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A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

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Introduction

Motivated by basic questions in Minkowski geometry, H. Busemann and C. M. Petty posed ten problems about convex bodies in 1956 (see [BP]). The first problem, now known as the Busemann-Petty problem, states:

If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and for each hyperplane H through the origin the volumes of their central slices satisfy

$$\mathrm{vol}_{n-1}(K \cap H) < \mathrm{vol}_{n-1}(L \cap H),$$

does it follow that the volumes of the bodies themselves satisfy

$$\mathrm{vol}_n(K) < \mathrm{vol}_n(L)?$$

The problem is trivially positive in $\mathbb{R}^2$. However, a surprising negative answer for $n \geq 12$ was given by Larman and Rogers [LR] in 1975. Subsequently, a series of contributions were made to reduce the dimensions to $n \geq 5$ by a number of authors (see [Ba], [Bo], [G2], [Gi], [Pa], and [Z1]). That is, the problem has a negative answer for $n \geq 5$. See [G3] for a detailed description. It was proved by Gardner [G1] that the problem has a positive answer for $n = 3$. The case of $n = 4$ was considered in [Z1]. But the answer to this case in [Z1] is not correct. This paper presents the correct solution, namely, the Busemann-Petty problem has a positive solution in $\mathbb{R}^4$, which, together with results of other cases, brings the Busemann-Petty problem to a conclusion.

A key step to the solution of the Busemann-Petty problem is the discovery of the relation of the problem to intersection bodies by Lutwak [Lu]. An origin-symmetric convex body K in $\mathbb{R}^n$ is called an intersection body if its radial function ρ_K is the spherical Radon transform of a nonnegative measure μ on the unit sphere S^{n-1}. The value of the radial function of K, $\rho_K(u)$, in the direction $u \in S^{n-1}$, is defined as the distance from the center of K to its boundary in that direction. When μ is a positive continuous function, K is

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called the intersection body of a star body. The notion of intersection body was introduced by Lutwak [Lu] who proved that the Busemann-Petty problem has a positive answer if K is an intersection body in $\mathbb{R}^n$. Based on this relation, a positive answer to the Busemann-Petty problem in $\mathbb{R}^3$ was given by Gardner [G1] who showed that all origin-symmetric convex bodies in $\mathbb{R}^3$ are intersection bodies.

The relation of the Busemann-Petty problem to intersection bodies proved by Lutwak can be formulated as: A negative answer to the Busemann-Petty problem is equivalent to the existence of convex nonintersection bodies (see [G2] and [Z2]). The author attempted in [Z1] to give a negative answer for all dimensions ≥ 4 by trying to show that cubes in $\mathbb{R}^n$ ($n \geq 4$) are not intersection bodies (see Theorem 5.3 in [Z1]). However, there is an error in Lemma 5.1 of [Z1]. It affects only Theorems 5.3 and 5.4 there. The correct version of Theorem 5.3 is that no cube in $\mathbb{R}^n$ ($n > 4$) is an intersection body. This follows immediately from Theorem 6.1 of [Z1] which says that no generalized cylinder in $\mathbb{R}^n$ ($n > 4$) is an intersection body. Note that the proof of Theorem 6.1 in [Z1] holds for intersection bodies, although the definition of intersection body of a star body was the one used in [Z1]. Therefore, Theorem 5.4 in [Z1] should have stated: The Busemann-Petty problem has a negative solution in $\mathbb{R}^n$ for $n > 4$.

In his important work [K1], Koldobsky applied the Fourier transform to the study of intersection bodies. In [K2], he showed that cubes in $\mathbb{R}^4$ are intersection bodies. It was this result that exposed the error mentioned above and led to the present paper, which presents the correct solution to the Busemann-Petty problem in $\mathbb{R}^4$. One of the key ideas in the proof, previously employed by Gardner [G1], is the use of cylindrical coordinates in computing the inverse spherical Radon transform.

1. The inverse Radon transform on S^3 and intersection bodies in $\mathbb{R}^4$

The radial function ρ_L of a star body L is defined by

$$\rho_L(u) = \max\{r \geq 0 : ru \in L\}, \quad u \in S^{n-1}.$$

It is required in this paper that the radial function is continuous and even. For basic facts about star bodies and convex bodies, see [G3] and [S].

For a continuous function f on S^{n-1}, the spherical Radon transform $\mathrm{R}f$ of f is defined by

$$(\mathrm{R}f)(u) = \int_{S^{n-1}\cap u^\perp} f(v)dv, \quad u \in S^{n-1},$$

where $u^{\perp}$ is the $(n-1)$-dimensional subspace orthogonal to the unit vector u. Since the spherical Radon transform is self-adjoint, one can define the Radon transform $\mathrm{R}\mu$ for a measure μ on S^{n-1} by

$$\langle \mathrm{R}\mu, f\rangle = \langle \mu, \mathrm{R}f\rangle.$$

The intersection body $\mathrm{I}L$ of star body L is defined by

$$\rho_{\mathrm{I}L}(u) = \mathrm{vol}_{n-1}(L \cap u^{-1}) = \mathrm{R}\left(\frac{1}{n-1}\rho_L^{n-1}\right)(u), \quad u \in S^{n-1}.$$

An origin-symmetric convex body K is called the *intersection body of a star body* if there exists a star body L so that $K = \mathrm{I}L$. That is, the inverse spherical Radon transform $\mathrm{R}^{-1}\rho_K$ is a positive continuous function. A slight extension of this definition is that an origin-symmetric convex body K is called an *intersection body* if the inverse spherical Radon transform $\mathrm{R}^{-1}\rho_K$ is a non-negative measure.

Let Δ be the Laplacian on the unit sphere S^3. Helgason's inversion formula for the Radon transform R on S^3 is (see [H, p. 161])

$$\frac{1}{16\pi^2}(1-\Delta)\mathrm{R}\mathrm{R} = 1.$$

It implies that

$$\mathrm{R}^{-1}\rho_K = \frac{1}{16\pi^2}\mathrm{R}(1-\Delta)\rho_K \tag{1}$$

for an origin-symmetric convex body K in $\mathbb{R}^4$. This formula shows that $\mathrm{R}^{-1}\rho_K$ is continuous when ρ_K is of class C^2. The following lemma provides an inversion formula which gives the positivity of $\mathrm{R}^{-1}\rho_K$.

Let K be an origin-symmetric convex body in $\mathbb{R}^4$, and let $A_u(z)$ be the volume of $K \cap (zu + u^{\perp})$, where z is real and $u \in S^3$.

Lemma 1. *If K is an origin-symmetric convex body in $\mathbb{R}^4$ whose boundary is of class C^2, then*

$$(\mathrm{R}^{-1}\rho_K)(u) = -\frac{1}{16\pi^2}A_u''(0), \quad u \in S^3. \tag{2}$$

Proof. By rotation, it suffices to prove (2) for the north pole of S^3. From Helgason's inversion formula (1), the inverse spherical Radon transform of ρ_K, $f = \mathrm{R}^{-1}\rho_K$, is a continuous function when ρ_K is of class C^2. Let

$$u = u(v,\phi) = (v\sin\phi, \cos\phi), \quad u \in S^3, \quad v \in S^2, \quad 0 \le \phi \le \pi,$$

and let $\rho_K(v,\phi) = \rho_K(u)$ be the radial function of K. Define

$$\begin{aligned}\bar\rho_K(\phi) &= \int_{S^2} \rho_K(v,\phi)dv,\\ \bar f(\phi) &= \int_{S^2} f(u)dv,\\ r(v,\phi) &= \rho_K(v,\phi)\sin\phi,\\ \bar r(\phi) &= \bar\rho_K(\phi)\sin\phi.\end{aligned}$$

Consider $\bar\rho_K$ and $\bar f$ as functions on S^3 which are SO(3) invariant. Since the spherical Radon transform is intertwining, we have $\bar\rho_K = \mathrm{R}\bar f$ (for a simple proof, see [G3, Th C.2.8]). From this and Lemma 2.1 in [Z1], or Theorem C.2.9 in [G3], we obtain

$$\bar\rho_K(\phi) = \frac{4\pi}{\sin\phi}\int_{\frac{\pi}{2}-\phi}^{\frac{\pi}{2}} \bar f(\psi)\sin\psi d\psi.$$

Taking the derivative on both sides of this equation gives

$$(\bar\rho_K(\phi)\sin\phi)' = 4\pi\bar f(\frac{\pi}{2}-\phi)\sin(\frac{\pi}{2}-\phi).$$

It follows that

$$4\pi\bar f(0) = \lim_{\phi\to\frac{\pi}{2}} \frac{(\bar\rho_K(\phi)\sin\phi)'}{\cos\phi} = -\bar r''(\frac{\pi}{2}).$$

Since $\frac{1}{4\pi}\bar f(0)$ is the value of f at the north pole, we obtain

$$f(u_0) = -\frac{1}{16\pi^2}\bar r''(\frac{\pi}{2}), \tag{3}$$

where u_0 is the north pole of S^3.

Consider the variable z defined by $z = \rho_K\cos\phi$. Then $\tan\phi = \frac{r}{z}$. Differentiating this equation and using $\frac{1}{\cos^2\phi} = 1+\tan^2\phi = 1+\frac{r^2}{z^2}$ give

$$z^2 + r^2 = z\frac{dr}{d\phi} - r\frac{dz}{d\phi}. \tag{4}$$

This yields

$$\left.\frac{dz}{d\phi}\right|_{\phi=\frac{\pi}{2}} = -r(v,\frac{\pi}{2}). \tag{5}$$

Differentiating (4) gives

$$2z\frac{dz}{d\phi} + 2r\frac{dr}{d\phi} = z\frac{d^2r}{d\phi^2} - r\frac{d^2z}{d\phi^2}. \tag{6}$$

From (5),

$$\left.\frac{dr}{d\phi}\right|_{\phi=\frac{\pi}{2}} = \left.\frac{dr}{dz}\frac{dz}{d\phi}\right|_{\phi=\frac{\pi}{2}} = -r\left.\frac{dr}{dz}\right|_{z=0}. \tag{7}$$

From (6) and (7),

$$\left.\frac{d^2z}{d\phi^2}\right|_{\phi=\frac{\pi}{2}} = 2r\left.\frac{dr}{dz}\right|_{z=0}. \tag{8}$$

From (5), (8), and

$$\frac{d^2r}{d\phi^2} = \frac{d^2r}{dz^2}\left(\frac{dz}{d\phi}\right)^2 + \frac{dr}{dz}\frac{d^2z}{d\phi^2},$$

we have

$$\begin{aligned}\left.\frac{d^2r}{d\phi^2}\right|_{\phi=\frac{\pi}{2}} &= \left.\frac{d^2r}{dz^2}\right|_{z=0} r(v,\frac{\pi}{2})^2 + 2r(v,\frac{\pi}{2})\left(\frac{dr}{dz}\right)^2_{z=0} \\ &= \left(r^2\frac{d^2r}{dz^2}\right)_{z=0} + \left(2r\left(\frac{dr}{dz}\right)^2\right)_{z=0} \\ &= \frac{1}{3}\left.\frac{d^2r^3}{dz^2}\right|_{z=0}.\end{aligned} \tag{9}$$

Integrating both sides of (9) over S^2 with respect to v gives

$$\int_{S^2}\left.\frac{d^2r}{d\phi^2}(v,\phi)\right|_{\phi=\frac{\pi}{2}} dv = \frac{1}{3}\int_{S^2}\left.\frac{d^2r^3}{dz^2}(v,z)\right|_{z=0} dv.$$

Since K has C^2 boundary, one can interchange the second order derivative and the integral on each side of the last equation. We obtain

$$\left.\frac{d^2}{d\phi^2}\bar{r}(\phi)\right|_{\phi=\frac{\pi}{2}} = \frac{d^2}{dz^2}\left(\frac{1}{3}\int_{S^2} r^3(v,z)dv\right)_{z=0}.$$

Note that the 3-dimensional volume of the intersection of the hyperplane $x_4 = z$ with the convex body K, denoted by $A_{u_0}(z)$, is given by

$$A_{u_0}(z) = \frac{1}{3}\int_{S^2} r^3(v,z)dv.$$

Therefore, we have

$$\bar{r}''(\frac{\pi}{2}) = A''_{u_0}(0). \tag{10}$$

Formula (2) follows from (3) and (10). □

Recently, Gardner, Koldobsky and Schlumprecht [GKS] have generalized the formula (2) to n dimensions by using techniques of the Fourier transform. A different proof of their formulas is given by Barthe, Fradelizi and Maurey [BFM].

THEOREM 2. *If K is an origin-symmetric convex body in $\mathbb{R}^4$ whose boundary is of class C^2 and has positive curvature, then K is an intersection body of a star body.*

Proof. By the Brunn-Minkowski inequality and the strict convexity of K, $A(z)^{\frac{1}{3}}$ is strictly concave. When one slices a symmetric convex body by parallel hyperplanes, the central section has maximal volume. Hence, $A'(0) = 0$. It follows that

$$A''(0) = 3A(0)^{\frac{2}{3}}\left(A(z)^{\frac{1}{3}}\right)''_{z=0} < 0.$$

By Lemma 1, $\mathrm{R}^{-1}\rho_K$ is a positive continuous function. Therefore, K is the intersection body of a star body. □

When a convex body is identified with its radial function, the class of intersection bodies is closed under the uniform topology. Since every origin-symmetric convex body can be approximated by origin-symmetric convex bodies whose boundaries are of class C^2 and have positive curvatures, we obtain:

THEOREM 3. *All origin-symmetric convex bodies in $\mathbb{R}^4$ are intersection bodies.*

Theorem 3 is proved for convex bodies of revolution by Gardner [G2] and by Zhang [Z1], and is proved for cubes and other special cases by Koldobsky [K2]. In higher dimensions, the situation is different. For example, it is proved by Zhang [Z1] that generalized cylinders in $\mathbb{R}^n$ $(n > 4)$ are not intersection bodies, and is proved by Koldobsky [K1] that the unit balls of finite dimensional subspaces of an L_p space, $1 \leq p \leq 2$, are intersection bodies. In three dimensions, Gardner [G1] proved that all origin-symmetric convex bodies in $\mathbb{R}^3$ are intersection bodies. One can also prove this by Theorem 3 and a result of Fallert, Goodey and Weil [FGW] which says that central sections of intersection bodies are again intersection bodies. An intersection body may not be the intersection body of a star body. It is shown by Zhang [Z4] that no polytope in $\mathbb{R}^n$ $(n > 3)$ is an intersection body of a star body. Campi [C] is able to prove a complete result which says that no polytope in $\mathbb{R}^n$ $(n > 2)$ is an intersection body of a star body.

2. A positive solution to the Busemann-Petty problem in $\mathbb{R}^4$

The following relation of the Busemann-Petty problem to intersection bodies was proved by Lutwak [Lu].

THEOREM 4 (Lutwak). *The Busemann-Petty problem has a positive solution if the convex body with smaller cross sections is an intersection body.*

From Theorems 3 and 4, we conclude:

THEOREM 5. *The Busemann-Petty problem in $\mathbb{R}^4$ has a positive solution.*

From Theorem 3 and Corollary 2.19 in [Z2], we have the following corollary about the maximal cross section of a convex body.

COROLLARY 6. *If K is an origin-symmetric convex body in $\mathbb{R}^4$, then*

$$\mathrm{vol}_4(K)^{\frac{3}{4}} \leq \frac{3}{8}(\sqrt{2}\pi)^{\frac{1}{2}} \max_{u\in S^3} \mathrm{vol}_3(K \cap u^{\perp}) \tag{11}$$

with equality if and only if K is a ball.

Inequality (11) implies that, in $\mathbb{R}^4$, balls attain the minmax of the volume of central hyperplane sections of origin-symmetric convex bodies with fixed volume. The corresponding inequality in $\mathbb{R}^3$ to inequality (11) was proved by Gardner (see [G3, Th. 9.4.11]). However, it is no longer the case in higher dimensions at least for $n \geq 7$. Ball [Ba] showed that cubes are counterexamples for $n \geq 10$. Giannopoulos [Gi] showed that certain cylinders are counterexamples for $n \geq 7$. The following question, known as the slicing problem, has been of interest (see [MP] for details):

Does there exist a positive constant c independent of the dimension n so that

$$\mathrm{vol}_n(K)^{\frac{n-1}{n}} \leq c \max_{u\in S^{n-1}} \mathrm{vol}_{n-1}(K \cap u^{\perp})$$

for every origin-symmetric convex body K in $\mathbb{R}^n$?

3. The generalized Busemann-Petty problem

Besides considering hyperplane sections, one can also consider intermediate sections of convex bodies. For a fixed integer $1 < i < n$, the Busemann-Petty problem has the following generalization (see Problem 8.2 in [G3]):

If K and L are origin-symmetric convex bodies in $\mathbb{R}^n$, and for every i-dimensional subspace H the volumes of sections satisfy

$$\mathrm{vol}_i(K \cap H) < \mathrm{vol}_i(L \cap H),$$

does it follow that the volumes of the bodies themselves satisfy

$$\mathrm{vol}_n(K) < \mathrm{vol}_n(L)?$$

When $i = n - 1$, this is the Busemann-Petty problem. It turns out that the solution to the generalized Busemann-Petty problem depends strongly on the dimension i of the sections of convex bodies. It is proved by Bourgain and Zhang [BoZ] that the solution is negative when $3 < i < n$. The generalized Busemann-Petty problem has a positive solution when K belongs to a certain class of convex bodies, called i-intersection bodies, which contains all intersection bodies (see Theorem 5 in [Z3] and Lemma 6.1 in [GrZ]). In particular, when K is an intersection body, the generalized Busemann-Petty problem has a positive solution. From this fact and Theorem 3, we have:

THEOREM 7. *The generalized Busemann-Petty problem in $\mathbb{R}^4$ has a positive solution.*

It might be still true that the generalized Busemann-Petty problem has a positive solution when $i = 2, 3$, and $n \geq 5$. This remains open.

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