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Subordination results for certain classes of analytic functions defined by a generalized differential operator

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Abstract. In this paper, we derive some subordination results for certain classes of analytic functions defined by a generalized differential operator using the principle of subordination and a subordination theorem. Relevant connections of the results presented here with those obtained in earlier works are also pointed out.

1 Introduction and preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. We denote by $\mathcal{S}$, $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{C}$, the class of all functions in $\mathcal{A}$ which are, respectively, univalent, starlike, convex and close-to-convex in $\mathcal{U}$. For functions f given by (1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

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the Hadamard product (or convolution) of f and g is defined by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Let $\mathcal{T}(\gamma, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the inequality

$$\Re\left(\frac{zf'(z) + \gamma z^2 f''(z)}{(1-\gamma)f(z) + \gamma z f'(z)}\right) > \alpha, \qquad z \in \mathcal{U},$$

for some α $(0 \le \alpha < 1)$ and γ $(0 \le \gamma < 1)$, and let $\mathcal{C}(\gamma, \alpha)$ denote the class of functions in $\mathcal{A}$ satisfying the inequality

$$\Re\left(\frac{\gamma z^3 f'''(z) + (2\gamma+1)z^2 f''(z) + zf'(z)}{\gamma z^2 f''(z) + zf'(z)}\right) > \alpha, \qquad z \in \mathcal{U},$$

for some α $(0 \le \alpha < 1)$ and γ $(0 \le \gamma < 1)$. We note that

$$f \in \mathcal{C}(\gamma, \alpha) \Longleftrightarrow zf' \in \mathcal{T}(\gamma, \alpha).$$

The classes $\mathcal{T}(\gamma, \alpha)$ and $\mathcal{C}(\gamma, \alpha)$ were introduced and investigated by O. Altıntaş [2], and M. Kamali and S. Akbulut [4], respectively.

Let $\mathcal{M}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta, \; z \in \mathcal{U},$$

for some β $(\beta > 1)$, and let $\mathcal{N}(\beta)$ be the subclass of $\mathcal{A}$ consisting of functions f which satisfy the inequality

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \beta, \; z \in \mathcal{U},$$

for some β $(\beta > 1)$. The classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ were introduced and investigated by S. Owa and H. M. Srivastava [6] (see also J. Nishiwaki and S. Owa [5], S. Owa and J. Nishiwaki [7], H. M. Srivastava and A. A. Attiya [9]).

Let $\alpha_1, \alpha_2, \ldots, \alpha_q$ and $\beta_1, \beta_2, \ldots, \beta_s$ $(q, s \in \mathbb{N} \cup \{0\}, q \le s+1)$ be complex numbers such that $\beta_k \ne 0, -1, -2, \ldots$ for $k \in \{1, 2, \ldots, s\}$. The generalized hypergeometric function ${}_qF_s$ is given by

$${}_qF_s(\alpha_1, \alpha_2, \ldots, \alpha_q; \beta_1, \beta_2, \ldots, \beta_s; z) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_q)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_s)_n} \frac{z^n}{n!}, \qquad (z \in \mathcal{U}),$$

where $(x)_n$ denotes the Pochhammer symbol defined by

$$(x)_n = x(x+1)(x+2)\cdots(x+n-1) \text{ for } n \in \mathbb{N} \text{ and } (x)_0 = 1.$$

Corresponding to a function $\mathcal{G}^p_{q,s}(\alpha_1;\beta_1;z)$ defined by

$$\mathcal{G}_{q,s}(\alpha_1,\beta_1;z) := z\,{}_qF_s(\alpha_1,\alpha_2,\ldots,\alpha_q;\beta_1,\beta_2,\ldots,\beta_s;z),$$

we now define the following generalized differential operator:

$$\begin{aligned} D^0_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z),\\ D^1_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= D_{\lambda\mu}(\alpha_1,\beta_1)f(z) = \lambda\mu z^2(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z))''+\\ &\quad + (\lambda-\mu)z(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z))'+\\ &\quad + (1-\lambda+\mu)(f(z)*\mathcal{G}_{q,s}(\alpha_1,\beta_1;z)), \text{ and}\\ D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) &= D_{\lambda\mu}(D^{m-1}_{\lambda}(\alpha_1,\beta_1)f(z)), \end{aligned}$$

where $0 \le \mu \le \lambda \le 1$ and $m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$.

If $f(z) \in \mathcal{A}$, then we have

$$D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z) = z + \sum_{n=2}^{\infty} \vartheta^m_n\,\sigma_n\,a_n z^n, \tag{2}$$

where

$$\vartheta_n = 1 + (\lambda\mu n + \lambda - \mu)(n-1) \tag{3}$$

and

$$\sigma_n = \frac{(\alpha_1)_{n-1}(\alpha_2)_{n-1}\ldots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}(\beta_2)_{n-1}\ldots(\beta_s)_{n-1}\,(n-1)!}. \tag{4}$$

It can be seen that, by specializing the parameters the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to many known and new differential operators. In particular, when $m = 0$ the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the well- known Dziok-Srivastava operator [3] and for $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$, it reduces to the operator introduced by F. M. Al-Oboudi [1]. Further we remark that, when $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, and $\alpha_2 = 1$ the operator $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$ reduces to the operator introduced by G. S. Sălăgean [8].

For simplicity, in the sequel, we will write $D^m_{\lambda\mu}f(z)$ instead of $D^m_{\lambda\mu}(\alpha_1,\beta_1)f(z)$.

Motivated by the above mentioned function classes, we now introduce the following subclasses of $\mathcal{A}$ involving the generalized differential operator $D^m_{\lambda\mu}f(z)$.

Definition 1 *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$ *if it satisfies the following inequality*

$$\Re\left\{ \frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)} \right\} > \alpha, \qquad z \in \mathcal{U},$$

where

$$m \in \mathbb{N}_0, \quad 0 \leq \gamma \leq 1, \quad 0 \leq \alpha < 1.$$

It is easy to see that the classes $\mathcal{T}(\gamma, \alpha)$ and $\mathcal{C}(\gamma, \alpha)$ are special cases of the class $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$.

Definition 2 *A function* $f \in \mathcal{A}$ *is said to be in the class* $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$ *if it satisfies the following inequality*

$$\Re\left\{ \frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)} \right\} < \beta, \qquad z \in \mathcal{U},$$

where

$$m \in \mathbb{N}_0, \quad 0 \leq \gamma \leq 1, \quad \beta > 1.$$

It is also easy to see that the classes $\mathcal{M}(\beta)$ and $\mathcal{N}(\beta)$ are special cases of the class $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$.

We now provide some coefficient inequalities associated with the function classes $\mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$ and $\mathcal{M}^m_{\lambda\mu}(\gamma, \beta)$.

2 Coefficient inequalities

Theorem 1 *Let* $0 \leq \alpha < 1$ *and* $0 \leq \gamma \leq 1$. *If* $f \in \mathcal{A}$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty} (1 - \gamma + \gamma\vartheta_n)(\vartheta_n - \alpha)\, \vartheta_n^m\, \sigma_n\, |a_n| \leq 1 - \alpha, \tag{5}$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively, then* $f \in \mathcal{S}^m_{\lambda\mu}(\gamma, \alpha)$.

Proof. It is suffices to show that

$$\left| \frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z) + \gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z) + \gamma D^{m+1}_{\lambda\mu}f(z)} - 1 \right| < 1 - \alpha, \qquad z \in \mathcal{U}.$$

Now we note that for any $z \in \mathcal{U}$,

$$\left|\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z)+\gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z)+\gamma D^{m+1}_{\lambda\mu}f(z)}-1\right| = \left|\frac{\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-1)\vartheta_n^m\sigma_n a_n z^{n-1}}{1+\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^m\sigma_n a_n z^{n-1}}\right|$$

$$\leq \frac{\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-1)\vartheta_n^m\sigma_n|a_n|}{1-\sum\limits_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^m\sigma_n|a_n|}.$$

It follows from (5) that the last expression is bounded by $1-\alpha$. This completes the proof of the theorem. □

Theorem 2 *Let* $\beta > 1$ *and* $0 \leq \gamma \leq 1$. *If* $f \in \mathcal{A}$ *satisfies the following coefficient inequality*

$$\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n+|\vartheta_n-2\beta|)\,\vartheta_n^m\,\sigma_n\,|a_n| \leq 2(\beta-1), \tag{6}$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively, then* $f \in \mathcal{M}^m_{\lambda\mu}(\gamma,\beta)$.

Proof. It is sufficient to show that

$$\left|\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z)+\gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z)+\gamma D^{m+1}_{\lambda\mu}f(z)}\right| < \left|\frac{(1-\gamma)D^{m+1}_{\lambda\mu}f(z)+\gamma D^{m+2}_{\lambda\mu}f(z)}{(1-\gamma)D^{m}_{\lambda\mu}f(z)+\gamma D^{m+1}_{\lambda\mu}f(z)}-2\beta\right|, \tag{7}$$

where $z \in \mathcal{U}$.

Now, we define $M \in \mathbb{R}$ by

$$\begin{aligned}M := {} & \left|(1-\gamma)D^{m+1}_{\lambda\mu}f(z)+\gamma D^{m+2}_{\lambda\mu}f(z)\right| - \\ & -\left|(1-\gamma)D^{m+1}_{\lambda\mu}f(z)+\gamma D^{m+2}_{\lambda\mu}f(z)-2\beta\big((1-\gamma)D^{m}_{\lambda\mu}f(z)+\gamma D^{m+1}_{\lambda\mu}f(z)\big)\right| = \\ = {} & \left|z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m+1}+\gamma\vartheta_n^{m+2}\right]\sigma_n a_n z^n\right| - \\ & -\left|z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m+1}+\gamma\vartheta_n^{m+2}\right]\sigma_n a_n z^n - \right. \\ & \left. -2\beta\left\{z+\sum_{n=2}^{\infty}\left[(1-\gamma)\vartheta_n^{m}+\gamma\vartheta_n^{m+1}\right]\sigma_n a_n z^n\right\}\right|.\end{aligned}$$

Thus, for $|z| = r < 1$, we have

$$M \leq r + \sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)\vartheta_n^{m+1}\sigma_n|a_n|r^n -$$

$$- \left[(2\beta-1)r - \sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)|\vartheta_n - 2\beta|\vartheta_n^m\sigma_n|a_n|r^n\right] <$$

$$< \left(\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n + |\vartheta_n - 2\beta|)\vartheta_n^m\sigma_n|a_n| - 2(\beta-1)\right)r.$$

It follows from (6) that $M < 0$, which implies that (7) holds. This completes the proof of the theorem. □

In view of Theorem (1) and Theorem (2), we now introduce the subclasses

$$\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\alpha) \subset \mathcal{S}_{\lambda\mu}^m(\gamma,\alpha) \qquad \text{and} \qquad \widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta) \subset \mathcal{M}_{\lambda\mu}^m(\gamma,\beta),$$

which consist of functions $f \in \mathcal{A}$ whose Taylor-Maclaurin coefficients satisfy the inequalities (5) and (6) respectively. We now derive some subordination results for the function classes $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\alpha)$ and $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma,\beta)$.

3 Subordination result for the class $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma,\beta)$

We will use of the following definitions and lemma to prove our result.

Definition 3 *(Subordination Principle) Let* $f(z)$ *and* $g(z)$ *be analytic in* $\mathcal{U}$*. Then we say that the function* $f(z)$ *is subordinate to* $g(z)$ *in* $\mathcal{U}$*, and write*

$$f \prec g \qquad \text{or} \qquad f(z) \prec g(z)$$

if there exists a Schwarz function $w(z)$*, analytic in* $\mathcal{U}$ *with*

$$w(0) = 0, \quad |w(z)| < 1 \quad (z \in \mathcal{U}),$$

such that

$$f(z) = g(w(z)) \qquad (z \in \mathcal{U}).$$

In particular, if the function $g(z)$ *is univalent in* $\mathcal{U}$*, then*

$$f(z) \prec g(z) \quad (z \in \mathcal{U}) \Longleftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathcal{U}) \subset g(\mathcal{U}).$$

Definition 4 *(Subordinating Factor Sequence) A sequence* $\{b_n\}_{n=1}^{\infty}$ *of complex numbers is said to be a subordinating factor sequence if, whenever* $f(z)$ *of the form (1) is analytic, univalent and convex in* $\mathcal{U}$*, we have the subordination given by*

$$\sum_{n=1}^{\infty} a_n b_n z^n \prec f(z) \qquad (z \in \mathcal{U};\ a_1 := 1).$$

Lemma 1 *(See Wilf [11]) The sequence* $\{b_n\}_{n=1}^{\infty}$ *is a subordinating factor sequence if and only if*

$$\Re\left(1 + 2\sum_{n=1}^{\infty} b_n z^n\right) > 0 \qquad (z \in \mathcal{U}).$$

Theorem 3 *Let the function* $f(z)$ *defined by (1) be in the class* $\widetilde{\mathcal{S}}_{\lambda\mu}^{m}(\gamma, \alpha)$*. If* $g(z) \in \mathcal{K}$*, then*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}(f * g)(z) \prec g(z) \tag{8}$$

$$(z \in \mathcal{U},\ m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1)$$

and

$$\Re(f) > -\frac{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}, \tag{9}$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively. The constant factor in the subordination result (8)*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}$$

cannot be replaced by a larger one.

Proof. Let $f(z) \in \widetilde{\mathcal{S}}_{\lambda\mu}^{m}(\gamma, \alpha)$ and suppose that

$$g(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{K}.$$

Then we readily have

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}(f * g)(z) =$$

$$= \frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}\left(z + \sum_{n=2}^{\infty} a_n c_n z^n\right).$$

Thus, by Definition 4, the subordination result (8) will holds if

$$\left\{\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2]}\,a_n\right\}_{n=1}^{\infty}$$

is a subordinating factor sequence, with $a_1 = 1$. In view of Lemma 1 this is equivalent to the following inequality:

$$\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,a_n z^n\right\}>0 \qquad (z\in\mathcal{U}). \tag{10}$$

Since $(1-\gamma+\gamma\vartheta_n)(\vartheta_n-\alpha)\,\vartheta_n^m\sigma_n \quad (n\geq 2,\ m\in\mathbb{N}_0)$ is an increasing function of n, we have

$$\begin{aligned}
&\Re\left\{1+\sum_{n=1}^{\infty}\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,a_n z^n\right\}\\
&\quad=\Re\left\{1+\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,z\right.\\
&\qquad+\frac{1}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\cdot\\
&\qquad\left.\cdot\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2\,a_n z^n\right\}\\
&\quad\geq 1-\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\qquad-\frac{1}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\cdot\\
&\qquad\cdot\sum_{n=2}^{\infty}(1-\gamma+\gamma\vartheta_n)(\vartheta_n-\alpha)\vartheta_n^m\sigma_n\,|a_n|\,r^n\\
&\quad> 1-\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\qquad-\frac{1-\alpha}{(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m\sigma_2}\,r\\
&\quad= 1-r>0 \qquad (|z|=r<1),
\end{aligned}$$

where we have also made use of the assertion (5) of Theorem 1. This evidently proves the inequality (10), and hence also the subordination result (8) asserted by Theorem 3. The inequality (9) follows from (8) upon setting

$$g(z) = \frac{z}{1-z} = z + \sum_{n=2}^{\infty} z^n \in \mathcal{K}.$$

Next we consider the function

$$q(z) := z - \frac{1-\alpha}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m \sigma_2} z^2, \tag{11}$$

$$(m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1),$$

where ϑ_n and σ_n are given by (3) and (4) respectively, which is a member of the class $\widetilde{\mathcal{S}}_{\lambda\mu}^m(\gamma, \alpha)$. Then, by using (8), we have

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m \sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m \sigma_2]} q(z) \prec \frac{z}{1-z} \qquad (z \in \mathcal{U}).$$

One can easily verify for the function $q(z)$ defined by (11) that

$$\min\left\{\Re\left(\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m \sigma_2}{2[(1-\alpha)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2-\alpha)\vartheta_2^m \sigma_2]} q(z)\right)\right\} = -\frac{1}{2} \qquad (z \in \mathcal{U}),$$

which completes the proof of Theorem 3. □

Remark 1 *Setting* $\gamma = 0$, $\lambda = 1$, $\mu = 0$, $q = 2$, $s = 1$, $\alpha_1 = \beta_1$, *and* $\alpha_2 = 1$ *in Theorem 3, we get the corresponding result obtained by S. Sümer Eker et al. [10].*

4 Subordination result for the class $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma, \beta)$

The proof of the following subordination result is similar to that of Theorem 3. We, therefore, omit the analogous details involved.

Theorem 4 *Let the function* $f(z)$ *defined by (1) be in the class* $\widetilde{\mathcal{M}}_{\lambda\mu}^m(\gamma, \beta)$. *If* $g(z) \in \mathcal{K}$, *then*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m \sigma_2}{2[2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m \sigma_2]} (f * g)(z) \prec g(z) \tag{12}$$

$$(z \in \mathcal{U},\ m \in \mathbb{N}_0,\ 0 \le \gamma \le 1,\ 0 \le \alpha < 1)$$

and

$$\Re(f) > -\frac{2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2}{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2},$$

where ϑ_n *and* σ_n *are given by (3) and (4) respectively. The constant factor*

$$\frac{(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2}{2[2(\beta-1)+(1-\gamma+\gamma\vartheta_2)(\vartheta_2+|\vartheta_2-2\beta|)\vartheta_2^m\sigma_2]}$$

in the subordination result (12) cannot be replaced by a larger one.

Remark 2 *Setting* $m = 0$, *or* $\gamma = 0, \lambda = 1, \mu = 0, q = 2, s = 1, \alpha_1 = \beta_1$ *and* $\alpha_2 = 1$ *in Theorem 3, we get the corresponding results obtained by H. M. Srivastava and A. A. Attiya. [9].*

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