Integral mappings and the principle of local reflexivity for noncommutative L^1-spaces

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Dedicated to the memory of Irving Segal

Abstract

The operator space analogue of the *strong form* of the principle of local reflexivity is shown to hold for any von Neumann algebra predual, and thus for any C^*-algebraic dual. This is in striking contrast to the situation for C^*-algebras, since, for example, $K(H)$ does not have that property. The proof uses the Kaplansky density theorem together with a careful analysis of two notions of integrality for mappings of operator spaces.

1. Introduction

Transcendental models, such as ultraproducts and second duals of non-reflexive spaces, arise quite naturally in Banach space theory. Despite their esoteric nature, these constructions have proved to be indispensable for the classification of von Neumann algebras and C^*-algebras (see, e.g., [3], [25], and [26]). Generally speaking, if one wishes to prove that a Banach space or a C^*-algebra has some approximate property, one begins by proving that an appropriate model has the corresponding exact property. One must then relate the exact property in the model to the approximate property in the original space. In Banach space theory, this is often accomplished by using the *principle of local reflexivity.*

In its weakest form, which was first proved by Schatten in his early monograph [32], the principle of local reflexivity states that any finite-dimensional subspace L of the second dual E^{**} of a Banach space E can be approximated by finite-dimensional subspaces of E in the weak* topology. The importance of this result became evident in Grothendieck's ground-breaking study of Banach

spaces in the 1950's (see, e.g., [14], [27]). His theory rested, in part, upon relating various canonical tensor products to corresponding mapping spaces. One of his key observations, which is equivalent to the principle of local reflexivity, is that if E and F are Banach spaces, then

$$(1.1) \qquad (E \otimes_\lambda F)^* = I(E, F^*),$$

where $\otimes_\lambda$ is the injective Banach space tensor product, and $I(E, F^*)$ is the space of integral mappings $\varphi : E \to F^*$. In the 1960's, Johnson, Lindenstrauss, Rosenthal, and Zippin (see [4], [17], and [24]) formulated a strong form of local reflexivity, which implies that the approximating subspaces of E are close to L in the Banach-Mazur distance.

In recent years it has become evident that one can adapt Banach space techniques to the study of linear spaces of Hilbert space operators, provided one replaces the bounded linear mappings of Banach space theory by the *completely bounded* linear mappings (see [31]). As a result, there has been a remarkable convergence of classical and "noncommutative" functional analysis. Much of operator space theory has been developed along the lines pioneered by Grothendieck (see, e.g., [2], [9], [10] and [11]).

The operator space analogue of the weak form of local reflexivity was introduced in [7], and was further studied in [9], [10], [11], [12] and [18]. An operator space V is defined to be *locally reflexive* if for each finite-dimensional operator space F, any complete contraction $\varphi : F \to V^{**}$ may be approximated in the point-weak* topology by a net of complete contractions $\varphi_\lambda : F \to V$. Equivalently, V is locally reflexive if and only if for every finite-dimensional operator space F, we have the isometry

$$(1.2) \qquad F^* \check{\otimes} V^{**} = (F^* \check{\otimes} V)^{**}$$

(this is essentially condition C'' introduced in [1], [7]), or what is the same, V is locally reflexive if and only if we have the natural isometry

$$(1.3) \qquad (F^* \check{\otimes} V)^* = F \hat{\otimes} V^*$$

for each finite-dimensional operator space F. All exact C^*-algebras are locally reflexive (see [21]). On the other hand, it was shown in [7] that some C^*-algebras are *not* locally reflexive. The strong version of local reflexivity does not seem to have an interesting C^*-algebraic analogue, since apparently few C^*-algebras have that property (see §6).

Turning to other operator spaces, the second author showed that the operator space $T(H)$ of trace class operators on a Hilbert space H is locally reflexive [18]. The argument is unexpectedly subtle. The proof used asymptotic techniques related to Pisier's ultraproduct theory (see [29]), as well as a novel application of the Kaplansky density theorem (see §2 and §7). Employing different methods, the first and third author extended this result to the preduals of injective von Neumann algebras [12].

In this paper, we prove that the predual of *any* von Neumann algebra is locally reflexive. We recall that the space $T(H)$ may be regarded as a "noncommutative ℓ^1-space", and in turn, the preduals of von Neumann algebras play the role of the "noncommutative L^1-spaces" mentioned in the title of this paper. What is even more surprising is that these operator spaces are locally reflexive in the *strong sense,* i.e., we can assume that the approximations are close in the sense of the Pisier-Banach-Mazur distance for operator spaces. The approach in this paper is rather different than that used in either [12] or [18], since it does not depend upon ultraproduct techniques.

As in [12] and [18], the Kaplansky density theorem plays a fundamental role in this paper. We begin in Section 2 by showing how that result implies an unexpected theorem about completely bounded mappings $\varphi : A^* \to B$ for C^*-algebras A and B. Our analysis of local reflexivity rests upon a careful study of the relationship between the completely nuclear, completely integral, and exactly integral mappings introduced in [10], [11], and [18], respectively. These results are presented in Section 3, Section 4, and Section 5, respectively. The notion of exactly integral mappings is the most novel of these definitions, and we have explored it in considerable detail in Section 5. As we have indicated in the text, much of the material in Section 5 is not needed in the subsequent sections.

The main theorem on local reflexivity is proved in Section 6 (Theorem 6.7). In Section 7 we apply Theorem 6.7 to show that the preduals of von Neumann algebras with the QWEP property of Kirchberg and Lance (see [21], [23]) are locally approximable by subspaces of dual matrix spaces T_n with $n \in \mathbb{N}$ (see below). This covers a remarkably large class of von Neumann algebras, and in fact it has been conjectured that *all C^*-algebras* have the QWEP. We conclude by showing that the main theorem also implies a factorization theorem that was used by the second author in his proof that $T(H)$ is locally reflexive (see above).

Given any Hilbert space H, we let $B(H)$, $T(H)$, and $K(H)$ denote the bounded, trace class, and compact operators on a Hilbert space H, and we let M_n, T_n, and K_n denote these operator spaces when $H = \mathbb{C}^n$ for $n < \infty$ and $H = l^2$ for $n = \infty$. We use the pairings

$$(1.4) \qquad\qquad \langle a, b \rangle = \sum a_{i,j} b_{i,j}$$

for $a \in K_\infty$ or M_∞, and $b \in T_\infty$. Given operator spaces V and W, we let $\mathcal{CB}(V, W)$ denote the operator space of completely bounded mappings $\varphi : V \to W$, with the norm

$$\|\varphi\|_{cb} = \sup \left\{ \|\mathrm{id} \otimes \varphi : M_n \otimes V \to M_n \otimes W\| \right\}.$$

If V and W are operator spaces, we have corresponding injective and projective operator space tensor products $V \check{\otimes} W$ and $V \hat{\otimes} W$. For the first, let us suppose

that we have concrete representations $V \subseteq \mathcal{B}(H)$ and $W \subseteq \mathcal{B}(K)$. Then $V \check{\otimes} W$ is defined to be the closure of $V \otimes W$ in $\mathcal{B}(H \otimes K)$. On the other hand, the operator space $V \hat{\otimes} W$ is uniquely determined by the fact that we have a complete isometry:

$$(V \hat{\otimes} W)^* \cong \mathcal{CB}(V, W^*).$$

We write $V \otimes_\vee W$ and $V \otimes_\wedge W$ for the algebraic tensor product together with the relative matrix norms.

We emphasize that although we have used Banach space notation for these tensor products, they generally do not coincide with the corresponding tensor products of Banach space theory. On the other hand, the properties of these operator space tensor products under mappings are quite analogous to their Banach space antecedents (see, e.g., [2] and [9]). We also appropriate the Banach space notation ν and ι for the *completely* nuclear and *completely* integral mapping norms (see §3 and §4).

We shall say that an operator space is a *matrix space* if it is completely isometric to a subspace of M_n for some $n \in \mathbb{N}$. Unless otherwise indicated, we consider only complete operator spaces. For our purposes it often suffices to regard various mapping spaces as Banach spaces rather than operator spaces, i.e., we do not need to consider the natural matrix norms on these spaces. Reflecting this, we will at times state that a mapping is a "(complete) contraction," or a "(complete) quotient mapping" to indicate that although it is true, there is no need to prove the stronger statement.

In order to make this paper more accessible to operator algebraists, we have largely avoided using the formal machinery of operator ideals. Given a pair of operator spaces, we identify the algebraic tensor product $V^* \otimes W$ with the vector space $\mathcal{F}(V, W)$ of continuous finite rank mappings $\varphi : V \to W$. We use the terminology "operator ideal" to mean an assignment to each pair of operator spaces V and W, a space of completely bounded mappings $\alpha(V, W) \supseteq \mathcal{F}(V, W)$, with a norm $\alpha(\varphi)$, such that

$$(1.5) \qquad\qquad \alpha(\tau \circ \varphi \circ \sigma) \leq \|\tau\|_{cb} \, \alpha(\varphi) \, \|\sigma\|_{cb}$$

whenever we are given a diagram of mappings

$$X \xrightarrow{\sigma} V \xrightarrow{\varphi} W \xrightarrow{\tau} Y.$$

We let $\alpha^0(V, W)$ denote $\mathcal{F}(V, W)$ with the relative norm in $\alpha(V, W)$.

Given a Banach space V, we have a corresponding linear mapping

$$\text{trace} : \mathcal{F}(V, V) = V^* \otimes V \to \mathbb{C}$$

defined by

$$\text{trace}\,(f \otimes x) = f(x)$$

for $x \in V$ and $f \in V^*$. Given Banach spaces V and W and bounded linear mappings $\varphi : V \to W$ and $\psi : W \to V$, with $\psi \in \mathcal{F}(W, V)$, we have the corresponding *trace duality pairing*

$$(1.6) \qquad \langle \varphi, \psi \rangle = \mathrm{trace}\,(\varphi \circ \psi) = \mathrm{trace}\,(\psi \circ \varphi).$$

If we let

$$\psi = \sum_{i=1}^{n} g_i \otimes v_i \in W^* \otimes V,$$

we have

$$(1.7) \qquad \langle \varphi, \psi \rangle = \mathrm{trace}\left(\sum_i g_i \otimes \varphi(v_i) \right) = \mathrm{trace}\,(\mathrm{id} \otimes \varphi)(\psi).$$

Finally we note that for any operator space V,

$$(1.8) \qquad \mathrm{trace} \colon V^* \otimes_\wedge V \to \mathbb{C}$$

is contractive since $(f, v) \mapsto f(v)$ is a completely contractive bilinear mapping.

2. Finite rank approximations and the Kaplansky density theorem

Given operator spaces V and W, we say that a completely bounded mapping $\varphi : V^* \to W$ satisfies the *weak* approximation property* (W*AP) if there exists a net of finite rank weak* continuous mappings $\varphi_\lambda : V^* \to W$ with $\|\varphi_\lambda\|_{cb} \leq \|\varphi\|_{cb}$ which converge to φ in the *point-norm* topology. If H is an infinite-dimensional Hilbert space, the identity mapping $I : B(H) \to B(H)$ does not have such approximations since $B(H)$ does not have the metric approximation property of Grothendieck [33]. Our object in this section is to show that by contrast, if A and B are C^*-algebras, *any* completely bounded mapping $\varphi : A^* \to B$ has the W*AP.

Given von Neumann algebras R and S, we let $R \overline{\otimes} S$ denote the von Neumann algebra tensor product of R and S. Then each function $f \in R_*$ determines a *slice mapping*

$$f \otimes \mathrm{id} : R \overline{\otimes} S \to S$$

where

$$\langle (f \otimes \mathrm{id})(u),\ g \rangle = (f \otimes g)(u)$$

for $u \in R \overline{\otimes} S$ and $g \in S_*$ (see [36]). As a result, each element u in $R \overline{\otimes} S$ determines a mapping $\varphi_u \in \mathcal{CB}(R_*, S)$ by

$$\varphi_u(f) = (f \otimes \mathrm{id})(u).$$

It was shown in [9] that this determines a complete isometry

$$(2.1) \qquad R \overline{\otimes} S \cong \mathcal{CB}(R_*, S).$$

LEMMA 2.1.   *Given C^*-algebras A and B, every complete contraction $\varphi : A^* \to B^{**}$ can be approximated by a net of finite rank weak* continuous complete contractions $\varphi_\lambda : A^* \to B$ in the point-weak* topology. Every completely bounded mapping $\varphi : A^* \to B$ satisfies the* $\mathrm{W}^*\mathrm{AP}$.

Proof. Using the universal representations of A and B, we may identify A^{**} and B^{**} with von Neumann algebras on Hilbert spaces H and K. The *-algebra $A \otimes B$ is weak operator dense in the von Neumann algebra $A^{**}\overline{\otimes}B^{**} = (A \otimes B)''$ on $H \otimes K$. From the Kaplansky density theorem, the unit ball of the *-algebra $A \otimes B$ is weak operator dense in that of $A^{**}\overline{\otimes}B^{**}$.

If $\varphi : A^* \to B^{**}$ is a complete contraction, we may assume that $\varphi = \varphi_u$ for some contractive element $u \in A^{**}\overline{\otimes}B^{**}$ since we have the isometry

$$A^{**}\overline{\otimes}B^{**} = \mathcal{CB}(A^*, B^{**})$$

by (2.1). There exists a net of contractive elements $u_\lambda \in A \otimes B$ converging to u in the weak operator topology on $B(H \otimes K)$. It follows that u_λ converges to u relative to the topology determined by the algebraic tensor product $A^* \otimes B^*$. We have that $\varphi_\lambda = \varphi_{u_\lambda}$ is a net of finite rank weak* continuous complete contractions from A^* into B which converges to $\varphi = \varphi_u$ in the point-weak* topology, i.e., for each $f \in A^*$,

$$\varphi_\lambda(f) = f \otimes id(u_\lambda) \in B \to \varphi(f) = f \otimes id(u) \in B^{**}$$

in the weak* topology.

If φ is a complete contraction from A^* into B, we have that φ_λ converges to φ in the point-weak topology. The usual convexity argument shows that we can find a net of finite rank weak* continuous complete contractions $\psi_\mu : A^* \to B$ in the convex hull of $\{\varphi_\lambda\}$, which converges to φ in the point-norm topology (see, e.g., [6, p. 477]).   □

3. Completely nuclear mappings

Given operator spaces V and W, there is a canonical mapping

$$(3.1) \qquad\qquad \Phi : V^* \hat{\otimes} W \to V^* \check{\otimes} W \subseteq \mathcal{CB}(V, W)$$

which extends the identity mapping on the algebraic tensor product $V^* \otimes W$. A linear mapping $\varphi : V \to W$ is said to be *completely nuclear* if it lies in the image of Φ (see [10]). Identifying the linear space $\mathcal{N}(V, W) = \Phi(V^* \hat{\otimes} W)$ with the quotient Banach space $V^* \hat{\otimes} W / \ker \Phi$, we call the corresponding norm ν the *completely nuclear norm* on $\mathcal{N}(V, W)$. If V or W is finite-dimensional, then Φ is one-to-one, and we have the isometry

$$(3.2) \qquad\qquad \mathcal{N}(V, W) = V^* \hat{\otimes} W.$$

If $\varphi : V \to W$ is a linear mapping which is not nuclear, we write $\nu(\varphi) = \infty$.

Turning to a "prototypical" example, let us suppose that a and b are infinite scalar matrices with Hilbert-Schmidt norms $\|a\|_2, \|b\|_2 < 1$. Then the mapping

$$M(a,b): M_\infty \to T_\infty : x \to axb,$$

satisfies $\nu(M(a,b)) < 1$. More generally, given any operator spaces V and W, a linear mapping $\varphi : V \to W$ satisfies $\nu(\varphi) < 1$ if and only if it factors through such a mapping via completely contractive mappings. Thus we have that $\nu(\varphi) < 1$ if and only if there is a commutative diagram

(3.3)
$$
\begin{array}{ccc}
M_\infty & \xrightarrow{M(a,b)} & T_\infty \\
r \uparrow & & \downarrow s \\
V & \xrightarrow{\varphi} & W
\end{array}
$$

where r and s are complete contractions, and $\|a\|_2, \|b\|_2 < 1$. It is also equivalent to assume that there is a commuting diagram

(3.4)
$$
\begin{array}{ccc}
K_\infty & \xrightarrow{M(a,b)} & T_\infty \\
r \uparrow & & \downarrow s \\
V & \xrightarrow{\varphi} & W
\end{array}
$$

with the same assumptions (see [10]).

LEMMA 3.1.   *Given operator spaces V and W, the canonical mapping*

$$\mathrm{id} \otimes \iota_W : V \hat{\otimes} W \to V \hat{\otimes} W^{**}$$

is a complete isometry.

Proof. Let $\iota_W : W \to W^{**}$ be the canonical embedding. It follows from the definition of the projective tensor product that $\mathrm{id} \otimes \iota_W$ is a complete contraction from $V \hat{\otimes} W$ into $V \hat{\otimes} W^{**}$. In order to show that $\mathrm{id} \otimes \iota_W$ is isometric, it suffices to show that its adjoint $(\mathrm{id} \otimes \iota_W)^*$ is a norm quotient mapping. Equivalently, since we have the commutative diagram

$$
\begin{array}{ccc}
(V \hat{\otimes} W^{**})^* & \xrightarrow{(\mathrm{id} \otimes \iota_W)^*} & (V \hat{\otimes} W)^* \\
\| & & \| \\
\mathcal{CB}(V, W^{***}) & \xrightarrow{\theta} & \mathcal{CB}(V, W^*) \ ,
\end{array}
$$

where $\theta(\varphi) = (\iota_W)^* \circ \varphi$, it suffices to prove that θ is a quotient mapping. If we are given a complete contraction $\psi : V \to W^*$, we have that

$$\psi = (\iota_W)^*(\iota_{W^*} \circ \psi),$$

where $\iota_{W^*} \circ \psi$ is a complete contraction in $\mathcal{CB}(V, W^{***})$. Thus θ is indeed a quotient mapping, and $\mathrm{id} \otimes \iota_W$ is isometric.

Applying this to the space $T_n \hat{\otimes} V$, and using associativity of the projective tensor product, it follows that we have an isometry

$$\mathrm{id} \otimes (\mathrm{id} \otimes \iota_W) : T_n \hat{\otimes} (V \hat{\otimes} W) \to T_n \hat{\otimes} (V \hat{\otimes} W^{**})$$

for each $n \in \mathbb{N}$. Taking the adjoint,

$$(\mathrm{id} \otimes \iota_W)_n^* : M_n((V \hat{\otimes} W^{**})^*) \to M_n((V \hat{\otimes} W)^*)$$

is a quotient mapping, and thus $(\mathrm{id} \otimes \iota_W)^*$ is a complete quotient mapping. It follows that $(\mathrm{id} \otimes \iota_W)^{**}$ is a complete isometry, and restricting it to $V \hat{\otimes} W$, we conclude that $\mathrm{id} \otimes \iota_W$ is a complete isometry. $\qquad \square$

LEMMA 3.2. *Given a nuclear mapping $\varphi : V \to W$, we have $\nu(\varphi^*) \leq \nu(\varphi)$. If V or W is finite-dimensional, then $\nu(\varphi^*) = \nu(\varphi)$.*

Proof. If we let $S(\varphi) = \varphi^*$, it is evident from the commutative diagram

(3.5)
$$\begin{array}{ccc} V^* \hat{\otimes} W & \xrightarrow{\mathrm{id} \otimes \iota_W} & V^* \hat{\otimes} W^{**} \\ \downarrow \Phi_1 & & \downarrow \Phi_2 \\ \mathcal{N}(V, W) & \xrightarrow{S} & \mathcal{N}(W^*, V^*) \end{array}$$

that S is a contraction. Even though the top row is isometric (Lemma 3.1), and the two columns are quotient mappings, it does not follow that S is isometric, since one might have that

$$\ker \Phi_2 \cap (V^* \hat{\otimes} W) \neq \ker \Phi_1.$$

On the other hand, if either V or W is finite-dimensional, then the mappings Φ_i are isometric, and thus the same is true for S. $\qquad \square$

We note that if V and W are both infinite-dimensional, we can have that $\nu(\varphi^*) < \nu(\varphi)$ even if φ is of finite rank (see [5, p. 67]). On the other hand if V^* has the operator approximation property (see [9]), the mappings Φ_i in (3.5) are one-to-one, and thus $\nu(\varphi^*) = \nu(\varphi)$.

We will also use a minor variation on the previous result.

LEMMA 3.3. *Suppose that L and W are operator spaces with L finite-dimensional, and let $\iota_W : W \to W^{**}$ denote the canonical complete isometry. Then for any mapping $\varphi : L \to W$, we have that*

$$\nu(\iota_W \circ \varphi) = \nu(\varphi).$$

Proof. We have a commutative diagram

(3.6)
$$
\begin{array}{ccc}
L^*\hat{\otimes}W & \to & L^*\hat{\otimes}W^{**} \\
\downarrow & & \downarrow \\
\mathcal{N}(L,W) & \to & \mathcal{N}(L,W^{**})
\end{array}
$$

where from Lemma 3.1, the top row is a completely isometric injection, and since L is finite-dimensional, the columns are (complete) isometries. It follows that the bottom row is a (completely) isometric injection. □

4. Completely integral mappings

As in Banach space theory, the completely nuclear norm is not local. By this we mean that if we are given operator spaces V and W, and a linear mapping $\varphi : V \to W$ such that $\nu(\varphi|_F) \leq 1$ for all finite-dimensional subspaces $F \subseteq V$, it need not follow that φ is completely nuclear. As we will see, this naturally leads to the more general class of completely integral mappings.

We recall from [10] that a linear mapping $\varphi : V \to W$ is *completely integral* with *completely integral norm* $\iota(\varphi) \leq 1$ if φ is in the point-norm closure of the set of finite rank mappings $\psi : V \to W$ such that $\nu(\psi) < 1$, or using a standard convexity argument (see [10, Prop. 3.2]), φ is in the point-weak closure of that set. We let $\mathcal{I}(V,W)$ denote the linear space of all completely integral mappings from V into W with the norm ι and, as usual, we write $\iota(\varphi) = \infty$ if $\varphi : V \to W$ is not completely integral. It is clear that we have that $\iota(\varphi) \leq \nu(\varphi)$ for any linear mapping $\varphi : V \to W$, and thus we have a natural contraction

$$\mathcal{N}(V,W) \to \mathcal{I}(V,W).$$

LEMMA 4.1. *If V is finite-dimensional, we have the isometry*

$$\mathcal{N}(V,W) = \mathcal{I}(V,W).$$

Proof. We must show that if $\varphi : V \to W$ satisfies $\iota(\varphi) \leq 1$, then $\nu(\varphi) \leq 1$. Therefore, let us suppose φ is a point-norm limit of mappings $\varphi_\lambda \in \mathcal{N}(V,W)$ with $\nu(\varphi_\lambda) < 1$. We fix a basis $x_1, \ldots, x_n$ for V and we let $f_1, \ldots, f_n$ be the corresponding dual basis for V^*; i.e., we define

$$f_i(\sum_{j=1}^{n} c_j x_j) = c_i.$$

Using the algebraic identification $\mathcal{CB}(V,W) = V^* \otimes W$, we have that

$$\varphi_\lambda = \sum_{i=1}^{n} f_i \otimes y_i^\lambda,$$

and

$$\varphi = \sum_{i=1}^{n} f_i \otimes y_i,$$

where $y_i^{\lambda} = \varphi_{\lambda}(x_i)$ and $y_i = \varphi(x_i) \in W$. Since φ_{λ} converges to φ in the point-norm topology, it follows that

$$\left\| y_i^{\lambda} - y_i \right\| = \| \varphi_{\lambda}(x_i) - \varphi(x_i) \| \to 0.$$

The operator projective tensor norm is a cross norm in the Banach space sense, and thus

$$\begin{aligned}
\nu(\varphi - \varphi_{\lambda}) &\leq \left\| \sum_{i=1}^{n} f_i \otimes (y_i^{\lambda} - y_i) \right\|_{V^* \hat{\otimes} W} \\
&\leq \sum_{i=1}^{n} \|f_i\| \left\| y_i^{\lambda} - y_i \right\| \to 0.
\end{aligned}$$

Since $\nu(\varphi_{\lambda}) < 1$ and

$$\nu(\varphi) \leq \nu(\varphi - \varphi_{\lambda}) + \nu(\varphi_{\lambda}),$$

we conclude that $\nu(\varphi) \leq 1$. $\qquad\square$

Given operator spaces V and W, the pairing (1.6) is given by

$$(4.1) \qquad \langle \cdot, \cdot \rangle : \mathcal{CB}(V, W) \times (V \otimes W^*) \to \mathbb{C} : (\varphi, (v \otimes g)) \to \langle \varphi(v), g \rangle,$$

and it thus determines a linear mapping

$$(4.2) \qquad\qquad \Psi : \mathcal{CB}(V, W) \hookrightarrow (V \otimes W^*)^*,$$

where we let $(V \otimes W^*)^*$ denote the space of linear functionals f for which $f(v \otimes g)$ is norm-continuous in each variable. In particular, since

$$(V \hat{\otimes} W^*)^* = \mathcal{CB}(V, W^{**}),$$

Ψ induces the (completely) isometric injection

$$(4.3) \qquad\qquad \Psi : \mathcal{CB}(V, W) \hookrightarrow (V \hat{\otimes} W^*)^*$$

corresponding to the usual inclusion mapping $\mathcal{CB}(V, W) \subseteq \mathcal{CB}(V, W^{**})$.

Modifying (3.12) in [10], we have a natural commutative diagram

$$\begin{array}{ccccc}
V^* \hat{\otimes} W & \xrightarrow{\Psi_0} & (V \check{\otimes} W^*)^* & \xrightarrow{\Phi_1^*} & (V \hat{\otimes} W^*)^* \\
\downarrow \Phi & & & & \uparrow \Psi \\
\mathcal{N}(V, W) & & \hookrightarrow & & \mathcal{CB}(V, W)
\end{array}$$

where Φ and $\Phi_1 : V \hat{\otimes} W^* \to V \check{\otimes} W^*$ are the canonical (complete) contractions, and the (complete) contraction Ψ_0 is determined by the relation

$$\Psi_0(f \otimes w)(v \otimes g) = f(v) g(w).$$

Since Φ_1 has dense range, Φ_1^* is one-to-one. It follows that $\ker \Phi \subseteq \ker \Psi_0$, and thus Ψ_0 determines a complete contraction $\Psi : \mathcal{N}(V, W) \to (V \check\otimes W^*)^*$, which is just the restriction of (4.3) to $\mathcal{N}(V, W)$.

THEOREM 4.2. *Suppose that V and W are operator spaces and that $\varphi : V \to W$ is a completely bounded linear mapping. Then the following are equivalent*:

(a) $\iota(\varphi) \leq 1$,

(b_1) $\nu(\varphi_{|E}) \leq 1$ *for all finite-dimensional subspaces $E \subseteq V$,*

(b_2) $\nu(\varphi \circ \psi) \leq 1$ *for all complete contractions $\psi : E \to V$, with E finite-dimensional,*

(c_1) $\|\mathrm{id} \otimes \varphi : F \check\otimes V \to F \hat\otimes W\| \leq 1$ *for finite-dimensional operator spaces F,*

(c_2) $\|\mathrm{id} \otimes \varphi : F \check\otimes V \to F \hat\otimes W\| \leq 1$ *for arbitrary operator spaces F,*

(d) $\|\Psi(\varphi) : V \otimes_\vee W^* \to \mathbb{C}\| \leq 1$.

Proof. (a)$\Leftrightarrow$(b_1). From [12, Prop. 2.1], we see that $\iota(\varphi) \leq 1$ if and only if $\iota(\varphi|_E) \leq 1$ for all finite-dimensional subspaces $E \subseteq V$. Thus the equivalence follows from Lemma 4.1.

(b_1)$\Leftrightarrow$(b_2). Given $\psi : E \to V$, we have that

$$\nu(\varphi \circ \psi) \leq \nu(\varphi_{|\psi(E)}) \, \|\psi\|_{cb}$$

and the equivalence is immediate.

(b_2)$\Leftrightarrow$(c_1). Given a finite-dimensional operator space F and letting $E = F^*$, we may identify $F \check\otimes V$ with $\mathcal{CB}(E, V)$. This equivalence is immediate from the commutative diagram

(4.4)
$$
\begin{array}{ccc}
F \check\otimes V & \overset{\mathrm{id} \otimes \varphi}{\longrightarrow} & F \hat\otimes W \\
\| & & \| \\
\mathcal{CB}(E, V) & \overset{\psi \mapsto \varphi \circ \psi}{\longrightarrow} & \mathcal{N}(E, W) \ .
\end{array}
$$

(c_1)$\Leftrightarrow$(c_2). Given an arbitrary operator space F and an element $u \in F \otimes_\vee V$, we have that $u \in F_0 \otimes_\vee V$ for some finite-dimensional subspace F_0 of V, and

$$\|u\|_{F \check\otimes V} = \|u\|_{F_0 \check\otimes V} \ .$$

Since $F_0 \hat\otimes W \to F \hat\otimes W$ is a contraction, it is evident that (c_1)$\Rightarrow$(c_2), and the converse is trivial.

(a)⇔(d).   Given $\varphi \in \mathcal{CB}(V, W)$ with $\iota(\varphi) \leq 1$ and an element $u \in V \otimes_\vee W^*$ with $\|u\|_\vee \leq 1$, we may assume that $u \in E \otimes_\vee W^*$, where E is a finite-dimensional subspace of V. It follows that

$$|\langle \Psi(\varphi), u \rangle| = \left| \langle \Psi(\varphi_{|E}), u \rangle \right| \leq \nu(\varphi_{|E}) \leq \iota(\varphi),$$

and thus $\|\Psi(\varphi)\| \leq 1$.

Let us suppose that $\varphi \in \mathcal{CB}(V, W)$ satisfies $\|\Psi(\varphi)\| \leq 1$. We have that

$$V \check{\otimes} W^* \cong W^* \check{\otimes} V \subseteq \mathcal{CB}(W, V^{**}) = (V^* \hat{\otimes} W)^*,$$

and thus $\Psi(\varphi)$ has a contractive extension $\bar{\Psi}_\varphi \in (V^* \hat{\otimes} W)^{**}$. From the bipolar theorem, we may choose a net of elements $u_\lambda \in V^* \hat{\otimes} W$ such that $\|u_\lambda\|_{V^* \hat{\otimes} W} < 1$ and u_λ converges to $\bar{\Psi}_\varphi$ in the weak* topology. Let $\varphi_\lambda = \Phi(u_\lambda) \in \mathcal{N}(V, W)$. Then $\nu(\varphi_\lambda) < 1$ and

$$\varphi_\lambda(v)(g) = \langle u_\lambda, v \otimes g \rangle \to \langle \bar{\Psi}_\varphi, v \otimes g \rangle = \langle \varphi, v \otimes g \rangle = \langle \varphi(v), g \rangle$$

for all $v \in V$ and $g \in W^*$. Therefore, φ_λ converges to φ in the point-weak topology, and $\iota(\varphi) \leq 1$.   □

COROLLARY 4.3.   *Given operator spaces V and W and a linear mapping $\varphi : V \to W$, we have*

$$(4.5) \quad \iota(\varphi) \;=\; \sup \left\{ \nu(\varphi_{|E}) : E \subseteq V \text{ finite-dimensional} \right\}$$

$$=\; \sup \left\{ \|\mathrm{id} \otimes \varphi : F \check{\otimes} V \to F \hat{\otimes} W\| : F \text{ finite-dimensional} \right\}$$

$$=\; \sup \left\{ \|\mathrm{id} \otimes \varphi : F \check{\otimes} V \to F \hat{\otimes} W\| : F \text{ arbitrary} \right\}.$$

Furthermore, the mapping

$$(4.6) \qquad\qquad \Psi : \mathcal{I}(V, W) \hookrightarrow (V \otimes_\vee W^*)^* = \left[\mathcal{CB}^0(W, V) \right]^*$$

is an isometric injection.

In particular, we see that ι is *local*. If W is finite-dimensional, we have from (4.6) the isometry

$$(4.7) \qquad\qquad \mathcal{I}(V, W) = (V \check{\otimes} W^*)^* = \mathcal{CB}(W, V)^*.$$

However, in contrast to the situation for Banach spaces (see (1.1)), we *need not have* that the natural mapping

$$\tilde{\Psi} : \mathcal{I}(V, W^*) \to (V \check{\otimes} W)^*$$

is isometric. Using the identification $\mathcal{N}(V, W) = V^* \hat{\otimes} W$, we obtain the following result from the discussion of (1.3).

PROPOSITION 4.4. *An operator space V is locally reflexive if and only if we have the isometry*

$$(4.8) \qquad\qquad \mathcal{N}(V, W) = \mathcal{I}(V, W)$$

for all finite-dimensional W.

It is shown in [7] that $C^*(\mathbb{F}_2)$, the full group C^*-algebra of 2-generators, is not locally reflexive, and thus there exists a finite-dimensional operator space W such that $\mathcal{N}(C^*(\mathbb{F}_2), W) \to \mathcal{I}(C^*(\mathbb{F}_2), W)$ is not isometric. Denoting the corresponding Banach mapping spaces with the subscript B, we always have the isometry

$$\mathcal{N}_B(V, W) = \mathcal{I}_B(V, W)$$

for finite-dimensional W.

We conclude this section with a factorization which characterizes the completely integral mappings.

PROPOSITION 4.5. *Given operator spaces V and W and a completely bounded mapping $\varphi : V \to W$, we have that $\iota(\varphi) \leq 1$ if and only if there exist Hilbert spaces H and K, a contractive functional $\omega \in B(H \otimes K)^*$ and completely contractive maps $r : V \to B(H)$, $t : W^* \to B(K)$ such that for all $v \in V$ and $g \in W^*$,*

$$(4.9) \qquad\qquad \langle \varphi(v), g \rangle = \langle \omega, r(v) \otimes t(g) \rangle.$$

Proof. Let us suppose that $\iota(\varphi) \leq 1$. We fix completely isometric embeddings $r : V \to B(H)$ and $s : W^* \to B(K)$. From (4.5)

$$\|\Psi(\varphi) : V \otimes_\vee W^* \to \mathbb{C}\| \leq 1,$$

hence we may extend $\Psi(\varphi)$ to an element $\omega \in B(H \otimes K)^*$ with $\|\omega\| \leq 1$. It follows that

$$\langle \varphi(v), g \rangle = \Psi(\varphi)(v \otimes g) = \omega(r(v) \otimes t(g)).$$

Conversely given such a factorization with $\|\omega\| \, \|r\|_{cb} \, \|t\|_{cb} \leq 1$, we have that for any $u \in V \otimes W^*$,

$$
\begin{aligned}
|\Psi(\varphi)(u)| &= |\langle \omega, (r \otimes t)(u) \rangle| \\
&\leq \|\omega\| \, \|(r \otimes t)(u)\|_{B(H \otimes K)} \\
&\leq \|\omega\| \, \|r\|_{cb} \, \|t\|_{cb} \, \|u\|_{V \otimes_\vee W^*} \\
&\leq \|u\|_{V \otimes_\vee W^*} \, .
\end{aligned}
$$

Therefore, we have $\iota(\varphi) = \|\Psi(\varphi)\| \leq 1$. $\square$

Given a bounded linear functional $\omega : B(H \otimes K) \to \mathbb{C}$, we define a linear mapping

$$M(\omega) : B(H) \to B(K)^*$$

by

$$M(\omega)(b)(g) = \omega(b \otimes g).$$

COROLLARY 4.6. *Let us suppose that V and W are operator spaces, and that $\varphi : V \to W$ is a linear mapping. We have that $\iota(\varphi) \leq 1$ if and only if there is a commutative diagram*

(4.10)
$$
\begin{array}{ccc}
B(H) & \xrightarrow{M(\omega)} & B(K)^* \\
{\scriptstyle r}\uparrow & & \searrow{\scriptstyle s} \\
V & \xrightarrow{\varphi} \quad W & \xhookrightarrow{\iota_W} \quad W^{**} ,
\end{array}
$$

where $\omega \in B(H \otimes K)^*$ satisfies $\|\omega\| \leq 1$, r and s are complete contractions, $\iota_W : W \to W^{**}$ is the canonical embedding, and $s : B(K)^* \to W^{**}$ is weak* continuous.

Proof. Letting $s = t^*$, this is immediate from Proposition 4.5. □

5. Exactly integral mappings

Weakening the characterization in Corollary 4.6, we say that a linear mapping $\varphi : V \to W$ is *exactly integral* if it has a factorization (4.10), where r and s are completely bounded and $\omega \in B(H \otimes K)^*$, *but we do not assume that s is weak* continuous.* We define the corresponding *exactly integral norm*

$$\iota^{\mathrm{ex}}(\varphi) = \inf\{\|r\|_{cb}\,\|\omega\|\,\|s\|_{cb}\}$$

where the infimum is taken over all such factorizations. It is trivial that if $\varphi : V \to W$ is completely integral, then φ is exactly integral and $\iota^{\mathrm{ex}}(\varphi) \leq \iota(\varphi)$. The fact that ι^{ex} is a norm follows from Theorem 5.5.

LEMMA 5.1. *Let us suppose that V and W are operator spaces. If $\varphi : V \to W$ is completely integral, then $\varphi^* : W^* \to V^*$ is exactly integral with $\iota^{\mathrm{ex}}(\varphi^*)$*
$\leq \iota(\varphi)$.

Proof. We may use (4.10) to construct a commutative diagram

$$
\begin{array}{ccc}
B(K) & \xrightarrow{M(\tilde{\omega})} & B(H)^* \\
{\scriptstyle s_*}\uparrow & & \searrow{\scriptstyle \iota_{V^*}\circ r^*} \\
W^* & \xrightarrow{\varphi^*} \quad V^* & \xhookrightarrow{\iota_{V^*}} \quad V^{***} ,
\end{array}
$$

where $s = (s_*)^*$, and $\tilde{\omega} : B(K \otimes H) \to \mathbb{C}$ is the obvious "flip" of ω. □

It will be noted that in the above proof, $\iota_{V^*} \circ r^*$ is generally not weak* continuous, and thus we cannot conclude that $\iota(\varphi^*) \leq \iota(\varphi)$.

LEMMA 5.2.   *If A is a C^*-algebra and V is an arbitrary operator space, then we have the isometric identification*

$$\mathcal{I}^{\mathrm{ex}}(V, A) = \mathcal{I}(V, A).$$

Proof. Let us assume that $\iota^{\mathrm{ex}}(\varphi) \leq 1$. Then we can find a factorization

$$
\begin{array}{ccccc}
B(H) & \xrightarrow{M(\omega)} & B(K)^* & & \\
r \uparrow & & & \searrow^{s} & \\
V & \xrightarrow{\varphi} & A & \xhookrightarrow{\iota_A} & A^{**},
\end{array}
$$

where r, s are complete contractions and ω is of norm one. From Theorem 2.1 we may approximate s in the point-weak* topology by a net of weak* continuous mappings $s_\lambda : B(K)^* \to A$ with $\|s_\lambda\|_{cb} \leq 1$. Fixing λ, and letting $\varphi_\lambda = \iota_A s_\lambda M(\omega) r$, we have the commutative diagram

$$
\begin{array}{ccccc}
B(H) & \xrightarrow{M(\omega)} & B(K)^* & & \\
r \uparrow & & \downarrow s_\lambda & \searrow^{\iota_A \circ s_\lambda} & \\
V & \xrightarrow{\varphi_\lambda} & A & \xhookrightarrow{\iota_A} & A^{**},
\end{array}
$$

where $\iota_A \circ s_\lambda : B(K)^* \to A^{**}$ is a weak*-continuous complete contraction. It follows from Corollary 4.6 that $\iota(\varphi_\lambda) \leq 1$. Since each s_λ and φ have range in A, φ_λ converges to φ in the point-weak topology. Thus we have from the definition of the completely integral norm that $\iota(\varphi) \leq 1$. $\qquad\square$

Although the definition of the exactly integral mappings might seem contrived, such mappings play a natural and important role in operator space theory. In order to substantiate this claim, we will provide several alternative characterizations. This material will not be needed in the subsequent sections.

The following is well-known:

LEMMA 5.3.   *Suppose that E is a matrix space (see §1). Then for any operator space W, we have the complete isometry*

(5.1) $$(E \check{\otimes} W)^* \cong E^* \hat{\otimes} W^*.$$

Proof. Let us suppose that $E \subseteq M_n$, and let $\rho : M_n^* \to E^*$ be the restriction mapping. We have that $E \check{\otimes} W \subseteq M_n \check{\otimes} W$, and this determines the restriction mapping $\bar{\rho}$ in the commutative diagram

$$
\begin{array}{ccc}
T_n \hat{\otimes} W^* & \cong & (M_n \check{\otimes} W)^* \\
\rho \otimes id \downarrow & & \bar{\rho} \downarrow \\
E^* \hat{\otimes} W^* & \to & (E \check{\otimes} W)^* \,.
\end{array}
$$

From the general theory, it follows that the top row is completely isometric, and the first column is a complete quotient mapping. On the other hand, since $E \check{\otimes} W \to M_n \check{\otimes} W$ is a complete isometry, the second column is a complete quotient mapping. It follows that the bottom row is a complete isometry. $\square$

By contrast to the situation for Banach spaces, if E is a finite-dimensional operator space, (5.1) need not hold in general. This is related to the fact that E need not be exact, i.e., approximable in the Pisier-Banach-Mazur sense by matrix spaces (see [28]). Nevertheless, it can be approximated in an asymptotic sense. We may identify E with a subspace of M_∞. For each $n \in \mathbb{N}$, we let

$$P_n : x \in M_\infty \to x^{(n)} \in M_n$$

be the usual truncation mapping. Restricting both the domain and the range, we let

(5.2) $$\tau_n = \tau_n^E = P_{n|E} : E \to P_n(E).$$

LEMMA 5.4.   *Given a finite-dimensional subspace E of M_∞, an integer $k > 0$, and $0 < \varepsilon < 1$, there exists an $n \in \mathbb{N}$ such that τ_n is invertible and*

$$\left\| (\tau_n)_k^{-1} \right\| \leq 1 + \varepsilon.$$

Proof. Let us fix elements x_i which are $\varepsilon/2$-dense in the unit sphere of E. Since

$$\lim_{n \to \infty} \|P_n(x_i)\| = \|x_i\|,$$

we may choose an n such that

$$\|P_n(x_i)\| \geq 1 - \varepsilon/2$$

for all i. If $x \in E$ and $\|x\| = 1$, we may find an i such that $\|x - x_i\| < \varepsilon/2$. It follows that

$$\|P_n(x)\| \geq \|P_n(x_i)\| - \|P_n(x_i) - P_n(x)\| \geq 1 - \varepsilon,$$

and thus

$$\|\tau_n(x)\| \geq (1 - \varepsilon) \|x\|.$$

It follows that τ_n is one-to-one, and $\|\tau_n^{-1}\| \leq (1 - \varepsilon)^{-1}$. We may apply this argument to the mappings

$$(P_n)_k : M_k(E) \to M_k(P_n(E)),$$

and the result follows. $\square$

We assume that readers are familiar with ultraproducts of Banach spaces and operator spaces (see [12], [15], [28], [29], and [35]).

Groh has proved that an ultrapower of von Neumann algebraic preduals is again the predual of a von Neumann algebra (see [13] and [30] — we are indebted to Ward Henson for bringing these papers to our attention). In order to make our discussion more accessible, we repeat his argument for the von Neumann algebra M_∞. Given an index set I, and a free ultrafilter $\mathcal{U}$ on I, we let $\prod_{\mathcal{U}} T_\infty$ denote the operator space ultrapower of T_∞. We have a natural completely isometric injection

$$(5.3) \qquad \theta : \prod_{\mathcal{U}} T_\infty \to \ell^\infty(I, M_\infty)^*$$

defined by the pairing

$$\langle \theta([\omega_\alpha]), (y_\alpha) \rangle = \lim_{\mathcal{U}} \langle \omega_\alpha, y_\alpha \rangle$$

(see, e.g., [12]). We may regard $\ell^\infty(I, M_\infty)^*$ as a bimodule over $\ell^\infty(I, M_\infty)$ or over $\ell^\infty(I, M_\infty)^{**}$ in the usual manner. The subspace $T = \theta(\prod_{\mathcal{U}} T_\infty)$ is a norm closed two-sided $\ell^\infty(I, M_\infty)$ submodule since if we are given $f = [(f_\alpha)] \in \prod_{\mathcal{U}} T_\infty$ and $x = (x_\alpha) \in \ell^\infty(I, M_\infty)$, we have that $(x_a f_\alpha) \in \ell^\infty(I, T_\infty)$ and thus

$$xf = \theta[(x_a f_\alpha)] \in T,$$

and the same argument shows that $fx \in T$. We conclude (see [34, Chap. III, Th. 2.7]) that the annihilator of T is a weak* closed two-sided ideal in the von Neumann algebra $\ell^\infty(I, M_\infty)^{**}$, and in particular, there is a central projection $e \in \ell^\infty(I, M_\infty)^{**}$ for which

$$(5.4) \qquad T = \ell^\infty(I, M_\infty)^* e = [\ell^\infty(I, M_\infty)^{**} e]_* \, .$$

It is useful to compare the following theorem with Theorem 4.2. Significant portions of this result may be found in [18], where a rather different approach is used. Condition (d) is related to Pisier's factorization theorem for completely 1-summing mappings [29].

THEOREM 5.5. *Given operator spaces V and W and a completely bounded mapping $\varphi : V \to W$, the following are equivalent:*

(a) $\iota^{\mathrm{ex}}(\varphi) \leq 1$,

(b) $\nu(\varphi \circ \psi) \leq 1$ *for all complete contractions $\psi : E \to V$ with E a matrix space,*

(c) $\|\mathrm{id} \otimes \varphi : E^* \check{\otimes} V \to E^* \hat{\otimes} W\| \leq 1$ *for all matrix spaces E,*

(d) *There exists an infinite index set I, a free ultrafilter $\mathcal{U}$ on I, and a commutative diagram*

(5.5)
$$
\begin{array}{ccccc}
\ell^\infty(I, M_\infty) & \xrightarrow{\;\mathcal{M}\;} & \prod_{\mathcal{U}} T_\infty & & \\
{\scriptstyle r}\uparrow & & & \searrow {\scriptstyle s} & \\
V & \xrightarrow{\;\varphi\;} & W & \overset{\iota_W}{\hookrightarrow} & W^{**} ,
\end{array}
$$

where r and s are complete contractions, and $\mathcal{M} = [M(a_\alpha, b_\alpha)]$ is determined by the multiplication operators $M(a_\alpha, b_\alpha) : M_\infty \to T_\infty$ with $\|a_\alpha\|_2, \|b_\alpha\|_2 < 1$.

Proof. (a)$\Rightarrow$(b). Let us suppose that we have a factorization (4.10) for the mapping φ, with ω contractive, and r and s completely contractive. Given a matrix space E and a complete contraction $\psi : E \to V$, we have from Lemma 5.3 that $\omega((r \circ \psi) \otimes id)$ is a strictly contractive element of

$$
(E \check{\otimes} B(K))^* = E^* \hat{\otimes} B(K)^*.
$$

The corresponding element of $\mathcal{N}(E, B(K)^*)$ is $M(\omega) \circ r \circ \psi$, since if $x \in E$, and $b \in B(K)$,

$$
M(\omega)(r(\psi(x))(b) = \omega(r(\psi(x)) \otimes b) = \omega((r \circ \psi) \otimes id)(x \otimes b)).
$$

Thus using Lemma 3.3 and the factorization (4.10),

$$
\nu(\varphi \circ \psi) = \nu(\iota_W \circ \varphi \circ \psi) = \nu(s \circ M(\omega) \circ r \circ \psi) \leq \|s\|_{cb}\, \nu(M(\omega) \circ r \circ \psi) \leq 1.
$$

(b)$\Leftrightarrow$(c) is immediate from the commutative diagram (4.4).

(c)$\Rightarrow$(d) We let I be the index set of all triples $\alpha = (E, F, k)$, where $E \subseteq V$ is finite-dimensional, $F \subseteq W$ is finite-codimensional, and $k \in \mathbb{N}$. Given such a triple α, we shall also use the notation $E = E_\alpha, F = F_\alpha$, and $k = k_\alpha$. We write $\iota_\alpha : E_\alpha \hookrightarrow V$ and $\pi_\alpha : W \to W/F_\alpha$ for the inclusion and quotient mappings. We define a partial order on I by $\alpha \preceq \alpha' = (E', F', k')$ if $E \subseteq E'$, $F' \subseteq F$, and $k \leq k'$. For each $\alpha \in I$, we let $I_\alpha = \{\alpha' \in I : \alpha \preceq \alpha'\}$, we write $\mathcal{F}_{\preceq}$ for the filter generated by these I_α's and we fix a free ultrafilter $\mathcal{U}$ on I containing $\mathcal{F}_{\preceq}$.

For each $\alpha = (E, F, k) \in I$, W/F is a finite-dimensional operator space, and thus we may identify it with a finite-dimensional subspace $G = G_\alpha$ of M_∞, and for each $n \in \mathbb{N}$, we let $\tau_n^{G_\alpha} = P_{n|G_\alpha}$ (see (5.2)). From Lemma 5.4, we may choose an integer $n(\alpha) \in \mathbb{N}$ with $\tau_{n(\alpha)}^{G_\alpha}$ invertible and with

(5.6)
$$
\left\|(\tau_{n(\alpha)}^{G_\alpha})^{-1}_{k(\alpha)}\right\| < 1 + \frac{1}{k(\alpha)}.
$$

We can choose a constant $0 < c_\alpha < 1$ so that $\tau_\alpha = c_\alpha \tau_{n(\alpha)}^{G_\alpha}$ also satisfies $\|(\tau_\alpha)^{-1}_{k(\alpha)}\| < 1 + \frac{1}{k(\alpha)}$. We have that

$$\varphi_\alpha = \tau_\alpha \circ \pi_\alpha \circ \varphi \circ \iota_\alpha$$

is a linear mapping from E_α onto a matrix space $N_\alpha \subseteq M_{n(\alpha)}$.

From (c) we see that

$$\left\| \mathrm{id} \otimes \varphi : N_\alpha^* \check{\otimes} V \to N_\alpha^* \hat{\otimes} W \right\| \le 1,$$

and thus

$$\left\| \mathrm{id} \otimes \varphi_\alpha : N_\alpha^* \check{\otimes} E_\alpha \to N_\alpha^* \hat{\otimes} N_\alpha \right\| < 1.$$

It follows that if we are given an element $\psi \in N_\alpha^* \check{\otimes} E_\alpha$, we have from (1.7) and (1.8) that

$$|\langle \varphi_\alpha, \psi \rangle| = |\mathrm{trace}\,(\mathrm{id} \otimes \varphi_\alpha)(\psi)| \le \|(\mathrm{id} \otimes \varphi_\alpha)(\psi)\| < \|\psi\|_{N_\alpha^* \check{\otimes} E_\alpha}.$$

We conclude that φ_α is a strictly contractive element in

$$(N_\alpha^* \check{\otimes} E_\alpha)^* = (N_\alpha \hat{\otimes} E_\alpha^*)^{**} = N_\alpha \hat{\otimes} E_\alpha^* = \mathcal{N}(E_\alpha, N_\alpha)$$

(the second dual of a finite-dimensional operator space coincides with itself), and from (3.3) we have a commutative diagram

$$
\begin{array}{ccc}
M_\infty & \xrightarrow{M(a_\alpha, b_\alpha)} & T_\infty \\
r_\alpha \uparrow & & \downarrow s_\alpha \\
E_\alpha & \xrightarrow{\varphi_\alpha} & N_\alpha,
\end{array}
$$

where r_α and s_α are complete contractions, and $\|a_\alpha\|_2, \|b_\alpha\|_2 \le 1$. Letting $\tilde{r}_\alpha : V \to M_\infty$ be a completely contractive extension of r_α to V, we obtain the following commutative diagram

$$
\begin{array}{ccccccccc}
 & & M_\infty & \xrightarrow{M(a_\alpha,b_\alpha)} & & & T_\infty & \xrightarrow{\tau_\alpha^{-1} \circ s_\alpha} & W/F_\alpha \\
 & \tilde{r}_\alpha \nearrow & r_\alpha \uparrow & & & & & \searrow s_\alpha & \uparrow \tau_\alpha^{-1} \\
V & \xleftrightarrow{\iota_\alpha} & E_\alpha & \xrightarrow{\varphi \circ \iota_\alpha} & W & \xrightarrow{\pi_\alpha} & W/F_\alpha & \xrightarrow{\tau_\alpha} & N_\alpha .
\end{array}
$$

We let $\tilde{r} = (\tilde{r}_\alpha) : V \to \ell^\infty(I, M_\infty)$, and $\mathcal{M} = [M(a_\alpha, b_\alpha)]$. The mappings s_α and τ_α^{-1} determine corresponding ultraproduct mappings

$$[\tau_\alpha^{-1} \circ s_\alpha] : \prod_{\mathcal{U}} T_\infty \xrightarrow{[s_\alpha]} \prod_{\mathcal{U}} N_\alpha \xrightarrow{[\tau_\alpha^{-1}]} \prod_{\mathcal{U}} W/F_\alpha,$$

where $[s_\alpha]$ is a complete contraction, and from (5.6), it is evident that the same is true for $[\tau_\alpha^{-1}]$. Finally, given $[w_\alpha + F_\alpha] \in \prod_{\mathcal{U}} W/F_\alpha$, we may assume that (w_α) is a uniformly bounded net of elements in W. Then the weak* ultralimit $\lim_{\mathcal{U}} w_\alpha$ exists in W^{**} (the unit ball is weak* compact), and we may define $\tilde{s}([w_\alpha + F_\alpha]) = \lim_{\mathcal{U}} w_\alpha$. It is easy to see that $\tilde{s} : \prod_{\mathcal{U}} W/F_\alpha \to W^{**}$ is a well-defined complete contraction such that the diagram

$$
\begin{array}{ccc}
 & \prod_{\mathcal{U}} W/F_\alpha & \\
[\pi_\alpha] \nearrow & & \searrow \tilde{s} \\
W & \xrightarrow{\iota_W} & W^{**}
\end{array}
$$

commutes. This gives us the following commutative diagram of complete contractions

$$
\begin{array}{ccccccc}
\ell^\infty(I, M_\infty) & \xrightarrow{\ \mathcal{M}\ } & \prod_{\mathcal{U}} T_\infty & \xrightarrow{[\tau_\alpha^{-1} \circ s_\alpha]} & \prod_{\mathcal{U}} W/F_\alpha \\
\scriptstyle\tilde{r} \uparrow & & & \scriptstyle[\pi_\alpha] \nearrow & \downarrow \scriptstyle\tilde{s} \\
V & \xrightarrow{\ \varphi\ } & W & \xrightarrow{\ \iota_W\ } & W^{**} \, .
\end{array}
$$

Letting $s = \tilde{s} \circ [\tau_\alpha^{-1} \circ s_\alpha]$, we obtain (5.5).

(d)⇒(a) Assuming that we have the factorization (5.5), our task is to construct from it a factorization of the form (4.10). From (5.4), we have a complete isometry

$$
\theta : \prod_{\mathcal{U}} T_\infty \to T = \ell^\infty(I, M_\infty)^* e,
$$

where e is a central projection in $\ell^\infty(I, M_\infty)^{**}$, and thus we may define a projection P_e of $\ell^\infty(I, M_\infty)^*$ onto $\prod_{\mathcal{U}} T_\infty$ by letting $P_e(f) = fe$. We may use this to elaborate (5.5) in the commutative diagram

$$
\begin{array}{ccccccc}
\ell^\infty(I, M_\infty) & \xrightarrow{\ \mathcal{M}\ } & \prod_{\mathcal{U}} T_\infty & \xrightarrow{\ \theta\ } & \ell^\infty(I, M_\infty)^* \\
\scriptstyle r \uparrow & & & \scriptstyle s \searrow & \downarrow \scriptstyle s \circ P_e \\
V & \xrightarrow{\ \varphi\ } & W & \xrightarrow{\ \iota_W\ } & W^{**} \, .
\end{array}
$$

Turning to the left side of this diagram, we may assume that V is an operator subspace of $B(H)$ and let $\iota : V \hookrightarrow B(H)$ be the inclusion mapping. We have that $r = (r_\alpha)$ where each $r_\alpha : V \to M_\infty$ is a complete contraction, and using the Arveson-Wittstock Hahn-Banach theorem, we may extend each r_α to a complete contraction $\tilde{r}_\alpha : B(H) \to M_\infty$. These determine the complete contraction $\tilde{r} = (\tilde{r}_\alpha) : B(H) \to \ell^\infty(I, M_\infty)$, and we have the commutative diagram.

$$
\begin{array}{ccccccccc}
B(H) & \xrightarrow{\ \tilde{r}\ } & \ell^\infty(I, M_\infty) & \xrightarrow{\ \mathcal{M}\ } & \prod_{\mathcal{U}} T_\infty & \xrightarrow{\ \theta\ } & \ell^\infty(I, M_\infty)^* \\
\scriptstyle \iota \uparrow & \scriptstyle r \nearrow & & & & \scriptstyle s \searrow & \downarrow \scriptstyle s \circ P_e \\
V & & & \xrightarrow{\ \varphi\ } & W & \xrightarrow{\ \iota_W\ } & W^{**}.
\end{array}
$$

For each $\alpha \in I$, we let $\omega_\alpha : B(H) \otimes M_\infty \to \mathbb{C}$ be the linear functional given by

$$
\omega_\alpha(x \otimes y) = \langle a_\alpha \tilde{r}_\alpha(x) b_\alpha, y \rangle \, .
$$

Since $\|a_\alpha\|_2, \|b_\alpha\|_2 \le 1$, it is clear that ω_α is a contractive linear functional on $B(H) \check{\otimes} M_\infty$. Then $[\omega_\alpha]$ is a contractive element in

$$
\prod_{\mathcal{U}} (B(H) \check{\otimes} M_\infty)^* \subseteq \ell^\infty(I, B(H) \check{\otimes} M_\infty)^*
$$

where we have used the corresponding identification of the ultraproduct with a subspace of $\ell^\infty(I, B(H) \check{\otimes} M_\infty)^*$ (see (5.3)). We can identify $B(H) \check{\otimes} \ell^\infty(I, M_\infty)$

with an operator subspace of $\ell^\infty(I, B(H)\check{\otimes}M_\infty)$, and we let ω be the restriction of $[\omega_\alpha]$ to $B(H) \check{\otimes} \ell^\infty(I, M_\infty)$. Then ω is a contractive linear functional on $B(H) \otimes_\vee \ell^\infty(I, M_\infty)$ such that for every $x \in B(H)$ and $(y_\alpha) \in \ell^\infty(I, M_\infty)$,

$$\omega(x \otimes (y_\alpha)) = \lim_{\mathcal{U}} \omega_\alpha(x \otimes y_\alpha) = \lim_{\mathcal{U}} \langle a_\alpha \tilde{r}_\alpha(x)b_\alpha, y_\alpha \rangle = \langle \theta \circ \mathcal{M} \circ \tilde{r}(x), (y_\alpha) \rangle.$$

This shows that $M(\omega) = \theta \circ \mathcal{M} \circ \tilde{r}$.

Finally, we let $J{:}\ell^\infty(I, M_\infty) \hookrightarrow B(K)$ be an identification of $\ell^\infty(I, M_\infty)$ with a von Neumann subalgebra of $B(K)$ for some Hilbert space K. Since $\ell^\infty(I, M_\infty)$ is injective, there is a completely contractive projection π from $B(K)$ onto $\ell^\infty(I, M_\infty)$. Taking adjoints we have that the composition

$$\ell^\infty(I, M_\infty)^* \xrightarrow{\pi^*} B(K)^* \xrightarrow{J^*} \ell^\infty(I, M_\infty)^*$$

is just the identity mapping, and we obtain the commutative diagram

$$\begin{array}{ccccc}
B(H) & \xrightarrow{M(\omega)} & \ell^\infty(I, M_\infty)^* & \xrightarrow{\pi^*} & B(K)^* \\
\iota \uparrow & & & & \downarrow s\circ P_e \circ J^* \\
V & \xrightarrow{\varphi} & W & \xstackrel{\iota_W}{\hookrightarrow} & W^{**}.
\end{array}$$

The composition $\omega \circ (\mathrm{id} \otimes \pi)$ is a contractive functional on $B(H) \check{\otimes} B(K)$; thus we may extend it to a contractive functional $\tilde{\omega}$ on $B(H \otimes K)$. For any $z \in B(K)$ we have that

$$M(\tilde{\omega})(x)(z) = \omega(x \otimes \pi(z)) = (M(\omega)(x))(\pi(z)) = (\pi^* M(\omega)(x))(z);$$

thus $\pi^* \circ M(\omega) = M(\tilde{\omega})$. We obtain the commutative diagram

$$\begin{array}{ccc}
B(H) & \xrightarrow{M(\tilde{\omega})} & B(K)^* \\
\iota \uparrow & & \searrow \tau \\
V \xrightarrow{\varphi} & W \xstackrel{\iota_W}{\hookrightarrow} & W^{**},
\end{array}$$

where $\tau = s \circ P_e \circ J^*$ is a complete contraction, and we conclude that φ is exactly integral with $\iota^{\mathrm{ex}}(\varphi) \leq 1$. □

Given operator spaces V and W, we let $\mathcal{I}^{\mathrm{ex}}(V, W)$ denote the space of all exactly integral mappings from V into W. It is easy to see from Theorem 5.5 that $\mathcal{I}^{\mathrm{ex}}(V, W)$ is an operator ideal, i.e., ι^{ex} satisfies (1.5). It was shown in [18] that ι^{ex} can also be characterized as a dual operator norm. We have from (c) of Theorem 5.5 and (5.1),

$$\iota^{\mathrm{ex}}(\varphi) = \sup\left\{ |\langle (\mathrm{id} \otimes \varphi)(u), v \rangle| \right\},$$

where the supremum is taken over all $u \in E^* \check{\otimes} V$ and $v \in E \check{\otimes} W^*$ and $\|u\|, \|v\| \leq 1$ with E an arbitrary matrix space. If we let u and v correspond to the functions $a \in \mathcal{CB}(E, V)$ and $b \in \mathcal{CB}(W, E)$, a simple calculation with elementary matrices leads to the formula

(5.7) $$\iota^{\mathrm{ex}}(\varphi) = \sup\left\{ \mathrm{trace}\,(\varphi \circ \psi) : \psi = a \circ b, \|a\|_{cb}, \|b\|_{cb} \leq 1 \right\}.$$

In general given a finite rank mapping $\psi : W \to V$, we define

$$\gamma_{SK}(\psi) = \inf \{\|a\|_{cb} \, \|b\|_{cb}\}$$

where the supremum is taken over all factorizations

$$
\begin{array}{ccc}
 & E & \\
{\scriptstyle b} \nearrow & & \searrow {\scriptstyle a} \\
W & \xrightarrow{\ \psi\ } & V
\end{array}
$$

with E a matrix space. It is easy to see that this determines a norm on $\mathcal{F}(W, V)$, and we let $\gamma_{SK}^0(W, V)$ denote the corresponding normed space. We conclude from (5.7) that we have an isometric injection

(5.8) $$\mathcal{I}^{\text{ex}}(V, W) \hookrightarrow \gamma_{SK}^0(W, V)^*,$$

and in particular, if W is finite-dimensional, we have that

(5.9) $$\mathcal{I}^{\text{ex}}(V, W) = \gamma_{SK}^0(W, V)^*.$$

We also have that the exactly integral norm is local.

PROPOSITION 5.6.   *A linear mapping $\varphi : V \to W$ is exactly integral with $\iota^{\text{ex}}(\varphi) \leq 1$ if and only if for every finite-dimensional subspace $E \subseteq V$ we have $\iota^{\text{ex}}(\varphi_{|E}) \leq 1$.*

Proof. Since $\mathcal{I}^{\text{ex}}(V, W)$ is an operator ideal, it is clear that $\iota^{\text{ex}}(\varphi_{|E}) \leq \iota^{\text{ex}}(\varphi)$. On the other hand, for any finite-dimensional subspace F of M_n and any complete contraction $\psi : F \to V$, let $E = \psi(F)$. We have

$$\nu(\varphi \circ \psi) = \nu(\varphi_{|E} \circ \psi) \leq \iota^{\text{ex}}(\varphi_{|E}).$$

It follows from Theorem 5.5 that

$$
\begin{aligned}
\iota^{\text{ex}}(\varphi) \;=\; & \sup\{\nu(\varphi \circ \psi) : \|\psi : F \to V \|_{cb} \leq 1, \; F \text{ a matrix space}\} \\
\leq\; & \sup\{\iota^{\text{ex}}(\varphi_{|E}) : E \subseteq V \text{ finite-dimensional}\}.
\end{aligned}
$$

$\square$

If V is a matrix space, it is immediate from Theorem 5.5 that $\mathcal{I}(V, W) = \mathcal{I}^{\text{ex}}(V, W)$. This is true more generally. We recall from Pisier [28] that an operator space V is called 1-*exact* if for every finite-dimensional operator space E of V and $\varepsilon > 0$ there is a matrix space F and a linear isomorphism $S : E \to F$ such that $\|S\|_{cb} \|S^{-1}\|_{cb} < 1 + \varepsilon$. The following result motivated the terminology "exact integral."

PROPOSITION 5.7.   *An operator space V is 1-exact if and only if for any operator space W,*

(5.10) $$\mathcal{I}(V, W) = \mathcal{I}^{\text{ex}}(V, W).$$

Proof. If V is 1-exact, then for any finite rank mapping $\psi : W \to V$,

$$\gamma_{SK}(\psi) = \|\psi\|_{cb},$$

since if we let $F = \psi(W)$, and we are given $\varepsilon > 0$, we may find a diagram

$$
\begin{array}{ccccc}
 & & E & & \\
 & & S \uparrow\downarrow S^{-1} & & \\
W & \xrightarrow{\psi} & F & \subseteq & V
\end{array}
$$

with E a matrix space and $\max\left\{\|S\|_{cb}, \|S^{-1}\|_{cb}\right\} < 1 + \varepsilon$. From (4.6) and (5.8) it follows that for any linear mapping $\varphi : V \to W$, we have $\iota(\varphi) = \iota^{\mathrm{ex}}(\varphi)$.

Let us suppose that (5.10) holds for all finite-dimensional subspaces $W \subseteq V$. Then fixing such a subspace, we have a norm decreasing linear isomorphism (both sides coincide with the vector space $W^* \otimes V$)

$$\theta : \gamma_{SK}^0(W, V) \to \mathcal{CB}^0(W, V).$$

But we are given that the adjoint mapping

$$\theta^* : \mathcal{I}(V, W) \to \mathcal{I}^{\mathrm{ex}}(V, W)$$

is isometric, and thus θ must itself be an isometry. Letting $j : W \hookrightarrow V$ be the inclusion mapping, it follows that for any $\varepsilon > 0$, we have a matrix space E and a commutative diagram

$$
\begin{array}{ccccc}
 & & E & & \\
 & a \nearrow & & \searrow b & \\
W & & \xrightarrow{j} & & V \,,
\end{array}
$$

where $\|a\|_{cb} \|b\|_{cb} < 1 + \varepsilon$. Thus W is 1-exact and we conclude V is exact. □

PROPOSITION 5.8.   *Given operator spaces V and W and a linear mapping $\varphi : V \to W$, we have that*

$$\iota(\varphi) \leq \iota(\varphi^*).$$

Moreover, V is locally reflexive if and only if we have the isometry

$$\iota(\varphi) = \iota(\varphi^*)$$

for all operator spaces W and linear mappings $\varphi : V \to W$.

Proof. For any finite-dimensional operator space E we have that

$$E^* \stackrel{\vee}{\otimes} V^* = (E \hat{\otimes} V)^*,$$

whereas the corresponding mapping

$$E^* \hat{\otimes} V^* \to (E \stackrel{\vee}{\otimes} V)^*$$

is norm-decreasing. From these we conclude that

$$
\begin{aligned}
\iota(\varphi) &= \sup\{\|\mathrm{id}\otimes\varphi : E^* \,\check{\otimes}\, V \to E^*\hat{\otimes}W\| : E \text{ finite-dimensional}\} \\
&= \sup\{\|(\mathrm{id}\otimes\varphi)^* : (E^*\hat{\otimes}W)^* \to (E^* \,\check{\otimes}\, V)^*\| : E \text{ finite-dimensional}\} \\
&\leq \sup\{\|\mathrm{id}\otimes\varphi^* : E \,\check{\otimes}\, W^* \to E\hat{\otimes}V^*\| : E \text{ finite-dimensional}\} \\
&= \iota(\varphi^*).
\end{aligned}
$$

If V is locally reflexive, then $E\hat{\otimes}V^* \to (E^* \,\check{\otimes}\, V)^*$ is isometric, and the above calculations show that $\iota(\varphi) = \iota(\varphi^*)$.

If W is a finite-dimensional operator space, we have the isometries

$$
\mathcal{CB}(V,W) \cong V^* \,\check{\otimes}\, W \cong W \,\check{\otimes}\, V^* = \mathcal{CB}(W^*,V^*),
$$

and from Lemma 4.1, we have the isometry

$$
\mathcal{I}(W^*,V^*) = \mathcal{N}(W^*,V^*).
$$

Therefore, if for every $\varphi : V \to W$, we have $\iota(\varphi) = \iota(\varphi^*)$, then from Lemma 3.2 we have the isometries

$$
\mathcal{I}(V,W) = \mathcal{I}(W^*,V^*) = \mathcal{N}(W^*,V^*) = \mathcal{N}(V,W),
$$

and we conclude from Proposition 4.4 that V is locally reflexive. □

6. The local reflexivity principle for von Neumann preduals

THEOREM 6.1.   *For any C^*-algebra A, A^* is a locally reflexive operator space.*

Proof. From Proposition 4.4, it suffices to show that we have the isometry

$$
\mathcal{I}(A^*,F) = \mathcal{N}(A^*,F)
$$

for all finite-dimensional operator spaces F. Given $\varphi : A^* \to F$, it is trivial from the definition that $\iota(\varphi) \leq \nu(\varphi)$. On the other hand, we have the mappings

$$
\mathcal{I}(A^*,F) \xrightarrow{\;S\;} \mathcal{I}^{\mathrm{ex}}(F^*,A^{**}) \cong \mathcal{I}(F^*,A^{**}) \cong \mathcal{N}(F^*,A^{**}) \overset{S^{-1}}{\cong} \mathcal{N}(A^*,F),
$$

where $S(\varphi) = \varphi^*$ is contractive (Lemma 5.1), and the second, third and fourth identifications are proved in Lemmas 5.2, 4.1, and 3.2. Therefore, we must have $\nu(\varphi) = \iota(\varphi)$. □

It was noted in [10, p. 185], that any subspace of a locally reflexive operator space is again locally reflexive. In particular, if R is a von Neumann algebra, then we may identify R_* with an operator subspace of R^*, and we obtain the following result.

COROLLARY 6.2.   *For any von Neumann algebra R, the predual R_* is a locally reflexive operator space.*

Guided by the classical theory, we may prove stronger versions of approximation. We begin by recalling "Helly's lemma" (see [5, p. 73]).

LEMMA 6.3.   *Suppose that E is a Banach space and that L is a finite-dimensional subspace of E^*. Then given any element $u \in E^{**}$ and $\varepsilon > 0$, there exists an element u_0 of E such that $\|u_0\| < (1 + \varepsilon)\|u\|$ and*

$$\langle u, h \rangle = \langle u_0, h \rangle$$

for all $h \in L$.

LEMMA 6.4.   *Suppose that V is a locally reflexive operator space, and that $F \subseteq V^{**}$ and $N \subseteq V^*$ are finite-dimensional. Then for each $\varepsilon > 0$ there exists a mapping $\varphi : F \to V$ such that $\|\varphi\|_{cb} < 1 + \varepsilon$, and*

$$\langle \varphi(v), f \rangle = \langle v, f \rangle$$

for all $f \in N$.

Proof. Local reflexivity implies that we have the isometry (1.2). We can regard the inclusion mapping $\iota : F \to V^{**}$ as a contractive element of $F^* \check{\otimes} V^{**}$, and $L = F \otimes N$ as a finite-dimensional subspace of $(F^* \check{\otimes} V)^*$. From Helly's lemma, we can choose an element $\varphi \in F^* \check{\otimes} V$ such that $\|\varphi\|_{cb} < 1 + \varepsilon$, and

$$\langle \varphi(v), f \rangle = \langle \varphi, v \otimes f \rangle = \langle \iota, v \otimes f \rangle = \langle v, f \rangle$$

for all $v \in F$ and $f \in N^*$.   □

The following result of Pisier (see [29, Lemma 7.1.4]), is the analogue of a well-known theorem in Banach space theory.

LEMMA 6.5.   *Suppose that V is an operator space, and that $v_i \in V$, $f_i \in V^*$ ($i = 1, \ldots, n$) are biorthogonal, i.e., $f_i(v_j) = \delta_{i,j}$. Then given $\varepsilon > 0$ and elements w_i such that*

(6.1)
$$\sum_i \|f_i\| \|v_i - w_i\| < \varepsilon,$$

it follows that there is a complete isomorphism $\varphi : V \to V$ such that $\varphi(v_i) = w_i$, where $\|\varphi\|_{cb} \leq 1 + \varepsilon$ and $\|\varphi^{-1}\|_{cb} \leq (1 - \varepsilon)^{-1}$.

We say that an operator space V is *strongly locally reflexive*, if given a finite-dimensional subspace $F \subseteq V^{**}$ and a finite-dimensional subspace $N \subseteq V^*$, there exists a complete isomorphism φ of F onto a subspace E of V such that

(a) $\|\varphi\|_{cb}, \|\varphi^{-1}\|_{cb} < 1 + \varepsilon$,

(b) $\langle \varphi(v), f \rangle = \langle v, f \rangle$ for all $v \in F$ and $f \in N$, and

(c) $\varphi(v) = v$ for all $v \in F \cap V$.

We will see in the next section that since the C^*-algebra $M_\infty = (K_\infty)^{**}$ contains every finite-dimensional operator space, K_∞ is not strongly locally reflexive.

THEOREM 6.6.   *Suppose that V is a locally reflexive operator space for which there exists a completely isometric injection*

$$\theta : V^{**} \to B(H)$$

which satisfies the $\mathrm{W}^*\mathrm{AP}$ *(see §2). Then V is strongly locally reflexive.*

Proof. In order to simplify the notation, we will assume that $V^{**} \subseteq B(H)$, and that we have a net of weak* continuous finite rank complete contractions $t_\lambda : V^{**} \to B(H)$ for which

$$\|t_\lambda(v^{**}) - v^{**}\| \to 0$$

for each $v^{**} \in V^{**}$.

We fix $0 < \varepsilon < 1$ and finite-dimensional subspaces $F \subseteq V^{**}$ and $N \subseteq V^*$, and we let $0 < \delta \leq \varepsilon/3n^3$, where $n = \dim F$. Then we can choose a complete contraction $t = t_\lambda$ such that

(6.2) $$\|t(v) - v\| < \delta \|v\|$$

for all $v \in F$. Letting W be the range of t, we have $t : V^{**} \to W$ and $t^* : W^* \to V^*$.

From Lemma 6.4, there exists a mapping $\varphi : F \to V$ such that

(6.3) $$\|\varphi\|_{cb} < 1 + \delta,$$

and

(6.4) $$\langle \varphi(v), f \rangle = \langle v, f \rangle$$

for all $v \in F$ and $f \in t^*(W^*) + N$. We let $C \subseteq \mathcal{CB}(F, V)$ be the convex set of all mappings $\varphi : F \to V$ satisfying (6.3) and (6.4). We let $F_0 = F \cap V$, and $\iota_0 : F_0 \to V$ be the inclusion mapping. We let $C_0 \subseteq \mathcal{CB}(F_0, V)$ denote the convex set of all mappings $\varphi \circ \iota_0$, where $\varphi \in C$. We claim that ι_0 is in the point-norm closure of C_0. This is apparent since if we are given an arbitrary finite-dimensional subspace $G \subseteq V^*$, then our previous argument shows that there is a mapping $\varphi' : F \to V$ satisfying

$$\|\varphi'\|_{cb} < 1 + \delta,$$

and

$$\langle \varphi'(v), f \rangle = \langle v, f \rangle$$

for all $v \in F$ and $f \in t^*(W^*) + N + G$. Since $\iota_0(F_0) \subseteq V$, we can suitably choose a net of φ' such that $\varphi' \circ \iota_0$ converges to ι_0 in the point-weak topology. The usual convexity argument, implies that ι_0 is in the point-norm closure of C_0, and since F_0 is finite-dimensional, we may choose a mapping $\varphi \in C$ satisfying (6.3) and (6.4), for which $\|\iota_0 - \varphi \circ \iota_0\| < \delta$ and thus

$$\|\iota_0 - \varphi \circ \iota_0\| < n\delta.$$

For all $v \in F$ and $f \in W^*$, we have that

$$\langle t(\varphi(v)), f \rangle = \langle \varphi(v), t^*(f) \rangle = \langle v, t^*(f) \rangle = \langle t(v), f \rangle;$$

thus

(6.5) $$t(\varphi(v)) = t(v)$$

for all $v \in F$.

We next perturb φ in order to satisfy (c). It follows from [11, Lemma 5.2] that there is a projection P of F onto $F_0 = V \cap F$ with $1 \le \|P\|_{cb} \le n^2$. Then

$$\varphi_1 = (\iota_0 - \varphi)P + \varphi : F \to V$$

is a completely bounded mapping such that $\varphi_1(v_0) = v_0$ for $v_0 \in F_0$, and if $v \in F$,

$$\langle \varphi P(v), f \rangle = \langle P(v), f \rangle$$

and thus

$$\langle \varphi_1(v), f \rangle = \langle \varphi(v), f \rangle = \langle v, f \rangle$$

for $f \in N$, i.e., φ_1 satisfies (b) and (c). We also have that

(6.6) $$\|\varphi_1\|_{cb} \le \|\iota_0 - \varphi \circ \iota_0\|_{cb} \|P\|_{cb} + (1 + \delta) \le \delta n^3 + (1 + \delta) < 1 + \varepsilon.$$

We let E be the range of φ_1. We must show that φ_1 is almost a complete isometry of F onto E. Let us assume that $v_1, \ldots, v_n$ is an Auerbach basis for F with bi-orthogonal dual basis f_i (i.e., $\|v_i\| = 1$, $\|f_i\| = 1$ and $f_i(v_j) = \delta_{ij}$). For each i, we have from (6.5) that

$$
\begin{aligned}
\|v_i - t\varphi_1(v_i)\| &\le \|v_i - t\varphi(v_i)\| + \|t\varphi(v_i) - t\varphi_1(v_i)\| \\
&\le \|v_i - t(v_i)\| + \|t\|_{cb} \|\varphi \circ \iota_0 - \iota_0\| \|P\|_{cb} \\
&\le \delta + (1 + \delta)\delta \|P\|_{cb} \\
&\le \delta + 2\delta \|P\|_{cb} \le 3\delta \|P\|_{cb} < 3\delta n^2.
\end{aligned}
$$

Thus we have that

$$\sum_i \|v_i - t\varphi_1(v_i)\| \|f_i\| \le 3\delta n^3 < \varepsilon.$$

From Lemma 6.5, we may find a mapping $s : B(H) \to B(H)$ for which

$$st\varphi_1(v_i) = v_i.$$

and $\|s\|_{cb} \leq (1 - \varepsilon)^{-1}$. It follows that $\varphi_1^{-1} = st|_E$, and since t is completely contractive,

$$(6.7) \qquad \left\|\varphi_1^{-1}\right\|_{cb} \leq (1 - \varepsilon)^{-1}.$$

Then φ_1 will also satisfy (a). $\qquad \square$

THEOREM 6.7. *If R is a von Neumann algebra, then R_* is strongly locally reflexive.*

Proof. We have from Theorem 2.1 that any complete isometry $R^* \to B(H)$ has the W*AP, and thus since R_* is locally reflexive (Cor. 6.2), the result follows from Theorem 6.6. $\qquad \square$

7. Finite representability and factorizations

The Banach space notion of *finite representability* was introduced by R.C. James in [16]. It has proved to be quite useful (for example, see Heinrich [15]), and it has an obvious analogue in operator space theory.

Given operator spaces E and F and $\varepsilon > 0$, we write $E \overset{1+\varepsilon}{\cong} F$ if E is $(1+\varepsilon)$-completely isomorphic to F; i.e., there is a linear isomorphism $T : E \to F$ such that $\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon$. This is equivalent to saying that the completely bounded Banach-Mazur distance introduced in [28] satisfies

$$d_{cb}(E, F) = \inf \left\{ \|S\|_{cb} \left\|S^{-1}\right\|_{cb} : S : E \cong F \right\} < 1 + \varepsilon.$$

We write $E \overset{1+\varepsilon}{\subseteq} F$ if there is a subspace $F_0 \subseteq F$ with $E \overset{1+\varepsilon}{\cong} F_0$.

Let us suppose that we are given a family of operator spaces $\mathcal{W}$. We say that an operator space V is *finitely representable in $\mathcal{W}$* if for every finite-dimensional subspace E of V and $\varepsilon > 0$, $E \overset{1+\varepsilon}{\subseteq} W$ for some $W \in \mathcal{W}$. If $\mathcal{W} = \{W\}$, we simply say that V is finitely representable in W. Two operator spaces V and W are *finitely equivalent* if each is finitely representable in the other. Turning to an important example, from the work of Kirchberg and Pisier ([20], [28]) we see that an operator space V is 1-exact if and only if it is finitely representable in $\{M_n\}_{n \in \mathbb{N}}$, or equivalently, it is finitely representable in K_∞. The following is an immediate consequence of Theorem 6.7.

COROLLARY 7.1. *If R is an arbitrary von Neumann algebra, then R^* is finitely equivalent to R_*, and if A is a C^*-algebra, A^{***} is finitely equivalent to A^*.*

We note that Corollary 7.1 is false for C^*-algebras. We have, for example, that although K_∞ is finitely representable in $\{M_n\}_{n \in \mathbb{N}}$, that is not the case for $M_\infty = K_\infty^{**}$.

We may use Corollary 7.1 to formulate other invariants for the preduals of von Neumann algebras which are preserved by taking second duals. To illustrate this, we again have from [11, Prop. 4.3] (or see below) that for any Hilbert space H, $T(H)$ is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$. Thus an operator space V is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$ if and only if V is finitely representable in T_∞. We have from Corollary 7.1 that the predual R_* of a von Neumann algebra R is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$ if and only if that is true for R^*.

A C^*-algebra A is said to have the *weak expectation property* (WEP) of Lance [23] if given any faithful representation $A \hookrightarrow B(H)$, there is a completely positive contraction P of $B(H)$ into the weak closure $\bar{A}$ such that $P(a) = a$ for all $a \in A$. It is well known that nuclear C^*-algebras and injective C^*-algebras have the WEP. A C^*-algebra A has the WEP if and only if given the universal representation $A \subseteq \mathcal{B}(H)$, there is a complete contraction $P : \mathcal{B}(H) \to \bar{A}$ such that $P(a) = a$ for all $a \in A$. A is said to have the QWEP if it is a C^*-algebraic quotient of a WEP algebra. It has been conjectured that *all* C^*-algebras have the QWEP. Kirchberg [21] has shown that this problem is equivalent to Connes' question of whether any II_1 factor on a separable Hilbert space can be realized as a subalgebra of the ultrapower N^ω, where N is the hyperfinite II_1 factor.

COROLLARY 7.2. *If A is a* QWEP *C^*-algebra, then A^* is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$.*

Proof. Let us suppose that A is a unital WEP C^*-algebra, and that $A \subseteq B(H)$. We let $\rho : B(H)^* \to A^*$ denote the restriction mapping. We have from [8] Theorem 6.3 (i) that there is a dilation for A^*, i.e., a completely positive state preserving mapping $\theta : A^* \to B(H)^*$ such that $\rho \circ \theta = \mathrm{id}$. The adjoint mapping $\theta^* : B(H)^{**} \to A^{**}$ is again a completely positive mapping and it preserves the identity. It follows that θ^* is completely contractive, and thus the same is true for θ. Therefore, θ is a complete isometry, and we may identify A^* with a (complemented) subspace of $B(H)^*$. Since $T(H)$ and thus $B(H)^*$ are finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, the same is true for A^*.

Given a nonunital WEP C^*-algebra A, we have from the discussion in [21, pp. 458–459], that the unital extension A_1 has the WEP. It follows that A_1^* is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, and since A^* may be identified with a subspace of A_1^*, the same is true for A^*.

Finally suppose that A is a QWEP C^*-algebra. If we let J be a closed ideal in a WEP C^*-algebra B with $A = B/J$, we may identify A^* with the annihilator $J^\perp \subseteq B^*$. Since B^* is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, the same is true for A^*. □

COROLLARY 7.3. *If R is a* QWEP *von Neumann algebra, then R_* is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$.*

As in the classical case, the theory of finite representability is related to ultraproducts.

PROPOSITION 7.4.   *Let $\mathcal{W}$ be a family of operator spaces. If V is finitely representable in $\mathcal{W}$, then there exists an index set I and an ultrafilter $\mathcal{U}$ on I such that for each $\alpha \in I$ there exists an operator space $W_\alpha \in \mathcal{W}$ such that V is completely isometric to a subspace of $\prod_{\mathcal{U}} W_\alpha$.*

Proof. Let I be the collection of all pairs $\alpha = (E, \varepsilon)$ with E a finite-dimensional subspace of V and $\varepsilon > 0$. For convenience, we write $\alpha = (E_\alpha, \varepsilon_\alpha)$. There is a canonical partial order on I given by $\alpha \preceq \alpha'$ if and only if $E_\alpha \subseteq E'_\alpha$ and $\varepsilon_\alpha \geq \varepsilon'_\alpha$. For each $\alpha \in I$, we let $I_\alpha = \{\alpha' \in I : \alpha \preceq \alpha'\}$. The collection $\mathcal{I}$ of all such sets I_α is a filter on I, and it is evident that

$$\bigcap_{\alpha \in I} I_\alpha = \emptyset.$$

We let $\mathcal{U}$ on I be a free ultrafilter containing $\mathcal{I}$.

For each $\alpha = (E_\alpha, \varepsilon_\alpha) \in I$, there exists an element, say W_α, in $\mathcal{W}$ such that E_α is $(1+\varepsilon_\alpha)$-completely bounded isomorphic to a finite-dimensional subspace F_α of W_α. For each such α, we choose a completely bounded isomorphism $s_\alpha : E_\alpha \to F_\alpha$ such that $\|s_\alpha\|_{cb} \|s_\alpha^{-1}\|_{cb} < 1 + \varepsilon_\alpha$. We extend s_α (nonlinearly) to V by letting $s_\alpha(v) = 0$ if $v \notin E_\alpha$. It is easy to verify that the mapping

$$J : V \to \prod_{\mathcal{U}} W_\alpha; \ v \mapsto (\widehat{s_\alpha(v)})$$

is linear and completely isometric from V into $\prod_{\mathcal{U}} W_\alpha$. $\square$

In contrast to Banach space theory, the converse of above theorem fails for operator spaces. Any separable operator space E may be realized as an operator subspace of $\prod_{\mathcal{U}} M_n$ for a free ultrafilter $\mathcal{U}$ on $\mathbb{N}$, whereas only exact operator subspaces of $\prod_{\mathcal{U}} M_n$ are finitely representable in $\{M_n\}_{n \in \mathbb{N}}$. On the other hand, we have a necessary and sufficient condition for the operator subspaces of the ultraproduct of $\{T_n\}_{n \in \mathbb{N}}$.

THEOREM 7.5.   *An operator space V is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$ if and only if there exists an index set I and a free ultrafilter $\mathcal{U}$ on I such that V is completely isometric to a subspace of $\prod_{\mathcal{U}} T_{n(\alpha)}$.*

Proof. If V is finitely representable in $\{T_n\}_{n \in \mathbb{N}}$, then from Proposition 7.4 there exists an index set I and a free ultrafilter $\mathcal{U}$ on I such that V is completely isometric to a subspace of $\prod_{\mathcal{U}} T_{n(\alpha)}$. Conversely, we have a sequence

of inclusions

$$\prod_{\mathcal{U}} T_{n(\alpha)} \hookrightarrow \prod_{\mathcal{U}} M^*_{n(\alpha)} \hookrightarrow \ell^\infty(I, M_{n(\alpha)})^*.$$

Since $\ell^\infty(I, M_{n(\alpha)})$ is an injective von Neumann algebra, its dual space is finitely representable in $\{T_n\}_{n\in\mathbb{N}}$, and thus the same for any subspace of $\prod_{\mathcal{U}} T_\infty$. $\qquad\square$

It follows from the above argument and [21, Prop. 1.3] that the converses of Corollaries 7.2 and 7.3 are also true. Thus the QWEP conjecture mentioned above is true if and only if the predual of any von Neumann algebra is finitely representable in $\{T_n\}_{n\in\mathbb{N}}$. We conclude with an application of strong local reflexivity to a factorization theorem. The following result was first demonstrated by the second author [18] in response to a question posed by G. Pisier. The proof used the Kaplansky density theorem, the fact that the completely 1-summing norm π_1 is in trace duality with the mapping norm γ_K defined by factorizations through K_∞, and Pisier's ultraproduct factorization characterization for π_1. In turn, this result was used to show that $T(H)$ is locally reflexive, the first instance of Theorem 6.7. Here we proceed in the reverse direction, using the strong local reflexivity theorem to prove the factorization result.

THEOREM 7.6. *Suppose that V and W are finite-dimensional operator spaces and that $\varphi : V \to W$ is a linear mapping. If φ has a completely bounded factorization*

(7.1)
$$
\begin{array}{ccc}
 & B(H) & \\
{\scriptstyle r}\nearrow & & \searrow{\scriptstyle s} \\
V & \xrightarrow{\ \varphi\ } & W,
\end{array}
$$

then for each $\varepsilon > 0$, there is a factorization

(7.2)
$$
\begin{array}{ccc}
 & M_n & \\
{\scriptstyle \tilde{r}}\nearrow & & \searrow{\scriptstyle \tilde{s}} \\
V & \xrightarrow{\ \varphi\ } & W
\end{array}
$$

for some M_n such that $\|\tilde{r}\|_{cb}\|\tilde{s}\|_{cb} < \|r\|_{cb}\|s\|_{cb} + \varepsilon$.

Proof. Assume that we have the commutative diagram of complete contractions (7.1). Taking adjoints, we obtain the completely contractive diagram

$$
\begin{array}{ccc}
 & B(H)^* & \\
{\scriptstyle s^*}\nearrow & & \searrow{\scriptstyle r^*} \\
W^* & \xrightarrow{\ \varphi^*\ } & V^* \, .
\end{array}
$$

Then $E = s^*(W^*)$ is a finite-dimensional subspace of $B(H)^*$, and $F = r(V)$ is a finite-dimensional subspace of $B(H)$. It follows from Theorem 6.7 that

there exists a completely bounded isomorphism ψ from E onto a subspace $E_\psi = \psi(E)$ of $T(H)$ such that $\|\psi\|_{cb} \|\psi^{-1}\|_{cb} < (1+\varepsilon)^{1/2}$ and

$$\langle \psi(s^*(f)), r(v) \rangle = \langle s^*(f), r(v) \rangle$$

for all $f \in W^*$ and $v \in V$.

It is shown in [11] (and it follows very easily by truncation from Lemma 6.5) that $T(H)$ is a $\mathcal{T}$-space, i.e., for any finite-dimensional subspace G of $T(H)$ and $\varepsilon > 0$, there exists an $n \in \mathbb{N}$ and a subspace $\tilde{G}$ of $T(H)$ containing G such that $\tilde{G} \overset{1+\varepsilon}{\cong} T_n$. Applying this to E_ψ, we have a subspace $\tilde{E}_\psi \supseteq E_\psi$ and a linear isomorphism $t : \tilde{E}_\psi \to T_n$ for which $\|t\|, \|t^{-1}\| < (1+\varepsilon)^{1/2}$. Then $t \circ \psi \circ s^* : W^* \to T_n$ and $r^* \circ t^{-1} : T_n \to V^*$ are completely bounded maps such that

$$\|t \circ \psi \circ s^*\|_{cb} \left\| r^* \circ t^{-1} \right\|_{cb} \leq \|t\|_{cb} \|\psi\|_{cb} \left\| t^{-1} \right\|_{cb} < 1 + \varepsilon.$$

Putting $\tilde{r} = (r^* \circ t^{-1})^* : V \to M_n$ and $\tilde{s} = (t \circ \psi \circ s^*)^* : M_n \to W$, we get

$$
\begin{aligned}
\langle f, \tilde{s} \circ \tilde{r}(v) \rangle &= \left\langle (r^* \circ t^{-1}) \circ (t \circ \psi \circ s^*)(f), v \right\rangle \\
&= \langle r^* \circ \psi \circ s^*(f), v \rangle \\
&= \langle \psi \circ s^*(f), r(v) \rangle \\
&= \langle s^*(f), r(v) \rangle \\
&= \langle f, s \circ r(v) \rangle
\end{aligned}
$$

for all $f \in W^*$ and $v \in V$. This shows that $\tilde{s} \circ \tilde{r} = s \circ r$ and that $\|\tilde{r}\|_{cb} \|\tilde{s}\|_{cb} < 1 + \varepsilon$. $\qquad\square$

THEOREM 7.7. *Suppose that V and W are finite-dimensional operator spaces and A is a C*-algebra having WEP. If $\varphi : V \to W$ has a completely bounded factorization*

(7.3)
$$
\begin{array}{ccc}
 & A & \\
{\scriptstyle r} \nearrow & & \searrow {\scriptstyle s} \\
V & \xrightarrow{\ \varphi\ } & W,
\end{array}
$$

then for each $\varepsilon > 0$, there is a factorization

(7.4)
$$
\begin{array}{ccc}
 & M_n & \\
{\scriptstyle \tilde{r}} \nearrow & & \searrow {\scriptstyle \tilde{s}} \\
V & \xrightarrow{\ \varphi\ } & W
\end{array}
$$

for some M_n such that $\|\tilde{r}\|_{cb} \|\tilde{s}\|_{cb} < \|r\|_{cb} \|s\|_{cb} + \varepsilon$.

Proof. Using the universal representation, we may identify A^{**} with a von Neumann algebra on a Hilbert space H, and we may fix a complete contraction $P : B(H) \to A^{**}$ such that $P(a) = a$ for all $a \in A$. Then we can assume that

$$\iota_A \circ r : V \to A \hookrightarrow A^{**} \subseteq B(H)$$

is a completely bounded mapping from V into $B(H)$. Taking the second adjoint of s, we get a completely bounded mapping

$$s^{**} : A^{**} \to W^{**} = W$$

such that $s^{**} \circ \iota_A = s$. This gives us a completely bounded factorization

$$
\begin{array}{ccc}
 & B(H) & \\
\iota_A \circ r \nearrow & & \searrow s^{**} \circ P \\
V & \xrightarrow{\ \varphi\ } & W,
\end{array}
$$

where $\|\iota_A \circ r\|_{cb} = \|r\|_{cb}$ and $\|s^{**} \circ P\|_{cb} = \|s\|_{cb}$. Then the result follows from Theorem 7.6. $\square$

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