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Certain non-linear differential polynomials sharing a non zero polynomial

Sujoy Majumder
Department of Mathematics,
Katwa College, Katwa,
West Bengal-713130, India
email: sujoy.katwa@gmail.com,
sr.koshigram@gmail.com,
smajumder05@yahoo.in

1 Introduction, definitions and results

In this paper by meromorphic functions we shall always mean meromorphic functions in the complex plane.

Let f and g be two non-constant meromorphic functions and let a be a finite complex number. We say that f and g share a CM, provided that $f-a$ and $g-a$ have same zeros with same multiplicities. Similarly, we say that f and g share a IM, provided that $f-a$ and $g-a$ have same zeros ignoring multiplicities. In addition we say that f and g share ∞ CM, if $1/f$ and $1/g$ share 0 CM, and we say that f and g share ∞ IM, if $1/f$ and $1/g$ share 0 IM.

We adopt the standard notations of value distribution theory (see [6]). We denote by $T(r)$ the maximum of $T(r,f)$ and $T(r,g)$. The notation $S(r)$ denotes any quantity satisfying $S(r) = o(T(r))$ as $r \longrightarrow \infty$, outside of a possible exceptional set of finite linear measure.

A meromorphic function $a(z)$ is called a small function with respect to f, provided that $T(r,a) = S(r,f)$.

Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. Let $a(z)$ be a small function with respect to $f(z)$ and $g(z)$. We say that $f(z)$ and $g(z)$ share

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$a(z)$ CM (counting multiplicities) if $f(z)-a(z)$ and $g(z)-a(z)$ have same zeros with same multiplicities and we say that $f(z)$, $g(z)$ share $a(z)$ IM (ignoring multiplicities) if we do not consider the multiplicities.

Throughout this paper, we need the following definition.

$$\Theta(a;f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}(r,a;f)}{T(r,f)},$$

where a is a value in the extended complex plane.

In 1959, W. K. Hayman (see [6], Corollary of Theorem 9) proved the following theorem.

Theorem A *Let* f *be a transcendental meromorphic function and* n (≥ 3) *is an integer. Then* $f^n f' = 1$ *has infinitely many solutions.*

Fang and Hua [3], Yang and Hua [16] got a unicity theorem respectively corresponding Theorem A.

Theorem B *Let* f *and* g *be two non-constant entire (meromorphic) functions,* $n \geq 6$ (≥ 11) *be a positive integer. If* $f^n f'$ *and* $g^n g'$ *share* 1 *CM, then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(c_1c_2)^{n+1}c^2 = -1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^{n+1} = 1$.

Noting that $f^n(z)f^{'}(z) = \frac{1}{n+1}(f^{n+1}(z))^{'}$, Fang [4] considered the case of k-th derivative and proved the following results.

Theorem C *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n > 2k+4$. *If* $(f^n)^{(k)}$ *and* $(g^n)^{(k)}$ *share* 1 *CM, then either* $f(z) = c_1 e^{cz}$, $g(z) = c_2 e^{-cz}$, *where* c_1, c_2 *and* c *are three constants satisfying* $(-1)^k (c_1c_2)^n (nc)^{2k} = 1$ *or* $f \equiv tg$ *for a constant* t *such that* $t^n = 1$.

Theorem D *Let* f *and* g *be two non-constant entire functions, and let* n, k *be two positive integers with* $n > 2k+8$. *If* $(f^n(z)(f(z)-1))^{(k)}$ *and* $(g^n(z)(g(z)-1))^{(k)}$ *share* 1 *CM, then* $f(z) \equiv g(z)$.

In 2008, X. Y. Zhang and W. C. Lin [21] proved the following result.

Theorem E *Let* f *and* g *be two non-constant entire functions, and let* n, m *and* k *be three positive integers with* $n > 2k+m+4$. *If* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share* 1 *CM, then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f,g) = 0$, *where* $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega - 1)^m$.

In 2001 an idea of gradation of sharing of values was introduced in ([7], [8]) which measures how close a shared value is to being share CM or to being shared IM. This notion is known as weighted sharing and is defined as follows.

Definition 1 [7, 8] *Let* k *be a nonnegative integer or infinity. For* $a \in \mathbb{C} \cup \{\infty\}$ *we denote by* $E_k(a; f)$ *the set of all* a*-points of* f*, where an* a*-point of multiplicity* m *is counted* m *times if* $m \le k$ *and* $k+1$ *times if* $m > k$*. If* $E_k(a; f) = E_k(a; g)$*, we say that* f*,* g *share the value* a *with weight* k*.*

The definition implies that if f, g share a value a with weight k then z_0 is an a-point of f with multiplicity m $(\le k)$ if and only if it is an a-point of g with multiplicity m $(\le k)$ and z_0 is an a-point of f with multiplicity m $(> k)$ if and only if it is an a-point of g with multiplicity n $(> k)$, where m is not necessarily equal to n.

We write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly if f, g share (a, k), then f, g share (a, p) for any integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞) respectively.

If $a(z)$ is a small function with respect to $f(z)$ and $g(z)$, we define that $f(z)$ and $g(z)$ share $a(z)$ IM or $a(z)$ CM or with weight l according as $f(z) - a(z)$ and $g(z) - a(z)$ share $(0, 0)$ or $(0, \infty)$ or $(0, l)$ respectively.

In 2008, L. Liu [12] proved the following.

Theorem F *Let* f *and* g *be two non-constant entire functions, and let* n*,* m *and* k *be three positive integers such that* $n > 5k + 4m + 9$*. If* $E_0(1, [f^n(f-1)^m]^{(k)}) = E_0(1, [g^n(g-1)^m]^{(k)})$ *then either* $f \equiv g$ *or* f *and* g *satisfy the algebraic equation* $R(f, g) = 0$*, where* $R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m$*.*

Recently P. Sahoo [14] proved the following result.

Theorem G *Let* f *and* g *be two transcendental meromorphic functions and* n (≥ 1)*,* k (≥ 1)*,* m (≥ 0) *and* $l(\ge 0)$ *be four integers. Let* $[f^n(f-1)^m]^{(k)}$ *and* $[g^n(g-1)^m]^{(k)}$ *share* (b, l) *for a nonzero constant* b*. Then*

(1) *when* $m = 0$*, if* $f(z) \ne \infty$*,* $g(z) \ne \infty$ *and* $l \ge 2$*,* $n > 3k + 8$ *or* $l = 1$*,* $n > 5k + 10$ *or* $l = 0$*,* $n > 9k + 14$*, then either* $f \equiv tg$*, where* t *is a constant satisfying* $t^n = 1$*, or* $f(z) = c_1 e^{cz}$*,* $g(z) = c_2 e^{-cz}$*, where* c_1*,* c_2 *and* c *are three constants satisfying* $(-1)^k (c_1 c_2)^n (nc)^{2k} = b^2$*,*

(2) *when* $m = 1$ *and* $\Theta(\infty; f) > \frac{2}{n}$ *then either* $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)} \equiv b^2$*, except for* $k = 1$ *or* $f \equiv g$*, provided one of* $l \ge 2$*,* $n > 3k + 11$ *or* $l = 1$*,* $n > 5k + 14$ *or* $l = 0$*,* $n > 9k + 20$ *holds; and*

(3) *when* $m \ge 2$*, and* $l \ge 2$*,* $n > 3k + m + 10$ *or* $l = 1$*,* $n > 5k + 2m + 12$ *or* $l = 0$*,* $n > 9k + 4m + 16$*, then either* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv b^2$ *except for* $k = 1$ *or* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation*

$R(f,g)=0$, *where*

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

It is quite natural to ask the following questions.
Question 1: Can lower bound of n be further reduced in Theorems F, G?
Question 2: Can one remove the condition $f\neq\infty$, $g\neq\infty$ when $m=0$ in Theorem G?

In this paper, taking the possible answer of the above questions into background we obtain the following results which improve and generalize Theorems F, G.

Theorem 1 *Let* f *and* g *be two transcendental meromorphic functions and let* $p(z)$ *be a nonzero polynomial with* $\deg(p)=l$. *Suppose* $[f^n(f-1)^m]^{(k)}-p$ *and* $[g^n(g-1)^m]^{(k)}-p$ *share* $(0,k_1)$, *where* $n(\geq 1)$, $k(\geq 1)$, $m(\geq 0)$ *are three integers. Now when one of the following conditions holds:*

(i) $k_1\geq 2$ *and* $n>3k+m+8(=s_2)$;
(ii) $k_1=1$ *and* $n>4k+\frac{3m}{2}+9(=s_1)$;
(iii) $k_1=0$ *and* $n>9k+4m+14(=s_0)$;

then the following conclusions occur

(1) *when* $m=0$, *then either* $f\equiv tg$, *where* t *is a constant satisfying* $t^n=1$, *or if* $p(z)$ *is not a constant and* $n>\max\{s_i,2k+2l-1\}$, $i=0,1,2$, *then* $f(z)=c_1e^{cQ(z)}$, $g(z)=c_2e^{-cQ(z)}$, *where* $Q(z)=\int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2(c_1c_2)^n=-1$, *if* $p(z)$ *is a nonzero constant* b, *then* $f(z)=c_3e^{dz}$, $g(z)=c_4e^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k(c_3c_4)^n(nd)^{2k}=b^2$;
(2) *when* $m=1$ *and* $\Theta(\infty;f)+\Theta(\infty;g)>\frac{4}{n}$, *then either* $[f^n(f-1)]^{(k)}[g^n(g-1)]^{(k)}\equiv p^2$, *except for* $k=1$ *or* $f\equiv g$;
(3) *when* $m\geq 2$, *then either* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2$ *except for* $k=1$ *or* $f\equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f,g)=0$, *where*

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

In addition, when f *and* g *share* $(\infty,0)$, *then the possibility* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)}\equiv p^2$ *does not occur for* $m\geq 1$.

Remark 1 *When* f *and* g *share* ∞ *IM then the conditions (i), (ii) and (iii) of Theorem 1 will be replaced by respectively* $l \geq 2$ *and* $n > 3k + m + 7$, $l = 1$ *and* $n > 4k + \frac{3m}{2} + 8$ *and* $l = 0$ *and* $n > 9k + 4m + 13$.

Theorem 2 *Let* f *and* g *be two transcendental entire functions and let* $p(z)$ *be a nonzero polynomial with* $\deg(p) = l$. *Suppose* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0, k_1)$, *where* $n\ (\geq 1)$, $k\ (\geq 1)$, $m\ (\geq 0)$ *are three integers. Now when one of the following conditions holds:*

(i) $k_1 \geq 2$ *and* $n > 2k + m + 4 (= s_2)$;

(ii) $k_1 = 1$ *and* $n > \frac{5k+3m+9}{2} (= s_1)$;

(iii) $k_1 = 0$ *and* $n > 5k + 4m + 7 (= s_0)$;

then the following conclusions occur

(1) *when* $m = 0$, *then either* $f \equiv tg$, *where* t *is a constant satisfying* $t^n = 1$, *or if* $p(z)$ *is not a constant and* $n > \max\{s_i, k + 2l\}$, $i = 0, 1, 2$, *then* $f(z) = c_1 e^{cQ(z)}$, $g(z) = c_2 e^{-cQ(z)}$, *where* $Q(z) = \int_0^z p(z)dz$, c_1, c_2 *and* c *are constants such that* $(nc)^2(c_1c_2)^n = -1$,
if $p(z)$ *is a nonzero constant* b, *then* $f(z) = c_3 e^{dz}$, $g(z) = c_4 e^{-dz}$, *where* c_3, c_4 *and* d *are constants such that* $(-1)^k (c_3c_4)^n (nd)^{2k} = b^2$;

(2) *when* $m = 1$ then $f \equiv g$;

(3) *when* $m \geq 2$, *then either* $f \equiv g$ *or* f *and* g *satisfying the algebraic equation* $R(f, g) = 0$, *where*

$$R(\omega_1, \omega_2) = \omega_1^n(\omega_1 - 1)^m - \omega_2^n(\omega_2 - 1)^m.$$

We now explain some definitions and notations which are used in the paper.

Definition 2 [10] *Let* p *be a positive integer and* $a \in \mathbb{C} \cup \{\infty\}$.

(i) $N(r, a; f \mid\geq p)$ $(\overline{N}(r, a; f \mid\geq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not less than* p.

(ii) $N(r, a; f \mid\leq p)$ $(\overline{N}(r, a; f \mid\leq p))$ *denotes the counting function (reduced counting function) of those* a*-points of* f *whose multiplicities are not greater than* p.

Definition 3 {11, cf.[18]} *For* $a \in \mathbb{C} \cup \{\infty\}$ *and a positive integer* p *we denote by* $N_p(r, a; f)$ *the sum* $\overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + \ldots \overline{N}(r, a; f \mid\geq p)$. *Clearly* $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 4 *Let* $a, b \in \mathbb{C} \cup \{\infty\}$. *Let* p *be a positive integer. We denote by* $\overline{N}(r, a; f \mid \geq p \mid g = b)$ *(*$\overline{N}(r, a; f \mid \geq p \mid g \neq b)$*) the reduced counting function of those* a*-points of* f *with multiplicities* $\geq p$*, which are the* b*-points (not the* b*-points) of* g.

Definition 5 {cf.[1], 2} *Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p*, a* 1*-point of* g *with multiplicity* q*. We denote by* $\overline{N}_L(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p > q$*, by* $N_E^{1)}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q = 1$ *and by* $\overline{N}_E^{(2}(r, 1; f)$ *the counting function of those* 1*-points of* f *and* g *where* $p = q \geq 2$*, each point in these counting functions is counted only once. In the same way we can define* $\overline{N}_L(r, 1; g)$*,* $N_E^{1)}(r, 1; g)$*,* $\overline{N}_E^{(2}(r, 1; g)$.

Definition 6 {cf.[1], 2} *Let* k *be a positive integer. Let* f *and* g *be two non-constant meromorphic functions such that* f *and* g *share the value* 1 *IM. Let* z_0 *be a* 1*-point of* f *with multiplicity* p*, a* 1*-point of* g *with multiplicity* q*. We denote by* $\overline{N}_{f>k}(r, 1; g)$ *the reduced counting function of those* 1*-points of* f *and* g *such that* $p > q = k$. $\overline{N}_{g>k}(r, 1; f)$ *is defined analogously.*

Definition 7 *[7, 8] Let* f*,* g *share a value* a *IM. We denote by* $\overline{N}_*(r, a; f, g)$ *the reduced counting function of those* a*-points of* f *whose multiplicities differ from the multiplicities of the corresponding* a*-points of* g.

Clearly $\overline{N}_*(r, a; f, g) \equiv \overline{N}_*(r, a; g, f)$ *and* $\overline{N}_*(r, a; f, g) = \overline{N}_L(r, a; f) + \overline{N}_L(r, a; g)$.

2 Lemmas

Let F and G be two non-constant meromorphic functions defined in $\mathbb{C}$. We denote by H the function as follows:

$$H = \left(\frac{F''}{F'} - \frac{2F'}{F-1}\right) - \left(\frac{G''}{G'} - \frac{2G'}{G-1}\right). \tag{1}$$

Lemma 1 [15] *Let* f *be a non-constant meromorphic function and let* $a_n(z)(\not\equiv 0)$*,* $a_{n-1}(z)$*, ... ,* $a_0(z)$ *be meromorphic functions such that* $T(r, a_i(z)) = S(r, f)$ *for* $i = 0, 1, 2, ..., n$. *Then*

$$T(r, a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f + a_0) = nT(r, f) + S(r, f).$$

Lemma 2 [20] *Let* f *be a non-constant meromorphic function, and* p, k *be positive integers. Then*

$$N_p\left(r,0;f^{(k)}\right) \le T\left(r,f^{(k)}\right) - T(r,f) + N_{p+k}(r,0;f) + S(r,f), \tag{2}$$

$$N_p\left(r,0;f^{(k)}\right) \le k\overline{N}(r,\infty;f) + N_{p+k}(r,0;f) + S(r,f). \tag{3}$$

Lemma 3 [9] *If* $N(r,0;f^{(k)} \mid f \ne 0)$ *denotes the counting function of those zeros of* $f^{(k)}$ *which are not the zeros of* f, *where a zero of* $f^{(k)}$ *is counted according to its multiplicity, then*

$$N(r,0;f^{(k)} \mid f \ne 0) \le k\overline{N}(r,\infty;f) + N(r,0;f \mid< k) + k\overline{N}(r,0;f \mid\ge k) + S(r,f).$$

Lemma 4 [11] *Let* f_1 *and* f_2 *be two non-constant meromorphic functions satisfying* $\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$ *for* $i = 1,2$. *If* $f_1^s f_2^t - 1$ *is not identically zero for arbitrary integers* s *and* $t(|s|+|t|>0)$, *then for any positive* ε, *we have*

$$N_0(r,1;f_1,f_2) \le \varepsilon T(r) + S(r;f_1,f_2),$$

where $N_0(r,1;f_1,f_2)$ *denotes the deduced counting function related to the common* 1*-points of* f_1 *and* f_2 *and* $T(r) = T(r,f_1) + T(r,f_2)$, $S(r;f_1,f_2) = o(T(r))$ *as* $r \longrightarrow \infty$ *possibly outside a set of finite linear measure.*

Lemma 5 [6] *Suppose that* f *is a non-constant meromorphic function,* $k \ge 2$ *is an integer. If*

$$N(r,\infty,f) + N(r,0;f) + N(r,0;f^{(k)}) = S(r,\frac{f'}{f}),$$

then $f(z) = e^{az+b}$, *where* $a \ne 0$, b *are constants.*

Lemma 6 [5] *Let* $f(z)$ *be a non-constant entire function and let* $k \ge 2$ *be a positive integer. If* $f(z)f^{(k)}(z) \ne 0$, *then* $f(z) = e^{az+b}$, *where* $a \ne 0, b$ *are constant.*

Lemma 7 [19] *Let* f *be a non-constant meromorphic function, and let* k *be a positive integer. Suppose that* $f^{(k)} \not\equiv 0$, *then*

$$N(r,0;f^{(k)}) \le N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 8 *Let* f *and* g *be two non-constant meromorphic functions. Let* n (≥ 1), k (≥ 1) *and* m (≥ 0) *be three integers such that* $n > 3k + m + 1$. *If* $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$, *then* $f^n(f-1)^m \equiv g^n(g-1)^m$.

Proof. We have $[f^n(f-1)^m]^{(k)} \equiv [g^n(g-1)^m]^{(k)}$. Integrating we get

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)} + c_{k-1}.$$

If possible suppose $c_{k-1} \neq 0$. Now in view of Lemma 2 for $p = 1$ and using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r,f) \\
\leq\ & T(r,[f^n(f-1)^m]^{(k-1)}) - \overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + N_k(r,0;f^n(f-1)^m) \\
& + S(r,f) \\
\leq\ & \overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + \overline{N}(r,\infty;f) + \overline{N}(r,c_{k-1};[f^n(f-1)^m]^{(k-1)}) \\
& -\overline{N}(r,0;[f^n(f-1)^m]^{(k-1)}) + N_k(r,0;f^n(f-1)^m) + S(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,0;[g^n(g-1)^m]^{(k-1)}) + k\overline{N}(r,0;f) + N(r,0;(f-1)^m) \\
& + S(r,f) \\
\leq\ & (k+1+m)\ T(r,f) + (k-1)\overline{N}(r,\infty;g) + N_k(r,0;g^n(g-1)^m) + S(r,f) \\
\leq\ & (k+1+m)\ T(r,f) + k\ \overline{N}(r,\infty;g) + k\ \overline{N}(r,0;g) + N(r,0;(g-1)^m) \\
& +S\ (r,f) \\
\leq\ & (k+1+m)\ T(r,f) + (2k+m)\ T(r,g) + S(r,f) + S(r,g) \\
\leq\ & (3k+2m+1)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g) \leq (3k+2m+1)\ T(r) + S(r).$$

Combining these we get

$$(n-m-3k-1)\ T(r) \leq S(r),$$

which is a contradiction since $n > 3k+m+1$. Therefore $c_{k-1} = 0$ and so

$$[f^n(f-1)^m]^{(k-1)} \equiv [g^n(g-1)^m]^{(k-1)}.$$

Proceeding in this way we obtain

$$[f^n(f-1)^m]' \equiv [g^n(g-1)^m]'.$$

Integrating we get

$$f^n(f-1)^m \equiv g^n(g-1)^m + c_0.$$

If possible suppose $c_0 \neq 0$. Now using second fundamental theorem we get

$$\begin{aligned}
& (n+m)T(r,f) \\
\leq\ & \overline{N}(r,0;f^n(f-1)^m) + \overline{N}(r,\infty;f^n(f-1)^m) + \overline{N}(r,c_0;f^n(f-1)^m) + S(r,f) \\
\leq\ & \overline{N}(r,0;f) + mT(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g^n(g-1)^m) + S(r,f) \\
\leq\ & (m+1)\ T(r,f) + \overline{N}(r,\infty;f) + \overline{N}(r,0;g) + m\ T(r,g) + S(r,f) \\
\leq\ & (3+2m)\ T(r) + S(r).
\end{aligned}$$

Similarly we get

$$(n+m)\ T(r,g) \leq (3+2m)\ T(r) + S(r).$$

Combining these we get

$$(n-3-m)\ T(r) \leq S(r),$$

which is a contradiction since $n > 4+m$. Therefore $c_0 = 0$ and so

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

This proves the Lemma. □

Lemma 9 *Let* f, g *be two transcendental meromorphic functions, let* $n(\geq 1)$, $m(\geq 0)$ *and* $k(\geq 1)$ *be three integers with* $n > k+2$. *If* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0,0)$, *where* $p(z)$ *is a non zero polynomial, then* $T(r,f) = O(T(r,g))$ *and* $T(r,g) = O(T(r,f))$.

Proof. In view of Lemmas 1, 2 for $p = 1$ and using second fundamental theorem for small function (see [17]) we get

$$\begin{aligned}
& (n+m)T(r,f) = T(r,f^n(f-1)^m) + O(1) \\
\leq\ & T(r,[f^n(f-1)^m]^{(k)}) - \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m) \\
& + S(r,f) \\
\leq\ & \overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)}) \\
& -\overline{N}(r,0;[f^n(f-1)^m]^{(k)}) + N_{k+1}(r,0;f^n(f-1)^m) + (\varepsilon + o(1))T(r,f) \\
\leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,p;[f^n(f-1)^m]^{(k)}) + (k+1)\overline{N}(r,0;f) + N(r,0;(f-1)^m) \\
& +(\varepsilon + o(1))T(r,f) \\
\leq\ & (k+2+m)\ T(r,f) + \overline{N}(r,p;[g^n(g-1)^m]^{(k)}) + (\varepsilon + o(1))T(r,f) \\
\leq\ & (k+2+m)\ T(r,f) + (k+1)(n+m)\ T(r,g) + (\varepsilon + o(1))T(r,f),
\end{aligned}$$

i.e.,

$$(n-k-2)\ T(r,f) \le (k+1)(n+m)\ T(r,g) + (\varepsilon + o(1))T(r,f),$$

for all $\varepsilon > 0$. Take $\varepsilon < 1$. Since $n > k+2$, we have $T(r,f) = O(T(r,g))$. Similarly we have $T(r,g) = O(T(r,f))$. This completes the proof. □

Lemma 10 *Let* f, g *be two transcendental meromorphic functions and let* $F = \frac{[f^n(f-1)^m]^{(k)}}{p}$, $G = \frac{[g^n(g-1)^m]^{(k)}}{p}$, *where* $p(z)$ *is a non zero polynomial and* $n(\ge 1)$, $k(\ge 1)$ *and* $m(\ge 0)$ *are three integers such that* $n > 3k+m+3$. *If* $H \equiv 0$, *then* $[f^n(f-1)^m]^{(k)} - p$ *and* $[g^n(g-1)^m]^{(k)} - p$ *share* $(0,\infty)$ *as well as one of the following conclusions occur*

(i) $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2$;

(ii) $f^n(f-1)^m \equiv g^n(g-1)^m$.

Proof. Let $P(w) = (w-1)^m$. Then $F = \frac{[f^nP(f)]^{(k)}}{p}$ and $G = \frac{[g^nP(g)]^{(k)}}{p}$. Since $H \equiv 0$, by integration we get

$$\frac{1}{F-1} \equiv \frac{BG + A - B}{G-1}, \tag{4}$$

where A, B are constants and $A \ne 0$. From (4) it is clear that F and G share $(1,\infty)$. We now consider following cases.

Case 1. Let $B \ne 0$ and $A \ne B$.

If $B = -1$, then from (4) we have

$$F \equiv \frac{-A}{G-A-1}.$$

Therefore

$$\overline{N}(r, A+1; G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned}
&(n+m)\ T(r,g)\\
\le\ & T(r,G) + N_{k+1}(r,0;g^nP(g)) - \overline{N}(r,0;G) + S(r,g)\\
\le\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r,A+1;G) + N_{k+1}(r,0;g^nP(g))\\
& - \overline{N}(r,0;G) + S(r,g)\\
\le\ & \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + \overline{N}(r,\infty;f) + S(r,g)\\
\le\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + N_{k+1}(r,0;g^n) + N_{k+1}(r,0;P(g)) + S(r,g)\\
\le\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + S(r,g)\\
\le\ & T(r,f) + (k+2+m)\ T(r,g) + S(r,f) + S(r,g).
\end{aligned}$$

Without loss of generality, we suppose that there exists a set I with infinite measure such that $T(r,f) \leq T(r,g)$ for $r \in I$.
So for $r \in I$ we have

$$(n-k-3)\, T(r,g) \leq S(r,g),$$

which is a contradiction since $n > k+3$.
If $B \neq -1$, from (4) we obtain that

$$F - (1 + \frac{1}{B}) \equiv \frac{-A}{B^2[G + \frac{A-B}{B}]}.$$

So

$$\overline{N}(r, \frac{(B-A)}{B}; G) = \overline{N}(r,\infty;F) = \overline{N}(r,\infty;f) + \overline{N}(r,0;p).$$

Using Lemmas 1, 2 and the same argument as used in the case when $B = -1$ we can get a contradiction.
Case 2. Let $B \neq 0$ and $A = B$.
If $B = -1$, then from (4) we have

$$FG \equiv 1,$$

i.e.,

$$[f^n P(f)]^{(k)} [g^n P(g)]^{(k)} \equiv p^2,$$

i.e.,

$$[f^n (f-1)^m][g^n (g-1)^m] \equiv p^2.$$

If $B \neq -1$, from (4) we have

$$\frac{1}{F} \equiv \frac{BG}{(1+B)G - 1}.$$

Therefore

$$\overline{N}(r, \frac{1}{1+B}; G) = \overline{N}(r,0;F).$$

So in view of Lemmas 1, 2 and the second fundamental theorem we get

$$\begin{aligned} & (n+m)\, T(r,g) \\ \leq\ & T(r,G) + N_{k+1}(r,0;g^n P(g)) - \overline{N}(r,0;G) + S(r,g) \\ \leq\ & \overline{N}(r,\infty;G) + \overline{N}(r,0;G) + \overline{N}(r, \frac{1}{1+B}; G) + N_{k+1}(r.0; g^n P(g)) \\ & - \overline{N}(r,0;G) + S(r,g) \\ \leq\ & \overline{N}(r,\infty;g) + (k+1)\overline{N}(r,0;g) + T(r,P(g)) + \overline{N}(r,0;F) + S(r,g) \end{aligned}$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;g)+(k+1)\overline{N}(r,0;g)+T(r,P(g))+(k+1)\overline{N}(r,0;f)+T(r,P(f)) \\
&\quad +k\overline{N}(r,\infty;f)+S(r,f)+S(r,g) \\
&\leq (k+2+m)\,T(r,g)+(2k+1+m)\,T(r,f)+S(r,f)+S(r,g).
\end{aligned}$$

So for $r \in I$ we have

$$(n-3k-3-m)\,T(r,g) \leq S(r,g),$$

which is a contradiction since $n > 3k+3+m$.
Case 3. Let $B = 0$. From (4) we obtain

$$F \equiv \frac{G+A-1}{A}. \tag{5}$$

If $A \neq 1$, then from (5) we obtain

$$\overline{N}(r,1-A;G) = \overline{N}(r,0;F).$$

We can similarly deduce a contradiction as in Case 2. Therefore $A = 1$ and from (5) we obtain

$$F \equiv G,$$

i.e.,

$$[f^nP(f)]^{(k)} \equiv [g^nP(g)]^{(k)}.$$

Then by Lemma 8 we have

$$f^nP(f) \equiv g^nP(g), \tag{6}$$

i.e.,

$$f^n(f-1)^m \equiv g^n(g-1)^m.$$

□

Lemma 11 *Let f, g be two transcendental meromorphic functions, $p(z)$ be a non-zero polynomial with $\deg(p(z)) = l$, n, k be two positive integers. Let $[f^n]^{(k)} - p$ and $[g^n]^{(k)} - p$ share $(0,\infty)$. Suppose $[f^n]^{(k)}[g^n]^{(k)} \equiv p^2$,*

(i) *if $p(z)$ is not a constant and $n > 2k+2l-1$, then $f(z) = c_1e^{cQ(z)}$, $g(z) = c_2e^{-cQ(z)}$, where $Q(z) = \int_0^z p(z)dz$, c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$,*

(ii) *if* $p(z)$ *is a nonzero constant* b *and* $n > 2k$*, then* $f(z) = c_3e^{cz}$*,* $g(z) = c_4e^{-cz}$*, where* c_3*,* c_4 *and* c *are constants such that* $(-1)^k(c_3c_4)^n(nc)^{2k} = b^2$.

Proof. Suppose

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2. \tag{7}$$

We consider the following cases.

Case 1: Let $deg(p(z)) = l(\geq 1)$.

Let z_0 be a zero of f with multiplicity q. Then z_0 be a zero of $[f^n]^{(k)}$ with multiplicity $nq - k$. Now one of the following possibilities holds.

(i) z_0 will be neither a zero of $[g^n]^{(k)}$ nor a pole of g; (ii) z_0 will be a zero of g; (iii) z_0 will be a zero of $[g^n]^{(k)}$ but not a zero of g and (iv) z_0 will be a pole of g.

We now explain only the above two possibilities (i) and (iv) because other two possibilities follow from these.

For the possibility (i): Note that since $n \geq 2k + 2l$, we must have

$$nq - k \geq n - k \geq k + 2l. \tag{8}$$

Thus z_0 must be a zero of $[f^n]^{(k)}$ with multiplicity at least $k + 2l$. But we see from (7) that z_0 must be a zero of $p^2(z)$ with multiplicity atmost $2l$. Hence we arrive at a contradiction and so f has no zero in this case.

For the possibility (iv): Let z_0 be a pole of g with multiplicity q_1. Clearly z_0 will be pole of $[g^n]^{(k)}$ with multiplicity $nq_1 + k$. Obviously $q > q_1$, or else z_0 is a pole of $p(z)$, which is a contradiction since $p(z)$ is a polynomial. Clearly $nq - k \geq nq_1 + k$. Now

$$nq - k = nq_1 + k$$

implies that

$$n(q - q_1) = 2k. \tag{9}$$

Since $n \geq 2k + 2l$, we get a contradiction from (9). Hence we must have

$$nq - k > nq_1 + k.$$

This shows that z_0 is a zero of $p(z)$ and we have $N(r, 0; f) = O(\log r)$. Similarly we can prove that $N(r, 0; g) = O(\log r)$. Thus in general we can take $N(r, 0; f) + N(r, 0; g) = O(\log r)$.

We know that

$$N(r, \infty; [f^n]^{(k)}) = n\, N(r, \infty; f) + k\, \overline{N}(r, \infty; f).$$

Also by Lemma 7 we have

$$\begin{aligned} N(r,0;[g^n]^{(k)}) &\leq n\,N(r,0;g)+k\,\overline{N}(r,\infty;g)+S(r,g)\\ &\leq k\,\overline{N}(r,\infty;g)+\ O(\log r)+S(r,g). \end{aligned}$$

From (7) we get

$$N(r,\infty;[f^n]^{(k)})=N(r,0;[g^n]^{(k)}),$$

i.e.,

$$n\,N(r,\infty;f)+k\,\overline{N}(r,\infty;f)\leq k\,\overline{N}(r,\infty;g)+O(\log r)+S(r,g). \tag{10}$$

Similarly we get

$$n\,N(r,\infty;g)+k\,\overline{N}(r,\infty;g)\leq k\,\overline{N}(r,\infty;f)+O(\log r)+S(r,f). \tag{11}$$

Since f and g are transcendental, it follows that

$$S(r,f)+O(\log r)=S(r,f),\quad S(r,g)+O(\log r)=S(r,g).$$

Combining (10) and (11) we get

$$N(r,\infty;f)+N(r,\infty;g)=S(r,f)+S(r,g).$$

By Lemma 9 we have $S(r,f)=S(r,g)$ and so we obtain

$$N(r,\infty;f)=S(r,f),\quad N(r,\infty;g)=S(r,g). \tag{12}$$

Let

$$F_1=\frac{[f^n]^{(k)}}{p},\quad G_1=\frac{[g^n]^{(k)}}{p}. \tag{13}$$

Note that $T(r,F_1)\leq n(k+1)T(r,f)+S(r,f)$ and so $T(r,F_1)=O(T(r,f))$. Also by Lemma 2 one can obtain $T(r,f)=O(T(r,F_1))$. Hence $S(r,F_1)=S(r,f)$. Similarly we get $S(r,G_1)=S(r,g)$. Also

$$F_1G_1\equiv 1. \tag{14}$$

If $F_1\equiv cG_1$, where c is a nonzero constant, then F_1 is a constant and so f is a polynomial, which contradicts our assumption. Hence $F_1\not\equiv cG_1$ and so in view of (14) we see that F_1 and G_1 share $(-1,0)$.
Now by Lemma 7 we have

$$N(r,0;F_1)\leq n\,N(r,0;f)+k\,\overline{N}(r,\infty;f)+S(r,f)\leq S(r,F_1).$$

Similarly we have

$$N(r,0;G_1) \leq n\, N(r,0;g) + k\,\overline{N}(r,\infty;g) + S(r,g) \leq S(r,G_1).$$

Also we see that

$$N(r,\infty;F_1) = S(r,F_1), \quad N(r,\infty;G_1) = S(r,G_1).$$

Here it is clear that $T(r,F_1) = T(r,G_1) + O(1)$. Let

$$f_1 = \frac{F_1}{G_1}.$$

and

$$f_2 = \frac{F_1 - 1}{G_1 - 1}.$$

Clearly f_1 is non-constant. If f_2 is a nonzero constant then F_1 and G_1 share (∞,∞) and so from (14) we conclude that F_1 and G_1 have no poles. Next we suppose that f_2 is non-constant. Also we see that

$$F_1 = \frac{f_1(1-f_2)}{f_1 - f_2}, \quad G_1 = \frac{1-f_2}{f_1 - f_2}.$$

Clearly

$$T(r,F_1) \leq 2[T(r,f_1) + T(r,f_2)] + O(1)$$

and

$$T(r,f_1) + T(r,f_2) \leq 4T(r,F_1) + O(1).$$

These give $S(r,F_1) = S(r;f_1,f_2)$. Also we see that

$$\overline{N}(r,0;f_i) + \overline{N}(r,\infty;f_i) = S(r;f_1,f_2)$$

for $i = 1,2$.
Next we suppose $\overline{N}(r,-1;F_1) \neq S(r,F_1)$, otherwise F_1 will be a constant. Also we see that

$$\overline{N}(r,-1;F_1) \leq N_0(r,1;f_1,f_2).$$

Thus we have

$$T(r,f_1) + T(r,f_2) \leq 4\, N_0(r,1;f_1,f_2) + S(r,F_1).$$

Then by Lemma 4 there exist two integers s and $t(|s|+|t|>0)$ such that

$$f_1^s f_2^t \equiv 1,$$

i.e.,

$$\left[\frac{F_1}{G_1}\right]^s \left[\frac{F_1-1}{G_1-1}\right]^t \equiv 1. \tag{15}$$

We now consider following cases.

Case (i) Let $s=0$ and $t\neq 0$. Then from (15) we get

$$(F_1-1)^t \equiv (G_1-1)^t.$$

This shows that F_1 and G_1 share (∞,∞) and so from (14) we conclude that F_1 and G_1 have no poles.

Case (ii) Suppose $s\neq 0$ and $t=0$. Then from (15) we get

$$F_1^s \equiv G_1^s$$

and so we arrive at a contradiction from (14).

Case (iii): Suppose $s>0$ and $t=-t_1$, where $t_1>0$. Then we have

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{t_1}. \tag{16}$$

If possible suppose F_1 has a pole. Let z_{p_1} be a pole of F_1 of multiplicity p_1. Then from (14) we see that z_{p_1} must be a zero of G_1 of multiplicity p_1. Now from (16) we get $2s=t_1$ and so

$$\left[\frac{F_1}{G_1}\right]^s \equiv \left[\frac{F_1-1}{G_1-1}\right]^{2s}.$$

This implies that

$$F_1^{s-1}+F_1^{s-2}G_1+F_1^{s-3}G_1^2+\ldots+F_1G_1^{s-2}+G_1^{s-1} \equiv G_1^s\frac{(F_1-1)^{2s}-(G_1-1)^{2s}}{(G_1-1)^{2s}(F_1-G_1)}. \tag{17}$$

If z_p is a zero of F_1-1 with multiplicity p then the Taylor expansion of F_1-1 about z_p is

$$F_1-1 = a_p(z-z_p)^p + a_{p+1}(z-z_p)^{p+1}+\ldots\ldots, \quad a_p\neq 0.$$

Since F_1-1 and G_1-1 share $(0,\infty)$,

$$G_1-1 = b_p(z-z_p)^p + b_{p+1}(z-z_p)^{p+1}+\ldots\ldots, \quad b_p\neq 0.$$

Let

$$\Phi_1 = \frac{F_1'}{F_1} - \frac{G_1'}{G_1} \quad \text{and} \quad \Phi_2 = \left(\frac{F_1'}{F_1}\right)^{2s} - \left(\frac{G_1'}{G_1}\right)^{2s}. \tag{18}$$

Since $F_1 \not\equiv cG_1$, where c is a nonzero constant, it follows that $\Phi_1 \not\equiv 0$ and $\Phi_2 \not\equiv 0$. Also

$$T(r, \Phi_1) = S(r, F_1) \quad \text{and} \quad T(r, \Phi_2) = S(r, F_1).$$

From (18) we find

$$\overline{N}_{(2}(r, 1; F_1) = \overline{N}_{(2}(r, 1; G_1) \leq N(r, 0; \Phi_1) = S(r, F_1).$$

Let $p = 1$. If $a_1 = b_1$, then by an elementary calculation gives that $\Phi_1(z) = O((z - z_1)^k)$, where k is a positive integer. This proves that z_1 is a zero of Φ_1. Next we suppose $a_1 \neq b_1$, but $a_1^{2s} = b_1^{2s}$. Then by an elementary calculation we get $\Phi_2(z) = O((z - z_1)^q)$ where q is a positive integer. This proves that z_1 is a zero of Φ_2.

Finally we suppose $a_1 \neq b_1$ and $a_1^{2s} \neq b_1^{2s}$. Therefore from (17) we arrive at a contradiction. Hence

$$N_{1)}(r, 1; F_1) = N_{1)}(r, 1; G_1) = S(r, F_1).$$

But this is impossible as $\overline{N}(r, 1; F_1) \sim T(r, F_1)$ and $\overline{N}(r, 1; G_1) \sim T(r, G_1)$. Hence F_1 has no pole. Similarly we can prove that G_1 also has no poles.

Case (iv): Suppose either $s > 0$ and $t > 0$ or $s < 0$ and $t < 0$. Then from (15) one can easily prove that F_1 and G_1 have no poles. Consequently from (14) we see that F_1 and G_1 have no zeros. We deduce from (13) that both f and g have no pole.

Since F_1 and G_1 have no zeros and poles, we have

$$F_1 \equiv e^{\gamma_1} G_1,$$

i.e.,

$$[f^n]^{(k)} \equiv e^{\gamma_1} [g^n]^{(k)},$$

where γ_1 is a non-constant entire function. Then from (7) we get

$$[f^n]^{(k)} \equiv c e^{\frac{1}{2}\gamma_1} p, \quad [g^n]^{(k)} \equiv c e^{-\frac{1}{2}\gamma_1} p, \tag{19}$$

where $c = \pm 1$. Since $N(r, 0; f) = O(\log r)$ and $N(r, 0; g) = O(\log r)$, so we can take

$$f(z) = P_1(z) e^{\alpha_1(z)}, \quad g(z) = Q_1(z) e^{\beta_1(z)}, \tag{20}$$

P_1, Q_1 are nonzero polynomials, α_1, β_1 are two non-constant entire functions. If possible suppose that $P_1(z)$ is not a constant. Let z_1 be a zero of f with multiplicity t. Then z_1 must be a zero of $[f^n]^{(k)}$ with multiplicity $nt-k$. Note that $nt-k \geq n-k \geq k+2l$, as $n \geq 2k+2l$. Clearly z_1 must be a zero of $p^2(z)$ with multiplicity at least $k+2l$, which is impossible since z_1 can be a zero of $p^2(z)$ with multiplicity at most $2l$. Hence $P_1(z)$ is a constant. Similarly we can prove that $Q_1(z)$ is a constant. So we can rewrite f and g as follows

$$f = e^{\alpha}, \quad g = e^{\beta}. \tag{21}$$

We deduce from (7) and (21) that either both α and β are transcendental entire functions or both α and β are polynomials. We now consider following cases.

Subcase 1.1: Let $k \geq 2$.

First we suppose both α and β are transcendental entire functions.

Note that

$$S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}), \quad S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}).$$

Moreover we see that

$$N(r, 0; [f^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

$$N(r, 0; [g^n]^{(k)}) \leq N(r, 0; p^2) = O(\log r).$$

From these and using (21) we have

$$N(r, \infty; f^n) + N(r, 0; f^n) + N(r, 0; [f^n]^{(k)}) = S(r, n\alpha) = S(r, \frac{[f^n]'}{f^n}) \tag{22}$$

and

$$N(r, \infty; g^n) + N(r, 0; g^n) + N(r, 0; [g^n]^{(k)}) = S(r, n\beta) = S(r, \frac{[g^n]'}{g^n}). \tag{23}$$

Then from (22), (23) and Lemma 5 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{24}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But these types of f and g do not agree with the relation (7).

Next we suppose α and β are both polynomials.
Clearly $\alpha+\beta\equiv C$ and $\deg(\alpha)=\deg(\beta)$. Also $\alpha^{'}\equiv\beta^{'}$. If $\deg(\alpha)=\deg(\beta)=1$, then we again get a contradiction from (7).
Next we suppose $\deg(\alpha)=\deg(\beta)\geq 2$.
We deduce from (21) that

$$\begin{aligned}
&(f^{n})^{'}=n\alpha^{'}e^{n\alpha}\\
&(f^{n})^{''}=[n^{2}(\alpha^{'})^{2}+n\alpha^{''}]e^{n\alpha}\\
&(f^{n})^{'''}=[n^{3}(\alpha^{'})^{3}+3n^{2}\alpha^{'}\alpha^{''}+n\alpha^{'''}]e^{n\alpha}\\
&(f^{n})^{(iv)}=[n^{4}(\alpha^{'})^{4}+6n^{3}(\alpha^{'})^{2}\alpha^{''}+3n^{2}(\alpha^{''})^{2}+4n^{2}\alpha^{'}\alpha^{'''}+n\alpha^{(iv)}]e^{n\alpha}\\
&(f^{n})^{(v)}=[n^{5}(\alpha^{'})^{5}+10n^{4}(\alpha^{'})^{3}\alpha^{''}+15n^{3}\alpha^{'}(\alpha^{''})^{2}+10n^{3}(\alpha^{'})^{2}\\
&\alpha^{'''}+10n^{2}\alpha^{''}\alpha^{'''}+5n^{2}\alpha^{'}\alpha^{(iv)}+n\alpha^{(v)}]e^{n\alpha}\\
&\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\quad\ldots\\
&[f^{n}]^{(k)}=[n^{k}(\alpha^{'})^{k}+K(\alpha^{'})^{k-2}\alpha^{''}+P_{k-2}(\alpha^{'})]e^{n\alpha},
\end{aligned}$$

where K is a suitably positive integer and $P_{k-2}(\alpha^{'})$ is a differential polynomial in $\alpha^{'}$.
Similarly we get

$$\begin{aligned}
[g^{n}]^{(k)} &= [n^{k}(\beta^{'})^{k}+K(\beta^{'})^{k-2}\beta^{''}+P_{k-2}(\beta^{'})]e^{n\beta}\\
&= [(-1)^{k}n^{k}(\alpha^{'})^{k}-K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''}+P_{k-2}(-\alpha^{'})]e^{n\beta}.
\end{aligned}$$

Since $\deg(\alpha)\geq 2$, we observe that $\deg((\alpha^{'})^{k})\geq k\deg(\alpha^{'})$ and so $(\alpha^{'})^{k-2}\alpha^{''}$ is either a nonzero constant or $\deg((\alpha^{'})^{k-2}\alpha^{''})\geq(k-1)\deg(\alpha^{'})-1$. Also we see that

$$\deg\left((\alpha^{'})^{k}\right)>\deg\left((\alpha^{'})^{k-2}\alpha^{''}\right)>\deg\left(P_{k-2}(\alpha^{'})\right)(\text{or }\ \deg\left(P_{k-2}(-\alpha^{'})\right)).$$

From (19), it is clear that the polynomials

$$n^{k}(\alpha^{'})^{k}+K(\alpha^{'})^{k-2}\alpha^{''}+P_{k-2}(\alpha^{'})$$

and

$$(-1)^{k}n^{k}(\alpha^{'})^{k}-K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''}+P_{k-2}(-\alpha^{'})$$

must be identical but this is impossible for $k\geq 2$. Actually the terms $n^{k}(\alpha^{'})^{k}+K(\alpha^{'})^{k-2}\alpha^{''}$ and $(-1)^{k}n^{k}(\alpha^{'})^{k}-K(-1)^{k-2}(\alpha^{'})^{k-2}\alpha^{''}$ can not be identical for

$k \geq 2$.

Subcase 2: Let $k = 1$. Then from (7) we get

$$AB\alpha^{'}\beta^{'}e^{n(\alpha+\beta)} \equiv p^2, \tag{25}$$

where $AB = n^2$. Let $\alpha + \beta = \gamma$. Suppose that α and β are both transcendental entire functions. From (25) we know that γ is not a constant since in that case we get a contradiction. Then from (25) we get

$$AB\alpha^{'}(\gamma^{'} - \alpha^{'})e^{n\gamma} \equiv p^2. \tag{26}$$

We have $T(r, \gamma^{'}) = m(r, \gamma^{'}) \leq m(r, \frac{(e^{n\gamma})^{'}}{e^{n\gamma}}) + O(1) = S(r, e^{n\gamma})$. Thus from (26) we get

$$\begin{aligned} T(r, e^{n\gamma}) &\leq T(r, \frac{p^2}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + O(\log r) + O(1) \\ &\leq 2\,T(r, \alpha^{'}) + S(r, \alpha^{'}) + S(r, e^{n\gamma}), \end{aligned}$$

which implies that $T(r, e^{n\gamma}) = O(T(r, \alpha^{'}))$ and so $S(r, e^{n\gamma})$ can be replaced by $S(r, \alpha^{'})$. Thus we get $T(r, \gamma^{'}) = S(r, \alpha^{'})$ and so $\gamma^{'}$ is a small function with respect to $\alpha^{'}$. In view of (26) and by the second fundamental theorem for small functions we get

$$\begin{aligned} T(r, \alpha^{'}) &\leq \overline{N}(r, \infty; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'}) + \overline{N}(r, 0; \alpha^{'} - \gamma^{'}) + S(r, \alpha^{'}) \\ &\leq O(\log r) + S(r, \alpha^{'}), \end{aligned}$$

which shows that $\alpha^{'}$ is a polynomial and so α is a polynomial, which contradicts that α is a transcendental entire function. Next suppose without loss of generality that α is a polynomial and β is a transcendental entire function. Thus γ is transcendental. So in view of (26) we can obtain

$$\begin{aligned} nT(r, e^{\gamma}) &\leq T(r, \frac{p^2}{\alpha^{'}(\gamma^{'} - \alpha^{'})}) + O(1) \\ &\leq T(r, \alpha^{'}) + T(r, \gamma^{'} - \alpha^{'}) + S(r, e^{\gamma}) \\ &\leq T(r, \gamma^{'}) + S(r, e^{\gamma}) = S(r, e^{\gamma}), \end{aligned}$$

which leads a contradiction. Thus α and β are both polynomials. Also from (25) we can conclude that $\alpha + \beta \equiv C$ for a constant C and so $\alpha^{'} + \beta^{'} \equiv 0$. Again from (25) we get $n^2e^{nC}\alpha^{'}\beta^{'} \equiv p^2$. By computation we get

$$\alpha^{'} = cp, \quad \beta^{'} = -cp. \tag{27}$$

Hence

$$\alpha = cQ + b_1, \quad \beta = -cQ + b_2, \tag{28}$$

where $Q(z) = \int_0^z p(z)dz$ and b_1, b_2 are constants. Finally f and g take the form

$$f(z) = c_1 e^{cQ(z)}, \quad g(z) = c_2 e^{-cQ(z)},$$

where c_1, c_2 and c are constants such that $(nc)^2(c_1c_2)^n = -1$.
Case 2: Let $p(z)$ be a nonzero constant b. Since $n > 2k$, one can easily prove that f and g have no zeros. Now proceeding in the same way as done in proof of **Case 1** we get $f = e^{\alpha}$ and $g = e^{\beta}$, where α and β are two non-constant entire functions.
We now consider following two subcases:
Subcase 2.1: Let $k \geq 2$.
We see that $f^n(z)[f^n(z)]^{(k)} \neq 0$ and $g^n(z)[g^n(z)]^{(k)} \neq 0$. Then by Lemma 6 we must have

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{29}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. But from (7) we see that $a + c = 0$.
Subcase 2.1: Let $k = 1$.
Considering **Subcase 1.2** one can easily get

$$f(z) = e^{az+b}, \quad g(z) = e^{cz+d}, \tag{30}$$

where $a \neq 0$, b, $c \neq 0$ and d are constants. Finally f and g take the form

$$f(z) = c_3 e^{dz}, \quad g(z) = c_4 e^{-dz},$$

where c_3, c_4 and d are nonzero constants such that $(-1)^k(c_3c_4)^n(nd)^{2k} = b^2$. This completes the proof. □

Lemma 12 *Let* f, g *be two transcendental meromorphic functions, let* n, m *and* k *be three positive integers such that* $n > k$. *If* f *and* g *share* $(\infty, 0)$ *then* $[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2$, *where* $p(z)$ *is a non zero polynomial.*

Proof. Suppose

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2. \tag{31}$$

Since f and g share $(\infty, 0)$ we have from (31) that f and g are transcendental entire functions. So we can take

$$f(z) = h(z)e^{\alpha(z)}, \tag{32}$$

where h is a nonzero polynomial and α is a non-constant entire function. We know that $(w-1)^m = a_m w^m + a_{m-1}w^{m-1} + \ldots + a_0$, where $a_i = (-1)^{m-i}\,{}^mC_{m-i}$, $i = 0,1,2,\ldots,m$. Since $f = he^{\alpha}$, then by induction we get

$$(a_i f^{n+i})^{(k)} = t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)})e^{(n+i)\alpha}, \tag{33}$$

where $t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)})$ $(i = 0,1,2,\ldots,m)$ are differential polynomials in
$\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)}$. Obviously

$$t_i(\alpha', \alpha'', \ldots, \alpha^{(k)}, h, h', \ldots, h^{(k)}) \not\equiv 0,$$

for $i = 0,1,2,\ldots,m$ and $[f^n(f-1)^m]^{(k)} \not\equiv 0$. Now from (31) and (33) we obtain

$$\overline{N}(r,0;t_m e^{m\alpha(z)} + \ldots + t_0) \leq N(r,0;p^2) = S(r,f). \tag{34}$$

Since α is an entire function, we obtain $T(r,\alpha^{(j)}) = S(r,f)$ for $j = 1,2,\ldots,k$. Hence $T(r,t_i) = S(r,f)$ for $i = 0,1,2,\ldots,m$. So from (34) and using second fundamental theorem for small functions (see [17]), we obtain

$$\begin{aligned} mT(r,f) &= T(r, t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{m\alpha} + \ldots + t_1 e^{\alpha}) + \overline{N}(r,0;t_m e^{m\alpha} + \ldots + t_1 e^{\alpha} + t_0) \\ &\quad + S(r,f) \\ &\leq \overline{N}(r,0;t_m e^{(m-1)\alpha} + \ldots + t_1) + S(r,f) \\ &\leq (m-1)T(r,f) + S(r,f), \end{aligned}$$

which is a contradiction. This completes the Lemma. □

Lemma 13 *Let* f *and* g *be two non-constant meromorphic functions and* $\alpha(\not\equiv 0,\infty)$ *be small function of* f *and* g*. Let* n*,* m *and* k *be three positive integers such that* $n \geq m+3$*. Then*

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv \alpha^2, \quad \text{for} \quad k = 1.$$

Proof. We omit the proof since it can be proved in the line of the proof of Lemma 3 [14]. □

Lemma 14 [1] *If* f, g *be two non-constant meromorphic functions such that they share* $(1,1)$*. Then*

$$2\overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>2}(r,1;g) \leq N(r,1;g) - \overline{N}(r,1;g).$$

Lemma 15 [2] *Let* f, g *share* $(1,1)$. *Then*

$$\overline{N}_{f>2}(r,1;g) \leq \frac{1}{2}\overline{N}(r,0;f) + \frac{1}{2}\overline{N}(r,\infty;f) - \frac{1}{2}N_0(r,0;f') + S(r,f),$$

where $N_0(r,0;f')$ *is the counting function of those zeros of* f' *which are not the zeros of* $f(f-1)$.

Lemma 16 [2] *Let* f *and* g *be two non-constant meromorphic functions sharing* $(1,0)$. *Then*

$$\begin{aligned} & \overline{N}_L(r,1;f) + 2\overline{N}_L(r,1;g) + \overline{N}_E^{(2}(r,1;f) - \overline{N}_{f>1}(r,1;g) - \overline{N}_{g>1}(r,1;f) \\ \leq\ & N(r,1;g) - \overline{N}(r,1;g). \end{aligned}$$

Lemma 17 [2] *Let* f, g *share* $(1,0)$. *Then*

$$\overline{N}_L(r,1;f) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) + S(r,f)$$

Lemma 18 [2] *Let* f, g *share* $(1,0)$. *Then*

$$\begin{aligned} &(i) \quad \overline{N}_{f>1}(r,1;g) \leq \overline{N}(r,0;f) + \overline{N}(r,\infty;f) - N_0(r,0;f') + S(r,f) \\ &(ii) \quad \overline{N}_{g>1}(r,1;f) \leq \overline{N}(r,0;g) + \overline{N}(r,\infty;g) - N_0(r,0;g') + S(r,g). \end{aligned}$$

3 Proof of the Theorem

Proof of Theorem 1. Let $F = \frac{[f^n P(f)]^{(k)}}{p}$ and $G = \frac{[g^n P(g)]^{(k)}}{p}$, where $P(w) = (w-1)^m$. It follows that F and G share $(1,k_1)$ except for the zeros of $p(z)$.
Case 1 Let $H \not\equiv 0$.
Subcase 1.1 $k_1 \geq 1$.
From (1) it can be easily calculated that the possible poles of H occur at (i) multiple zeros of F and G, (ii) those 1 points of F and G whose multiplicities are different, (iii) poles of F and G, (iv) zeros of $F'(G')$ which are not the zeros of $F(F-1)(G(G-1))$.
Since H has only simple poles we get

$$\begin{aligned} N(r,\infty;H) \leq\ & \overline{N}(r,\infty;f) + \overline{N}(r,\infty;g) + \overline{N}_*(r,1;F,G) + \overline{N}(r,0;F|\geq 2) \\ & + \overline{N}(r,0;G|\geq 2) + \overline{N}_0(r,0;F') + \overline{N}_0(r,0;G'), \end{aligned} \tag{35}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F' which are not the zeros of $F(F-1)$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Let z_0 be a simple zero of $F-1$ but $p(z_0)\neq 0$. Then z_0 is a simple zero of $G-1$ and a zero of H. So

$$N(r,1;F|=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,f)+S(r,g). \tag{36}$$

While $k_1\geq 2$, using (35) and (36) we get

$$\begin{aligned}
&\overline{N}(r,1;F)\\
&\leq N(r,1;F|=1)+\overline{N}(r,1;F|\geq 2)\leq \overline{N}(r,\infty;f)\\
&\quad+\ \overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\ \overline{N}(r,1;F|\geq 2)+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g).
\end{aligned}\tag{37}$$

Now in view of Lemma 3 we get

$$\begin{aligned}
&\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F\,|\geq 2)+\overline{N}_*(r,1;F,G)\\
&\leq \overline{N}_0(r,0;G^{'})+\overline{N}(r,1;F|\geq 2)+\overline{N}(r,1;F|\geq 3)\\
&\quad=\overline{N}_0(r,0;G^{'})+\overline{N}(r,1;G|\geq 2)+\overline{N}(r,1;G|\geq 3)\\
&\leq \overline{N}_0(r,0;G^{'})+N(r,1;G)-\overline{N}(r,1;G)\\
&\leq N(r,0;G^{'}\mid G\neq 0)\leq \overline{N}(r,0;G)+\overline{N}(r,\infty;g)+S(r,g),
\end{aligned}\tag{38}$$

Hence using (37), (38), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad-N_0(r,0;F^{'})\\
&\leq 2\,\overline{N}(r,\infty,f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F)+N_{k+2}(r,0;f^nP(f))\\
&\quad+\overline{N}(r,0;F|\geq 2)+\ \overline{N}(r,0;G|\geq 2)+\overline{N}(r,1;F|\geq 2)+\overline{N}_*(r,1;F,G)\\
&\quad+\overline{N}_0(r,0;G^{'})-N_2(r,0;F)+S(r,f)+S(r,g)\\
&\leq 2\,\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\}+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)\\
&\quad+S(r,f)+S(r,g)
\end{aligned}\tag{39}$$

$$\le 2\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+N_{k+2}(r,0;f^nP(f))+k\,\overline{N}(r,\infty;g)$$
$$+\ N_{k+2}(r,0;g^nP(g))+S(r,f)+S(r,g)$$
$$\le 2\left\{\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\right\}+(k+2)\,\overline{N}(r,0;f)+T(r,P(f))$$
$$+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+k\,\overline{N}(r,\infty;g)+S(r,f)+S(r,g)$$
$$\le (k+4+m)\,T(r,f)+(2k+4+m)\,T(r,g)+S(r,f)+S(r,g)$$
$$\le (3k+8+2m)\,T(r)+S(r).$$

In a similar way we can obtain

$$(n+m)\,T(r,g)\le (3k+8+2m)\ T(r)+S(r). \tag{40}$$

Combining (39) and (40) we see that

$$(n+m)\,T(r)\le (3k+8+2m)\ T(r)+S(r),$$

i.e.,

$$(n-3k-8-m)\ T(r)\le S(r). \tag{41}$$

Since $n>3k+8+m$, (41) leads to a contradiction.
While $k_1=1$, using Lemmas 3, 14, 15, (35) and (36) we get

$$\overline{N}(r,1;F) \tag{42}$$
$$\le\ N(r,1;F|=1)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)$$
$$\le\ \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_*(r,1;F,G)$$
$$+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')$$
$$+S(r,f)+S(r,g)$$
$$\le\ \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+2\overline{N}_L(r,1;F)$$
$$+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)$$
$$\le\ \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\overline{N}(r,0;G|\ge 2)+\overline{N}_{F>2}(r,1;G)$$
$$+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,f)+S(r,g)$$
$$\le\ \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\ge 2)$$
$$+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G')+\overline{N}_0(r,0;F')+S(r,f)+S(r,g)$$
$$\le\ \frac{3}{2}\,\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+\overline{N}(r,0;G|\ge 2)$$
$$+N(r,0;G'|G\ne 0)+\overline{N}_0(r,0;F')+S(r,f)+S(r,g)$$
$$\le\ \frac{3}{2}\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\ge 2)+\frac{1}{2}\,\overline{N}(r,0;F)+N_2(r,0;G)$$
$$+\overline{N}_0(r,0;F')+S(r,f)+S(r,g).$$

Hence using (42), Lemmas 1 and 2 we get from second fundamental theorem that

$$
\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F)+N_{k+2}(r,0;f^{n}P(f))-N_{2}(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^{n}P(f))-N_{2}(r,0;F)\\
&\quad -N_{0}(r,0;F^{'})\\
&\leq \frac{5}{2}\,\overline{N}(r,\infty,f)+2\overline{N}(r,\infty;g)+N_{2}(r,0;F)+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +N_{k+2}(r,0;f^{n}P(f))+N_{2}(r,0;G)-N_{2}(r,0;F)+S(r,f)+S(r,g)\\
&\leq \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^{n}P(f))+\frac{1}{2}\overline{N}(r,0;F)\\
&\quad +N_{2}(r,0;G)+S(r,f)+S(r,g)\\
&\leq \frac{5}{2}\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^{n}P(f))+k\,\overline{N}(r,\infty;g) \qquad (43)\\
&\quad +N_{k+2}(r,0;g^{n}P(g))+\frac{1}{2}\{k\overline{N}(r,\infty;f)\\
&\quad +N_{k+1}(r,0;f^{n}P(f))\}+S(r,f)+S(r,g)\\
&\leq \frac{5+k}{2}\,\overline{N}(r,\infty;f)+(k+2)\overline{N}(r,\infty;g)+\frac{3k+5}{2}\,\overline{N}(r,0;f)\\
&\quad +\frac{3}{2}\,T(r,P(f))+(k+2)\,\overline{N}(r,0;g)+T(r,P(g))+S(r,f)+S(r,g)\\
&\leq \left(2k+5+\frac{3m}{2}\right)\,T(r,f)+(2k+4+m)\,T(r,g)+S(r,f)+S(r,g)\\
&\leq \left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r).
\end{aligned}
$$

In a similar way we can obtain

$$(n+m)\,T(r,g)\leq \left(4k+9+\frac{5m}{2}\right)\,T(r)+S(r). \tag{44}$$

Combining (43) and (44) we see that

$$\left(n-4k-9-\frac{3m}{2}\right)\,T(r)\leq S(r). \tag{45}$$

Since $n>4k+9+\frac{3m}{2}$, (45) leads to a contradiction.
Subcase 1.2 $k_{1}=0$. Here (36) changes to

$$N_{E}^{1)}(r,1;F\mid=1)\leq N(r,0;H)\leq N(r,\infty;H)+S(r,F)+S(r,G). \tag{46}$$

Using Lemmas 3, 16, 17, 18, (35) and (46) we get

$$\begin{aligned}
&\overline{N}(r,1;F)\\
&\leq N_E^{1)}(r,1;F)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\
&\quad+\overline{N}_*(r,1;F,G)+\overline{N}_L(r,1;F)+\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)+\overline{N}_0(r,0;F^{'})\\
&\quad+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\
&\quad+2\overline{N}_L(r,1;F)+2\overline{N}_L(r,1;G)+\overline{N}_E^{(2}(r,1;F)\\
&\quad+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)+\overline{N}(r,0;F|\geq 2)+\overline{N}(r,0;G|\geq 2)\\
&\quad+\overline{N}_{F>1}(r,1;G)+\overline{N}_{G>1}(r,1;F)+\overline{N}_L(r,1;F)+N(r,1;G)-\overline{N}(r,1;G)\\
&\quad+\overline{N}_0(r,0;F^{'})+\overline{N}_0(r,0;G^{'})+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,1;G)-\overline{N}(r,1;G)+\overline{N}_0(r,0;G^{'})+\overline{N}_0(r,0;F^{'})\\
&\quad+S(r,f)+S(r,g)\\
&\leq 3\,\overline{N}(r,\infty;f)+2\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+N(r,0;G^{'}|G\neq 0)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g)\\
&\leq 3\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+\overline{N}(r,0;F)+N_2(r,0;G)\\
&\quad+\overline{N}(r,0;G)+\overline{N}_0(r,0;F^{'})+S(r,f)+S(r,g).
\end{aligned}\tag{47}$$

Hence using (47), Lemmas 1 and 2 we get from second fundamental theorem that

$$\begin{aligned}
&(n+m)T(r,f)\\
&\leq T(r,F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)+S(r,f)\\
&\leq \overline{N}(r,0;F)+\overline{N}(r,\infty;F)+\overline{N}(r,1;F)+N_{k+2}(r,0;f^nP(f))-N_2(r,0;F)\\
&\quad-N_0(r,0;F^{'})\\
&\leq 4\overline{N}(r,\infty,f)+3\overline{N}(r,\infty;g)+N_2(r,0;F)+2\,\overline{N}(r,0;F)\\
&\quad+N_{k+2}(r,0;f^nP(f))+N_2(r,0;G)+\overline{N}(r,0;G)-N_2(r,0;F)\\
&\quad+S(r,f)+S(r,g)\\
&\leq 4\overline{N}(r,\infty;f)+3\overline{N}(r,\infty;g)+N_{k+2}(r,0;f^nP(f))+2\,\overline{N}(r,0;F)
\end{aligned}$$

$$\begin{aligned}
&+ N_2(r,0;G) + \overline{N}(r,0;G) + S(r,f) + S(r,g)\\
&\le 4\overline{N}(r,\infty;f) + 3\overline{N}(r,\infty;g) + N_{k+2}(r,0;f^nP(f)) + 2k\overline{N}(r,\infty;f)\\
&+ 2\, N_{k+1}(r,0;f^nP(f)) + k\,\overline{N}(r,\infty;g) + N_{k+2}(r,0;g^nP(g))\\
&+ k\overline{N}(r,\infty;g) + N_{k+1}(r,0;g^nP(g)) + S(r,f) + S(r,g)\\
&\le (2k+4)\,\overline{N}(r,\infty;f) + (2k+3)\overline{N}(r,\infty;g) + (3k+4)\overline{N}(r,0;f)\\
&+ 3T(r,P(f)) + (2k+3)\,\overline{N}(r,0;g) + 2T(r,P(g)) + S(r,f) + S(r,g)\\
&\le (5k+8+3m)\,T(r,f) + (4k+6+2m)\,T(r,g) + S(r,f) + S(r,g)\\
&\le (9k+14+5m)\,T(r) + S(r).
\end{aligned} \tag{48}$$

In a similar way we can obtain

$$(n+m)\,T(r,g) \le (9k+14+5m)\;T(r) + S(r). \tag{49}$$

Combining (48) and (49) we see that

$$(n-9k-14-4m)\;T(r) \le S(r). \tag{50}$$

Since $n > 9k+14+4m$, (50) leads to a contradiction.
Case 2. Let $H \equiv 0$. Then by Lemma 10 we get either

$$f^n(f-1)^m \equiv g^n(g-1)^m \tag{51}$$

or

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \equiv p^2. \tag{52}$$

We now consider following two subcases.
Subcase 2.1: Let $m = 0$.
Now from (51) we get $f^n \equiv g^n$ and so $f \equiv tg$, where t is a constant satisfying $t^n = 1$.
Also from (52) we get

$$[f^n]^{(k)}[g^n]^{(k)} \equiv p^2.$$

Then by Lemma 11 we get the conclusion (1).
Subcase 2.2: Let $m \ge 1$.
Applying Lemma 13, from (52) we see that

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2,$$

for $k = 1$.
In addition, when f and g share $(\infty, 0)$, then by Lemma 12 we must have

$$[f^n(f-1)^m]^{(k)}[g^n(g-1)^m]^{(k)} \not\equiv p^2.$$

Next we consider the relation (51) and let $h = \frac{g}{f}$.
First we suppose that h is non-constant.
For $m = 1$: Then from (51) we get $f \equiv \frac{1-h^n}{1-h^{n+1}}$, i.e.,

$$f \equiv \left(\frac{h^n}{1+h+h^2+\ldots+h^n} - 1\right).$$

Hence by Lemma 1 we get

$$T(r,f) = T(r, \sum_{j=0}^{n} \frac{1}{h^j}) + O(1) = n\, T(r, \frac{1}{h}) + S(r,h) = n\, T(r,h) + S(r,h).$$

Similarly we have $T(r,g) = nT(r,h) + S(r,h)$. Therefore $S(r,f) = S(r,g) = S(r,h)$.
Also it is clear that

$$\sum_{j=1}^{n} \overline{N}(r, u_j; h) \le \overline{N}(r, \infty; f),$$

where $u_j = exp(\frac{2j\pi i}{n+1})$ and $j = 1, 2, \ldots, n$.
Then by the second fundamental theorem we get

$$(n-2)\, T(r,h) \le \sum_{j=1}^{n} \overline{N}(r, u_j; h) + S(r,f) \le \overline{N}(r, \infty; f) + S(r,f).$$

Similarly we have

$$(n-2)\, T(r,h) \le \overline{N}(r, \infty; g) + S(r,g).$$

Adding and simplifying these we get

$$2(n-2)T(r,h) \le n(2 - \Theta(\infty; f) - \Theta(\infty; g) + \varepsilon)T(r,h) + S(r,h),$$

where $0 < \varepsilon < \Theta(\infty; f) + \Theta(\infty; g)$. This leads to a contradiction as $\Theta(\infty; f) + \Theta(\infty; g) > \frac{4}{n}$.

For $m \geq 2$: Then from (51) we can say that f and g satisfying the algebraic equation $R(f,g)=0$, where

$$R(\omega_1,\omega_2)=\omega_1^n(\omega_1-1)^m-\omega_2^n(\omega_2-1)^m.$$

Next we suppose that h is a constant.
Then from (51) we get

$$f^n\sum_{i=0}^{m}(-1)^i\,{}^mC_{m-i}\,f^{m-i}\equiv g^n\sum_{i=0}^{m}(-1)^i\,{}^mC_{m-i}g^{m-i}. \tag{53}$$

Now substituting $g=fh$ in (53) we get

$$\sum_{i=0}^{m}(-1)^i\,{}^mC_{m-i}\,f^{n+m-i}(h^{n+m-i}-1)\equiv 0,$$

which implies that $h=1$. Hence $f\equiv g$. This completes the proof. □

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