ and let $a^{**}(x,y)$ be the nonincreasing rearrangement of the function $k \to a(k,y)$ (clearly $a^{**} : (0,1]\times(0,1] \to \mathbb{R}_+$). The following lemma summarizes some of the well-known properties of nonincreasing rearrangements (see for example [11, Chapter V]):

LEMMA 7. (a) *If $a : K \to \mathbb{R}_+$ is a measurable function then*

$$\left[\int_K a(k)^2\,dk\right]^{1/2} = \left[\int_{(0,1]} a^*(x)^2\,dx\right]^{1/2}.$$

(b) *If $a : K \times K \to \mathbb{R}_+$ is a measurable function then*

(i) $$\int_K \int_K a(k,k')\,dk'\,dk = \int_0^1 \int_0^1 a^{**}(x,y)\,dy\,dx.$$

(ii) *The function a^{**} is nonincreasing: $a^{**}(x,y) \le a^{**}(x',y')$ whenever $x \ge x'$ and $y \ge y'$.*

(iii) *For any measurable sets $D, E \subset K$ with measures $|D|$ and $|E|$*

$$\int_D \int_E a(k,k')\,dk'\,dk \le \int_0^{|D|} \int_0^{|E|} a^{**}(x,y)\,dy\,dx.$$

Returning to our setting, let F^* and H^* be the nonincreasing rearrangements of F and H, let $\tilde{g} : K \times K \times \mathbb{R}_+ \to \{0,1\}$ be given by $\tilde{g}(k,k',u) = g(k^{-1}a(u)k')$ and let $G^{**} : (0,1] \times (0,1] \times \mathbb{R}_+ \to \{0,1\}$ be the double rearrangement of the function $\tilde{g}$ (i.e., $G^{**}(.,.,u)$ is the double rearrangement of $\tilde{g}(.,.,u)$ for all $u \ge 0$). Recall that we assumed that the function g is the characteristic function of a set included in $\bigcup_{u>1} Ka(u)K$; therefore

$$||g||_{L^{2,1}(G)} \approx \left[\int_{\mathbb{R}_+} \int_0^1 \int_0^1 G^{**}(x,y,u)e^{2|\rho|u}\,dy\,dx\,du\right]^{1/2}. \tag{5.15}$$

We will now show how to use these rearrangements to dominate the two expressions in (5.13). For any integers m, n let $D_m = \{k \in K : F(k) \in [e^{|\rho|m}, e^{|\rho|(m+1)}]\}$, $E_n = \{k' \in K : H(k') \in [e^{|\rho|n}, e^{|\rho|(n+1)}]\}$ and let $D_{-\infty} = \{k \in K : F(k) = 0\}$, $E_{-\infty} = \{k' \in K : H(k') = 0\}$ such that $K = \bigcup_m D_m = \bigcup_n E_n$. Let δ_m, respectively ε_n, be the measures of the sets D_m, respectively E_n, as subsets of K. The first of the two expressions in (5.13) is dominated by

$$C\sum_{m,n} \int_{D_m} \int_{E_n} \int_{s\le(n-m+1)} \int_{u\ge|s|} e^{2|\rho|(m+1)} G_1(k,k',u,s)\psi(u,s)e^{|\rho|s}\,du\,ds\,dk'\,dk. \tag{5.16}$$

Combining the definition (5.6) of the function G_1 (recall that the surfaces $T_{u,s}$ are defined as the set of points $P \in \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ with the property that $\overline{n}(P)a(s) \in Ka(u)K$), the fact that dk is a Haar measure on K and the last statement of Lemma 7, we conclude that

$$\int_{D_m} \int_{E_n} G_1(k,k',u,s)\,dk'\,dk \le \int_0^{\delta_m} \int_0^{\varepsilon_n} G^{**}(x,y,u)\,dy\,dx$$

for any s with the property that $|s| \leq u$. Substituting this inequality in (5.16), we find that the expression in (5.16) is dominated by

$$C \sum_{m,n} \int_{\mathbb{R}_+} e^{2|\rho|m} \left[\int_0^{\delta_m} \int_0^{\varepsilon_n} G^{**}(x,y,u)\, dy\, dx \right] \left[\int_{s \leq (n-m+1), |s| \leq u} \psi(u,s) e^{|\rho|s}\, ds \right] du. \tag{5.17}$$

The formula (2.5) shows that the last of the integrals in the expression above is dominated by $Ce^{|\rho|u}e^{|\rho|(n-m)}$; therefore the first of the two expressions in (5.13) is dominated by

$$C \int_{\mathbb{R}_+} \sum_{m,n} \left[e^{|\rho|(m+n)} \int_0^{\delta_m} \int_0^{\varepsilon_n} G^{**}(x,y,u)\, dy\, dx \right] e^{|\rho|u}\, du. \tag{5.18}$$

Let

$$S(x,y) = \sum_{m,n} \left[e^{|\rho|(m+n)} \chi_{\delta_m}(x) \chi_{\varepsilon_n}(y) \right],$$

where χ_{δ_m}, χ_{ε_n} are the characteristic functions of sets $(0, \delta_m)$, respectively $(0, \varepsilon_n)$. If $m_x = \max\{m : \delta_m > x\}$ and $n_y = \max\{n : \varepsilon_n > y\}$ then $S(x,y) \leq Ce^{|\rho|(m_x+n_y)}$. Clearly $F^*(x) \geq e^{|\rho|m_x}$, $H^*(y) \geq e^{|\rho|n_y}$; therefore the expression (5.18) is dominated by

$$C \int_{\mathbb{R}_+} \int_0^1 \int_0^1 F^*(x) H^*(y) G^{**}(x,y,u) e^{|\rho|u}\, dy\, dx\, du.$$

One can deal with the second of the two expressions in (5.13) in a similar way; therefore

$$\iint_{G\times G} f(z) g(z^{-1}z') h(z')\, dz'\, dz \leq C \int_{\mathbb{R}_+} \int_0^1 \int_0^1 F^*(x) H^*(y) G^{**}(x,y,u) e^{|\rho|u}\, dy\, dx\, du. \tag{5.19}$$

Step 4. *Final estimates.* Let $\mathcal{K}$ be a suitable constant (to be chosen later) and let $\mathcal{U} = \{(x,y,u) : F^*(x)H^*(y) \leq \mathcal{K}e^{|\rho|u}\}$ and $\mathcal{V} = \{(x,y,u) : F^*(x)H^*(y) \geq \mathcal{K}e^{|\rho|u}\}$. By (5.15),

$$\int_{\mathcal{U}} F^*(x)H^*(y)G^{**}(x,y,u)e^{|\rho|u}\, dy\, dx\, du \leq \int_{\mathbb{R}_+} \int_0^1 \int_0^1 \mathcal{K} G^{**}(x,y,u) e^{2|\rho|u}\, dy\, dx\, du \leq C\mathcal{K}||g||^2_{L^{2,1}}.$$

Using Lemma 7(a), (5.14) and the fact that $G^{**}(x,y,u) \leq 1$ one has

$$\int_{\mathcal{V}} F^*(x)H^*(y)G^{**}(x,y,u)e^{|\rho|u}\,dy\,dx\,du \leq C\int_0^1\int_0^1 \frac{[F^*(x)H^*(y)]^2}{\mathcal{K}}\,dy\,dx$$
$$\leq C\frac{||f||^2_{L^{2,1}}||h||^2_{L^{2,1}}}{\mathcal{K}}.$$

Finally one lets $\mathcal{K} = (||g||_{L^{2,1}})^{-1}(||f||_{L^{2,1}}||h||_{L^{2,1}})$ and the theorem follows.

6. A general rearrangement inequality

We will now extend the rearrangement inequality (5.19) to the case when f, g, h are arbitrary measurable functions (not just characteristic functions of sets). For any measurable function $f : G \to \mathbb{R}_+$ we define the function $F^* : (0,1] \to \mathbb{R}_+$ by the following procedure: first, let $\tilde{f} : K \times (0,\infty) \to \mathbb{R}_+$ be defined, for almost every $k \in K$, as the usual nonincreasing rearrangement of the function $f_k : \overline{N} \times A \to \mathbb{R}_+$, $f_k(\overline{n}a) = f(\overline{n}ak)$ with respect to the measure $e^{2\rho(\log a)}d\overline{n}da$. Using the function $\tilde{f}$ we define the function $\tilde{F} : (0,1] \times (0,\infty) \to \mathbb{R}_+$: for each $r > 0$ fixed, the function $\tilde{F}(.,r)$ is the usual the nonincreasing rearrangement of the function $k \to \tilde{f}(k,r)$. Finally let

$$F^*(x) = \frac{1}{2}\int_0^\infty \tilde{F}(x,r)r^{-1/2}\,dr \tag{6.1}$$

be the $L^{2,1}$ norm of the function $r \to \tilde{F}(x,r)$. Notice that this definition of the function F^* agrees with our earlier definition if f is a characteristic function.

Theorem 8. *If $f, g, h : G \to R_+$ are measurable functions then*

$$\iint_{G\times G} f(z)g(z^{-1}z')h(z')\,dz'\,dz \tag{6.2}$$
$$\leq C\int_{\mathbb{R}_+}\int_0^1\int_0^1 F^*(x)H^*(y)G^{**}(x,y,u)\phi(u)\,dy\,dx\,du,$$

where $G^{**} : (0,1] \times (0,1] \times \mathbb{R}_+ \to \mathbb{R}_+$ is the double rearrangement of the function $(k,k',u) \to g(k^{-1}a(u)k')$ (the same definition as before), F^* and H^* are defined in the previous paragraph and $\phi(u) = u^{m_1+m_2}$ if $u \leq 1$ and $\phi(u) = e^{|\rho|u}$ if $u \geq 1$.

Proof of Theorem 8. Notice that

$$\phi(u) \approx \sup_{r\in[-u,u]} e^{-|\rho|r}\int_{s\leq r,|s|\leq u}\psi(u,s)e^{|\rho|s}\,ds.$$

Notice also that if f and h are characteristic functions of sets then (6.2) is equivalent to (5.19). If f, h are simple positive functions, one can write (uniquely up to sets of measure zero) $f = \sum_1^{M_1} c_i f_i$, $h = \sum_1^{M_2} d_j h_j$, where $c_i, d_j > 0$ and f_i

and h_j, are characteristic functions of sets U_i and V_j with the property that for all i and j one has $U_{i+1} \subset U_i$ and $V_{j+1} \subset V_j$. Simple manipulations involving rearrangements show that $F^* = \sum_1^{M_1} c_i F_i^*$ and $H^* = \sum_1^{M_2} d_j H_j^*$ (this explains the reason why we chose the apparently complicated definition of the function F^* in (6.1)), and (6.2) follows by summation. Finally, a standard argument shows that (6.2) holds for arbitrary measurable functions f, g and h for which the right-hand side integral in (6.2) converges. □

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