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Performance evaluation of two Markovian retrial queueing model with balking and feedback

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Abstract. In this paper, we consider the performance evaluation of two retrial queueing system. Customers arrive to the system, if upon arrival, the queue is full, the new arriving customers either move into one of the orbits, from which they make a new attempts to reach the primary queue, until they find the server idle or balk and leave the system, these later, and after getting a service may comeback to the system requiring another service. So, we derive for this system, the joint distribution of the server state and retrial queue lengths. Then, we give some numerical results that clarify the relationship between the retrials, arrivals, balking rates, and the retrial queue length.

1 Introduction

In the parlance of queueing theory, such a mechanism in which ejected (or rejected) customers return at random intervals until they receive service is called a retrial queue. Retrial queues have an important application in a wide variety of fields, they are likewise prevalent in the evaluation and design of

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computer networks as they are in telecommunications, computer networks, and particularly wireless networks.

A retrial queue is similar to any ordinary queueing system in that there is an arrival process and one or more servers. The fundamental differences are firstly, the entities who enter during a down or busy period of the server or servers may reattempt service at some random time in the future, and secondly a waiting room, which is known as a primary queue in the context of retrial queues, is not mandatory. In place of the ordinary waiting room is a buffer called an orbit to which entities proceed after an unsuccessful attempt at service, and from which they retry service according to a given probabilistic or deterministic policy.

Owing to the utility and interesting mathematical properties of retrial queueing models, a vast literature on the subject has emerged over the past several decades. For a general survey of retrial queues and a summary of many results, the reader is directed to the works of [6, 8, 7, 5, 12, 15] and references therein.

In [4] Choi and Kim considered the $M/M/c$ retrial queues with geometric loss and feedback when $c = 1, 2$, they found the joint generating function of the number of busy servers and the queue length by solving Kummer differential equation for $c = 1$, and by the method of series solution for $c = 1, 2$. Retrial queueing model $MMAP/M_2/1$ with two orbits was studied by Avrachenkov, Dudin and Klimenok [3], in their paper, authors considered a retrial single-server queueing model with two types of customers. In case of the server occupancy at the arrival epoch, the customer moves to the orbit depending on the type of the customer. One orbit is infinite while the second one is a finite. Joint distribution of the number of customers in the orbits and some performance measures are computed. An $M/M/1$ queue with customers balking was proposed by Haight [9], Sumeet Kumar Sharma [10] studied the $M/M/1/N$ queuing system with retention of reneged customers, Kumar and Sharma [11] studied a single server queueing system with retention of reneged customers and balking. Kumar and Sharma [14] consider a single server, finite capacity Markovian feedback queue with balking, balking and retention of reneged customers in which the inter-arrival and service times follow exponential distribution. In our paper, we consider a retrial queueing model with two orbits O_1 and O_2, balking and feedback. In case of the server occupancy at the arrival epoch, the arriving customers have to choose between the two orbits depending on their thresholds if they decide to stay for an attempt to get served or leave the system (balk), and after getting a service, customers may comeback to the system requiring another service. The main result in this work consists in deriving the approximate analysis of the system.

2 Mathematical model

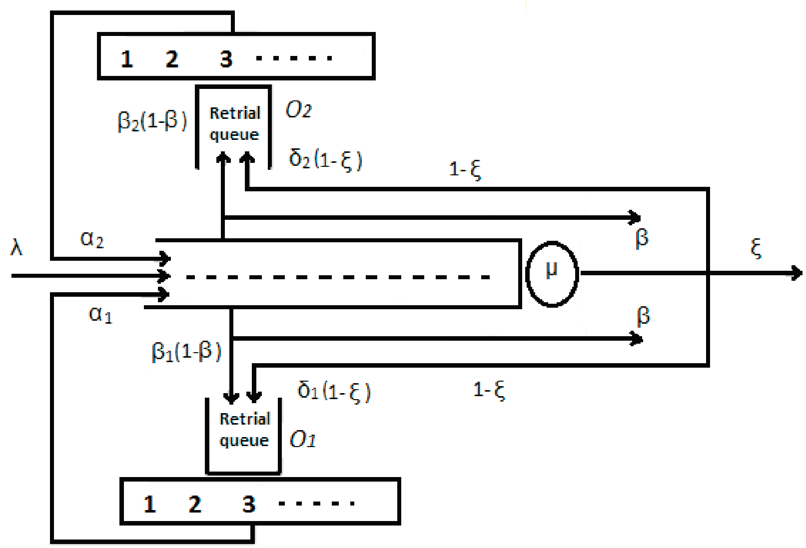

Figure 1: Retrial queueing model with balking and feedback

We consider a retrial queueing model with two orbits O_1 and O_2, new customers arrive from outside to the service node according to a poisson process with rate λ. If the queue is not full upon primary call arrivals, then the customers wait in the queue, thus will be served according to the FIFO order, where service times $B(t)$ are assumed to be independent and exponentially distributed with mean $1/\mu$. However, if upon arrival, the customers find the queue full, then they decide to stay for an attempt to get served with probability $\bar{\beta} = 1 - \beta$ or leave the system with probability β, $0 \leq \beta \leq 1$. The arriving customers who decide to stay for an attempt, they have to choose one of the orbits O_1, O_2; depending on their thresholds; if the number of customers in orbit O_1 is quite larger than that of orbit O_2, the customer will move into the orbit O_2 with probability $\bar{\beta}\beta_2$; $0 \leq \beta_2 \leq 1$, otherwise he/she removes into orbit O_1 with probability $\bar{\beta}\beta_1$; $0 \leq \beta_1 \leq 1$.

Notice that if the threshold of customers in orbit O_1 is quite larger than that of orbit O_2, the customers in orbit O_1 will make the attempts firstly and vice versa. Afterward, customers go in the retrial queues and make attempts to reach the primary queue, where the attempt times are assumed also to be independent and exponentially distributed with mean $1/\alpha_i$, $i = 1, 2$. Finally, after the customer is served completely, he/she may decide either to join the retrial groups O_1 or O_2 again for another service with probability $\bar{\xi}\delta_1$; (δ_1 is the probability that the customer chooses orbit O_1), with $0 \leq \delta_1 \leq 1$, or $\bar{\xi}\delta_2$; (δ_2 is the probability that the customer chooses orbit O_2), with $0 \leq \delta_2 \leq 1$, or leaves the system forever with probability ξ, $0 \leq \xi \leq 1$.

This sort of system abstracts and characterizes different practical situations in the telecommunication networks. For example, the mechanism based automatic repeat request protocol in data transmission systems may be modeled by a retrial queue system with feedback, since lost packets are negatively acknowledged by the receivers, then the senders send them once again.

In this paper we provide approximate expressions for our queueing performance measures; we investigate the joint distribution of the server state and queue length under the system steady state assumption. The condition of system stability is assumed to be hold, Further analysis around the stability of retrial queues can be found in [2], where E. Altman and A. A. Borovkov provided the general conditions under which ρ (system's load) < 1 is a sufficient condition for the stability of retrial queuing systems.

3 Main result

Theorem 1 *For our retrial queueing model with two orbits, balking and feedback in the steady state:*

1. *The average of the queue length along the idle period of the server is expressed by*

$$\mathbb{E}(N_i, S = 0) = m_i \left(\frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + \alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right) \right.$$
$$\left(\frac{\alpha_i\beta + \mu(1 - \delta_i\bar{\xi})}{\lambda} - \bar{\beta}\beta_i \right)$$
$$F\left\{ \frac{\beta\bar{\beta}_i(\lambda + 2\alpha_i) + \bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i + \lambda) + \mu(1 - \delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} \right\} + \left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2} \right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + \alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right) \left(\frac{\bar{\beta}\beta_i(\lambda + 2\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda + 2\alpha_i) + \mu(1 - \delta_i\bar{\xi})} \right)$$
$$\left. -\bar{\beta}\beta_i F\left\{ \frac{\beta_i\bar{\beta}(\lambda + \alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i + \lambda) + \mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta.} \right\} \right).$$

2. *The average of the queue length along the busy period of the server is expressed by*

$$\mathbb{E}(N_i, S=1) = m_i\left(\frac{\beta}{\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.$$
$$\left. F\left\{\frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right).$$

3. *The average of the queue length is given by*

$$\mathbb{E}(N, S=0)+\mathbb{E}(N, S=1) = \sum_{i=1}^{2} m_i\left(\frac{\beta}{\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.$$
$$\left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda}+1-\bar{\beta}\beta_i\right)$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}$$
$$+\left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2}\right)\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i}, \frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}$$
$$\left.-\bar{\beta}\beta_i F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}, \frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i}, \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right).$$

Proof. To prove the theorem, we should firstly introduce the system statistical equilibrium equations for the system, so let us denote $N_1(t)$, $N_2(t)$ the number of repeated calls in the the retrial queue O_1 respectively O_2 at time t, and $S(t)$ represents the server state, where it takes two values 1 or 0 at time t when the server is busy or idle respectively. Thus, a process $\{S(t), N_1(t), N_2(t)\}$ which describes the number of customers in the system is the simplest and simultaneously the most important process associated with the retrial queueing system described in Fig.1.

To simplify our analysis, we suppose that the service time function $B(t)$ is exponentially distributed. Thus, $\{S(t), N_1(t), N_2(t)\}$ forms a markov process, where we can consider the markov chain of this process representing this system is embedded at jump customers arrival times rather than a chain embedded at service completion epochs. Hence, the process $\{(S(t), N_1(t), N_2(t)) : t \succ$

$0\}$ forms a Markov chain with a state space $\{0,1\}\times\{0,1,...,N_1\}\times\{0,1,...,N_2\}$, where $\{S,N_1,N_2\}\approx\lim_{t\to\infty}\{S(t),N_1(t),N_2(t)\}$ in the steady state.
As a result, in the steady state the joint probabilities of server state S and the retrial queue lengths N_1, N_2, $P_{0n_1n_2}=\mathbb{P}\{S=0,N_1=n_1,N_2=n_2\}$, and $P_{1n_1n_2}=\mathbb{P}\{S=1,N_1=n_1,N_2=n_2\}$, can be characterized through the corresponding partial generating functions for $|z_1|\leq 1$, $|z_2|\leq 1$ by $P_0(z_1)=\sum_{n_1=0}^{\infty}P_{0n_1n_2}z_1^{n_1}$, $P_0(z_2)=\sum_{n_2=0}^{\infty}P_{0n_1n_2}z_2^{n_2}$ and $P_1(z_1)=\sum_{n_1=0}^{\infty}P_{1n_1n_2}z_1^{n_1}$, $P_1(z_2)=\sum_{n_2=0}^{\infty}P_{1n_1n_2}z_2^{n_2}$. Consequently, we can describe the set of statistical equilibrium equations for these probabilities $(P_{0n_1n_2},P_{1n_1n_2})$ as follows:

$$(\lambda+n_1\alpha_1)P_{0n_1n_2}=\xi\mu P_{1n_1n_2}+\bar{\xi}\delta_1\mu P_{1n_1-1n_2} \tag{1}$$

$$\begin{aligned}(\lambda\beta\beta_1+\mu+n_1\beta\alpha_1)P_{1n_1n_2}=&\bar{\beta}\beta_1\lambda P_{1n_1-1n_2}+(n_1+1)\beta\alpha_1P_{1n_1+1n_2}\\&+(n_1+1)\alpha_1P_{0n_1+1n_2}+\lambda P_{0n_1n_2}\end{aligned} \tag{2}$$

$$(\lambda+n_2\alpha_2)P_{0n_1n_2}=\xi\mu P_{1n_1n_2}+\bar{\xi}\delta_2\mu P_{1n_1n_2-1} \tag{3}$$

$$\begin{aligned}(\lambda\bar{\beta}\beta_2+\mu+n_2\beta\alpha_2)P_{1n_1n_2}=&\bar{\beta}\beta_2\lambda P_{1n_1n_2-1}+(n_2+1)\beta\alpha_2P_{1n_1n_2+1}\\&+(n_2+1)\alpha_2P_{0n_1n_2+1}+\lambda P_{0n_1n_2}.\end{aligned} \tag{4}$$

Now to continue in deriving the joint distribution, we multiply the equations (1), (2), (3) and (4) by $\sum_{i=n_i}^{\infty}z_i^{n_i}$, $i=1,2$ which yields to the following equations :

$$\lambda P_0(z_i)+\alpha_iz_iP_0'(z_i)=\xi\mu P_1(z_i)+\bar{\xi}\delta_i\mu z_iP_1(z_i) \tag{5}$$

$$\left[\lambda\bar{\beta}\beta_i(1-z_i)+\mu)\right]P_1(z_i)+\alpha_i\beta(z_i-1)P_1'(z_i)=\alpha_iP_0'(z_i)+\lambda P_0(z_i). \tag{6}$$

By taking the sum of equation (5) and (6), then divide the sum by (z_i-1) we obtain

$$\alpha_iP_0'(z_i)+\alpha_i\beta P_1'(z_i)=(\bar{\beta}\beta_i\lambda+\bar{\xi}\delta_i\mu)P_1(z_i). \tag{7}$$

By substituting equation (7) into (6), we can express $P_0(z_i)$ in terms of $P_1'(z_i)$, $P_1(z_i)$ as follows:

$$P_0(z_i)=(\tfrac{\alpha_i\beta}{\lambda})z_iP_1'(z_i)+\left(\tfrac{\mu}{\lambda}(1-\delta_i\bar{\xi})-\bar{\beta}\beta_iz_i\right)P_1(z_i). \tag{8}$$

By differentiating equation (8), we get

$$P_0'(z_i)=\tfrac{\alpha_i\beta}{\lambda}z_iP_1''(z_i)+\left(\tfrac{\mu(1-\delta_i\bar{\xi})+\alpha_i\beta}{\lambda}-\bar{\beta}\beta_iz_i\right)P_1'(z_i)-\bar{\beta}\beta_iP_1(z_i). \tag{9}$$

By substituting equations (8) and (9) into (5), we obtain a differential equation of $P_1(z_i)$

$$
\begin{aligned}
& z_i P_1''(z_i) + \left(\frac{\mu(1-\delta_i\bar{\xi}) + (\lambda+\alpha_i)\beta}{\alpha_i\beta} - \frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta} z_i \right) P_1'(z_i) \\
& - \frac{\lambda\left(\bar{\beta}\beta_i(\lambda+\alpha_i) + (1-\bar{\xi}\delta_i)\mu\right)}{\alpha_i^2\beta} P_1(z_i) = 0.
\end{aligned} \tag{10}
$$

Consequently, we transform the equation (10) into Kummer's differential equation, since it has already a solution.
Let

$$
Y(x_i) = P_1(z_i(x_i)) \text{ and } z_i = \frac{\beta\alpha_i}{\bar{\beta}\beta_i\lambda} x_i, \quad i = 1, 2
$$

which transforms (10) into

$$
x_i Y_i''(x_i) + \left(\tfrac{(\lambda+\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\beta\alpha_i} - x_i \right) Y_i'(x_i) - \left(\tfrac{\beta\bar{\beta}(\alpha_i+\lambda) + \mu\delta_i\bar{\xi}}{\alpha_i\beta\bar{\beta}} \right) Y_i(x_i) = 0. \tag{11}
$$

The equation (11) can be rewritten as follows

$$
x_i Y_i''(x_i) + (d_i - x_i) Y'(x) - a_i Y(x_i) = 0 \tag{12}
$$

such that $a_i = \frac{\beta\bar{\beta}(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\alpha_i\beta\bar{\beta}}$ and $d_i = \frac{(\lambda+\alpha_i)\beta+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i}$. Referring to [1], [13], the equation (12) has a regular singular point at $x_i = 0$, and an irregular singularity at $x_i = \infty$. Furthermore, the solution of equation (12) is found analytically in a unite circle, $U = \{x : |x| \prec 1\}$ which represents in turn the solution of kummer's function $Y(x_i)$ and expressed by $Y(x_i) = m_i \times F(a_i; d_i; x_i)$, $m_i \neq 0$ so, equation (10) is solved for $P_1(z_i)$ as follows

$$
\begin{aligned}
P_1(z_i) = m_i \times F \Bigg\{ & \frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \\
& \frac{(\lambda+\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\alpha_i\beta}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} z_i \Bigg\}, \quad |z_i| \leq 1.
\end{aligned} \tag{13}
$$

Referring to [13], the first derivative of Kummer's function $F(a_i; d_i; x_i)$ is defined as follows: $\frac{dF}{dx_i} = \frac{a_i}{d_i} F(a_i+1; d_i+1; x_i)$, hence $P_1'(z_i)$ is expressed as follows:

$$
\begin{aligned}
P_1'(z_i) = m_i \Bigg\{ & \frac{\beta}{\bar{\beta}\beta_i} \left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i) + \bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i) + \mu(1-\delta_i\bar{\xi})} \right) F \Bigg\{ \frac{\bar{\beta}\beta_i(\lambda+2\alpha_i) + \bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i}; \\
& \frac{(\lambda+2\alpha_i)\beta + \mu(1-\delta_i\bar{\xi})}{\alpha_i\beta}; \frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta} z_i \Bigg\} \Bigg\}, \quad |z_i| \leq 1.
\end{aligned} \tag{14}
$$

Then we replace into equation (8) for $P_0(z_i)$, $P_1(z_i)$ and $P_1'(z_i)$ by their equivalence in equations (13) and (14), and hence $P_0(z_i)$ is expressed as follows:

$$\begin{aligned}P_0(z_i) = m_i &\left[\frac{\alpha_i\beta^2}{\lambda\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.\\ &F\left\{\frac{\beta\bar{\beta_i}(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\\ &+\left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda}-\bar{\beta}\beta_i\right)\\ &\left.F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right]\end{aligned}\tag{15}$$

Then at the boundary condition, where $z_i = 1, i = 1, 2$ we can ge the value of m through $P_0(1) + P_1(1) = 1$

$$\begin{aligned}m_i = &\left[\frac{\alpha_i\beta^2}{\lambda\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.\\ &F\left\{\frac{\beta\bar{\beta_i}(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\\ &+\left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda}-\bar{\beta}\beta_i+1\right)\\ &\left.F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right]^{-1}\end{aligned}\tag{16}$$

So, the generating functions of the joint distribution of server state S and queue length N_i are given by

$$\begin{aligned}P_0(z_i) = \mathbb{E}(z_i^{N_i}, S=0) = m_i &\left[\frac{\alpha_i\beta^2}{\lambda\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.\\ &F\left\{\frac{\beta\bar{\beta_i}(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\\ &+\left(\frac{\mu(1-\delta_1\bar{\xi})}{\lambda}-\bar{\beta}\beta_i\right)\\ &\left.F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}z_i\right\}\right]\\ P_1(z_i) = \mathbb{E}(z_i^{N_i} : S=1) = m_i\cdot F&\left\{\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i};\right.\\ &\left.\frac{\beta(\lambda+\alpha_i)+(1-\delta_i\bar{\xi})\mu}{\beta\alpha_i};\frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta}z_i\right\}, |z_i|\leq 1.\end{aligned}$$

Consequently, the average of the queue length along the idle period of the server is equivalent to $P_0'(1)$, which is expressed by

$$\mathbb{E}(N_i, S=0) = m_i\left(\frac{\beta}{\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda}-\bar{\beta}\beta_i\right)\right.$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i},\frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}+\left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2}\right)$$
$$\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i},\frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}$$
$$\left.-\bar{\beta}\beta_i F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right)$$

And the average of the queue length along the busy period of the server is equivalent to $P_1'(1)$, which is expressed by

$$\mathbb{E}(z_i^{N_i}:S=1) = m_i\left\{\frac{\beta}{\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.$$
$$\left.F\left\{\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i};\frac{(\lambda+2\alpha_i)\beta+\mu(1-\delta_i\bar{\xi})}{\alpha_i\beta};\frac{\bar{\beta}\beta_i\lambda}{\alpha_i\beta}\right\}\right\} \tag{17}$$

Thus the average of the queue length in the retrial queuing system is the sum of $P_0'(1)$ and $P_1'(1)$, which is given by

$$\mathbb{E}(N, S=0)+\mathbb{E}(N,S=1) = \sum_{i=1}^{2} m_i\left(\frac{\beta}{\bar{\beta}\beta_i}\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\right.$$
$$\left(\frac{\alpha_i\beta+\mu(1-\delta_i\bar{\xi})}{\lambda}+1-\bar{\beta}\beta_i\right)$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i},\frac{\beta(2\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}$$
$$+\left(\frac{\alpha_i\beta^3}{\lambda\bar{\beta}^2\beta_i^2}\right)\left(\frac{\bar{\beta}\beta_i(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)\left(\frac{\bar{\beta}\beta_i(\lambda+2\alpha_i)+\bar{\xi}\delta_i\mu}{\beta(\lambda+2\alpha_i)+\mu(1-\delta_i\bar{\xi})}\right)$$
$$F\left\{\frac{\beta\bar{\beta}_i(\lambda+3\alpha_i)+\bar{\xi}\delta_i\mu}{\beta\beta_i\alpha_i},\frac{\beta(3\alpha_i+\lambda)+\mu(1-\delta_i\bar{\xi})}{\beta\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}$$
$$\left.-\bar{\beta}\beta_i F\left\{\frac{\beta_i\bar{\beta}(\lambda+\alpha_i)+\bar{\xi}\delta_i\mu}{\bar{\beta}\beta_i\alpha_i},\frac{\bar{\beta}\beta_i(\alpha_i+\lambda)+\mu\delta_i\bar{\xi}}{\bar{\beta}\beta_i\alpha_i},\frac{\lambda\bar{\beta}\beta_i}{\alpha_i\beta}\right\}\right)$$

□

4 Numerical results

The average waiting time W in the steady state is often considered to be the most important of performance measures in retrial queuing systems. However, W is an average over all primary calls, including those calls which receive immediate service and really do not wait at all. A better grasp of understanding the waiting time process can be obtained by studying first the relationship between the retrial queue length $\mathbb{E}(N) = \mathbb{E}(N_1) + \mathbb{E}(N_2)$ and other inputs,outputs and feedback parameters. We have conducted some preliminary analysis through some simulations done on the queue lengths, in order to show the impact of the different parameters and its relationship with the retrial queue length $\mathbb{E}(N)$. The primary objective behind this was to understand what does happen at some telecommunication systems where redials or connection retrials arise naturally.

These analysis involved three scenarios "figure 2-figure 4" in order to clarify the relations in different situations among the input, output, balk and feedback parameters. These scenarios are realized through simulations via Matlab program. To begin, we chose a significant values for the parameters so as to meet the requirements of the phase-merging algorithm.

For the first figure, for each value of $\bar{\xi}$ ($\bar{\xi}$ = 0; 0.2; 0.4; 0.6; 0.8; 1) selected, we vary μ from 0 to 1 in increments of 0.1, where we evaluate $\mathbb{E}(N)$ at different values of service completion probability while $\beta_1 = \beta_2 = \alpha_1 = \alpha_2 = \delta_1 = \delta_2 = 0.5$, $\beta = 0.7$, $\lambda = 0.7$. The numerical results are summarized in the following table:

μ	Average Retrial Queue Length	$\xi = 1$	$\xi = 0.8$	$\xi = 0.6$	$\xi = 0.4$	$\xi = 0.2$	$\xi = 0$
0	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2.2963	2.2963	2.2963	2.2963	2.2963	2.2963
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2.6852	2,5577	2,4302	2,3026	2,1749	2,0472
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.0167	2,7798	2,5434	2,3077	2,0726	1,8383
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.3060	2,9723	2,6409	2,3119	1,9857	1,6625
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.5629	3,1417	2,7258	2,3154	1,9111	1,5135
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3.7937	3,2926	2,8005	2,3183	1,8467	1,3864
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.0031	3,4280	2,8670	2,3208	1,7905	1,2771
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.1945	3,5506	2,9264	2,3229	1,7413	1,1826
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.3705	3,6623	2,9800	2,3247	1,6977	1,1002
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.5332	3,7645	3,0284	2,3262	1,6591	1,0278
1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4.6843	3,8586	3,0726	2,3276	1,6245	0,9639

The first figure shows that along the increase of μ the retrial queue lengths increase when the values of $\bar{\xi}$ become larger; for instance when $\bar{\xi}$ = 1; 0.8; 0.6 and decrease when $\bar{\xi}$ become smaller; for instance when $\bar{\xi}$ = 0; 0.2; 0.4. Obviously, this refers to the possibility of accepting repeated and primary calls becomes large. This figure shows us also that when ξ becomes greater than 0.6 or the feedback probability becomes less than 0.6, then $\mathbb{E}(N)$ is not affected remarkably or it decreases very slowly.

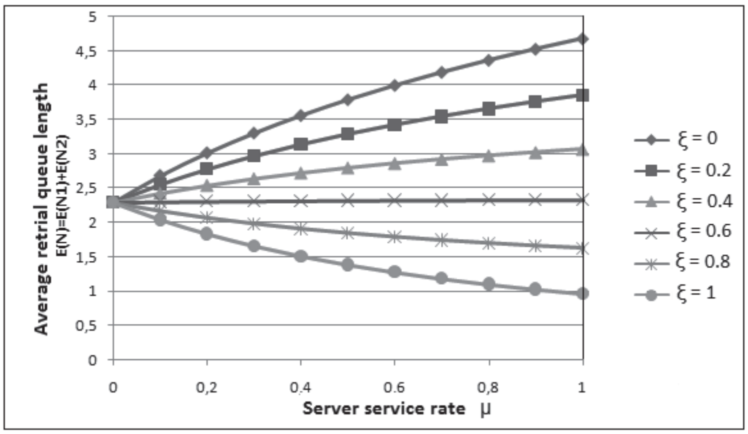

Figure 2: Average retrial queue length $\mathbb{E}(N)$ & service server rate μ

For the second figure, for each value of $\bar{\beta}$ such that

- For $\beta = 0,5$ we choose a significant parameters $\alpha_1 = \alpha_2 = 0.7$, $\delta_1 = 0.1$, $\delta_2 = 0.9$ and $\beta_1 = \beta_2 = 0.5$.
- For $\beta = 0.7$, we choose a significant parameters $\alpha_1 = \alpha_2 = 0.7$, $\delta_1 = 0.1$, $\delta_2 = 0.9$, and $\beta_1 = 0.6, \beta_2 = 0.4$,

we vary ξ from 0 to 1 in increments of 0.1, where we evaluate $E(N)$ at different values of balking probability β, while $\mu = 0.8$ and $\lambda = 0.7$. The numerical results are summarized in the following table:

ξ	Average Retrial Queue Length	$\bar{\beta} = 0.3$	$\bar{\beta} = 0.5$
0	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,8845	2,9873
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,4893	2,7386
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	4,1030	2,4980
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3,7245	2,2662
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	3,3533	2,0440
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,9887	1,8319
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,6304	1,6302
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	2,2782	1,4391
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,9322	1,2588
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,5926	1,0889
1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	1,2598	0,9294

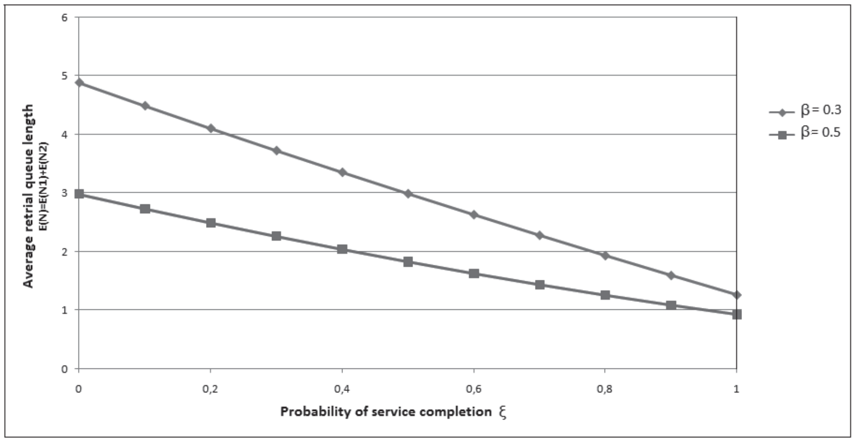

Figure 3: Average retrial queue length $\mathbb{E}(N)$ & probability of service completion ξ

The second figure shows that $\mathbb{E}(N)$ for our model with balking and feedback is not affected by feedback probability $\bar{\xi}$ when the probability $\bar{\beta}$ of non-balking or returning to retrial group after customer attempt's failure becomes less than 0.5. However, $\mathbb{E}(N)$ increases rapidly as $\bar{\xi}$ and $\bar{\beta}$ become high.

For the third figure, For each value of α_i ($\alpha_1 = \alpha_2 = 0.1$ and $\alpha_1 = \alpha_2 = 0.8$) selected, we vary $\bar{\beta}$ from 0.1 to 0.9 in increments of 0.1, such that for a good requirement we choose

for $\bar{\beta} = 0.1$	$\beta_1 = 0.7$	$\beta_2 = 0.3$
for $\bar{\beta} = 0.2$	$\beta_1 = 0.9$	$\beta_2 = 0.1$
for $\bar{\beta} = 0.3$	$\beta_1 = 0.95$	$\beta_2 = 0.05$
for $\bar{\beta} = 0.4$	$\beta_1 = 0.97$	$\beta_2 = 0.03$
for $\bar{\beta} = 0.5$	$\beta_1 = 0.98$	$\beta_2 = 0.02$
for $\bar{\beta} = 0.6$	$\beta_1 = 0.99$	$\beta_2 = 0.01$
for $\bar{\beta} = 0.7$	$\beta_1 = 0.993$	$\beta_2 = 0.007$
for $\bar{\beta} = 0.8$	$\beta_1 = 0.996$	$\beta_2 = 0.004$
for $\bar{\beta} = 0.9$	$\beta_1 = 0.998$	$\beta_2 = 0.002$

Then, we evaluate $\mathbb{E}(N)$ at different values of retrial probability α_i, while $\delta_1 = \delta_2 = 0.5$, $\xi = 0.5$, $\mu = 0.8$ and $\lambda = 0.7$. The numerical results are summarized in the following table:

$\bar{\beta}$	Average Retrial Queue Length	$\alpha_1 = \alpha_2 = 0.1$	$\alpha_1 = \alpha_2 = 0.8$
0.1	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	7.3610	5,8755
0.2	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	7,8458	6,1070
0.3	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	8,9262	6,7783
0.4	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	9,7914	7,3964
0.5	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	10,2588	7,8443
0.6	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	14,2576	10,7133
0.7	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	14,3527	11,1251
0.8	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	15,7566	13,0344
0.9	$\mathbb{E}(N, C = 0) + \mathbb{E}(N, C = 1)$	16,3376	13,9861

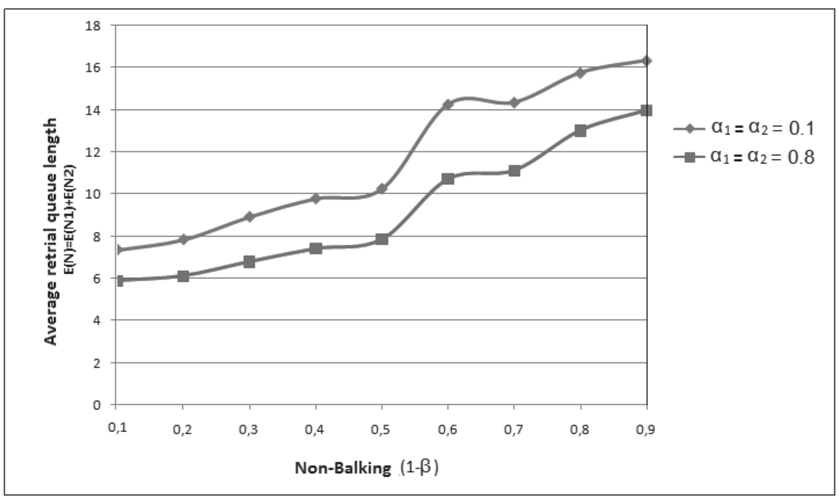

Figure 4: Average retrial queue length $\mathbb{E}(N)$ & non-balking rate $\bar{\beta}$

Figure 4 shows that along the design of retrial queuing system, we have to assign equivalent values for the non-balking probability $\bar{\beta}$ and the retrial probability α_i in order to keep the retrial queue length as short as possible. This can be concluded from the figure since when α_i takes values greater or equal to 0.5, and $\bar{\beta}$ gets values less than 0.5 $\mathbb{E}(N)$ becomes small.

As a conclusion, we conclude that Figures 2 through 4 indicate that the phase-merging algorithm is reasonably effective in approximating $\mathbb{E}(N)$, for all values of $\mu, \xi, \beta,$ and α.

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