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COEFFICIENT BOUNDS FOR ω-QUASI-CONVEX FUNCTIONS DEFINED ON THE UNIT DISC

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Abstract. Main aim of this paper is to introduce a generalized class of ω-quasi-convex functions $f(z)$ defined on the unit disk $E := \{z/|z| < 1\}$ normalized by the conditions $f(0) = 0 = f'(0) - 1$ and we obtain several sharp bounds for $f(z)$, its inverse $f^{-1}(w)$, $log(\frac{f(z)}{z})$ and the Second Hankel determinant $|a_2a_4 - a_3^2|$.

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1. Introduction And Basic Results

Denote by S the family of regular and univalent functions in the unit disk E with the series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Let us designate by C and K the well-known sub-classes of convex and close-to-convex functions respectively. In the year 1980, K.I.Noor and D.K.Thomas introduced the concept of quasi convexity and investigated various properties by defining a new subclass of quasi-convex functions(C^*) in [9]. Moreover, $f(z)$ is quasi-convex if and only if $zf'(z)$ is close-to-convex. It was further generalized to α-quasi-convex functions by K.I.Noor and F.M.Al-oboudi in [8].

For $\alpha \geq 0$, if the real part of arithmetic mean of $\frac{f'(z)}{g'(z)}$ and $\frac{(zf'(z))'}{g'(z)}$ is positive, where $z \in E$ and $g(z) \in C$, then $f(z)$ is said to be α-quasi-convex. In the year 2018, D.K.Thomas in [12] introduced and investigated the subclass M^γ of γ-starlike functions by considering the geometric mean of the quantities $\frac{zf'(z)}{f(z)}$ and $\frac{(zf'(z))'}{f'(z)}$ for functions $f(z)$ of the form (1). Motivated by their work, we in this paper, define a subclass Q^{ω} of ω-quasi-convex functions. A function $f(z)$ of the form (1) is said to be in Q^{ω} if there is a convex function $g(z)$ in C such that

$$Re[\{\frac{(zf'(z))'}{g'(z)}\}^{\omega}\{\frac{f'(z)}{g'(z)}\}^{1-\omega}\}] \geq 0 \tag{2}$$

for $z \in E$.

We observe that when $\omega = 1$, $Q^{\omega} = C^{*}$, the class of quasi-convex functions.When $\omega = 1$ and $f(z) = g(z)$, Q^{ω} reduces to the familiar class of convex functions. Thus every ω-quasi-convex function is convex and hence univalent in E. If $\omega = 0$, Q^{ω} becomes K, the class of close-to-convex functions studied by Kaplan [4]. For $0 \leq \omega \leq 1$, the transition from close-to-convexity to quasi-convexity is smooth.

2. PRELIMINARIES

We need the following lemmas which will be used as tools to prove our results. Denote by $\mathbb{P}$ the class of Caratheodory functions $p(z)$ analytic in E for which $Re(p(z)) > 0$, $p(0) = 1$ and

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

Lemma 1. *[2] If $p(z) \in \mathbb{P}$, is of the form (3), then*

$$2p_2 = p_1^2 + y(4 - p_1^2) \tag{4}$$

for some y, $|y| \leq 1$ and

$$4p_3 = p_1^3 + 2(4 - p_1^2)p_1 y - p_1(4 - p_1^2)y^2 + 2(4 - p_1^2)(1 - |y|^2)\xi \tag{5}$$

for some $\xi, |\xi| \leq 1$.

Lemma 2. *[1] If $p(z) \in \mathbb{P}$, then the sharp estimate*

$$|p_n| \leq 2 \tag{6}$$

holds for n=1,2,....

Lemma 3. *[3] If $p(z) \in \mathbb{P}$,then the following estimate holds for n, k ∈ {1,2,...}*

$$|p_n - \mu_k p_{n-k}| \leq max\{2, 2\big|2\mu - 1\big|\}. \tag{7}$$

Lemma 4. *[6] If $0 \leq \beta \leq 1$ and $\beta(2\beta - 1) \leq \alpha \leq \beta$ then*

$$|p_3 - 2\beta p_1 p_2 + \alpha p_1^3| \leq 2. \tag{8}$$

3. THE COEFFICIENTS OF $f(z)$

In this section we state and prove a theorem that yields sharp coefficient bounds for functions belonging to Q^{ω} and in the sequel we obtain the results proved in [4] and [9].

Theorem 5. *Let $f(z) \in Q^{\omega}$ for $\omega \geq 0$ and given by (1). Then*

$$|a_2| \leq \frac{2}{(1+\omega)}$$

$$|a_3| \leq \begin{cases} \frac{(\omega^2+26\omega+9)}{3(1+\omega)^2(1+2\omega)} & , \omega \leq 1 \\ \frac{(\omega^2+8\omega+3)}{(1+\omega)^2(1+2\omega)} & , \omega \geq 1 \end{cases}$$

$$|a_4| \leq \begin{cases} \frac{2\omega^4+37\omega^3+160\omega^2+77\omega+12}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , 0 \leq \omega \leq 0.2 \\ \frac{2(\omega^4+11\omega^3+89\omega^2+37\omega+6)}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , 0.2 \leq \omega \leq 1 \\ \frac{4(\omega^4+11\omega^3+38\omega^2+19\omega+3)}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , \omega \geq 1 \end{cases}$$

Proof. Let g(z) be a convex function, with the Taylor series expansion

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

From (2), we have

$$\{\frac{(zf'(z))'}{g'(z)}\}^{\omega}\{\frac{f'(z)}{g'(z)}\}^{1-\omega} = p(z)$$

where $p(z) \in \mathbb{P}$. Then equating the coefficients we have

$$\begin{aligned}
a_2 &= \frac{p_1+2b_2}{2(1+\omega)} \\
a_3 &= \frac{p_2}{3(1+2\omega)} + \frac{2(1+3\omega)b_2p_1}{3(1+\omega)^2(1+2\omega)} + \frac{b_3}{(1+2\omega)} - \frac{\omega(\omega-1)p_1^2}{6(1+\omega)^2(1+2\omega)} \\
&\quad - \frac{2\omega(\omega-1)b_2^2}{3(1+\omega)^2(1+2\omega)} \\
a_4 &= \frac{1}{4(1+3\omega)}(p_3 + \frac{2(1+5\omega)b_2p_2}{(1+\omega)(1+2\omega)} + \frac{3(1+5\omega)b_3p_1}{(1+\omega)(1+2\omega)} - \frac{12\omega(\omega-1)b_2b_3}{(1+\omega)(1+2\omega)} \\
&\quad + \frac{2\omega(\omega-1)(1-5\omega)b_2^2p_1}{(1+\omega)^3(1+2\omega)} - \frac{2\omega(\omega-1)p_1p_2}{(1+\omega)(1+2\omega)} + \frac{\omega(\omega-1)(1-5\omega)b_2p_1^2}{(1+\omega)^3(1+2\omega)} \\
&\quad + \frac{\omega(\omega-1)(4\omega^2+3\omega+5)p_1^3}{6(1+\omega)^3(1+2\omega)} + \frac{4\omega(\omega-1)(4\omega^2+3\omega+5)b_2^3}{3(1+\omega)^3(1+2\omega)} + 4b_4)
\end{aligned}$$

Since $g(z) \in$ C,

$$\frac{(zg'(z))'}{g'(z)} = c(z)$$

where $c(z) \in \mathbb{P}$ and let $c(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$. Then $b_2 = \frac{c_1}{2}$, $b_3 = \frac{c_2}{6} + \frac{c_1^2}{6}$ and $b_4 = \frac{c_3}{12} + \frac{c_1c_2}{8} + \frac{c_1^3}{24}$. Thus we have,

$$\begin{aligned}
a_2 &= \frac{p_1 + c_1}{2(1+\omega)} \\
a_3 &= \frac{1}{3(1+2\omega)}[p_2 - \frac{\omega(\omega-1)p_1^2}{2(1+\omega)^2} + \frac{(1+3\omega)p_1c_1}{(1+\omega)^2} + \frac{1}{2}(c_2 + \frac{(1+3\omega)c_1^2}{(1+\omega)^2})] \\
a_4 &= \frac{1}{4(1+3\omega)}[p_3 - \frac{2\omega(\omega-1)p_1p_2}{(1+\omega)(1+2\omega)} + \frac{\omega(4\omega^3-\omega^2+2\omega-5)p_1^3}{6(1+\omega)^3(1+2\omega)} \\
&\quad + \frac{(1+5\omega)}{(1+\omega)(1+2\omega)}\{c_1(p_2 - \frac{\omega(5\omega^2-6\omega+1)p_1^2}{2(1+\omega)^2(1+5\omega)}) + \frac{p_1}{2}(c_2 \\
&\quad + \frac{(17\omega^2+6\omega+1)c_1^2}{(1+\omega)^2(1+5\omega)})\} + \frac{1}{3}(c_3 + \frac{3(1+5\omega)c_1c_2}{2(1+\omega)(1+2\omega)} + \frac{\omega(17\omega^2+6\omega+1)c_1^3}{2(1+\omega)^3(1+2\omega)})]
\end{aligned}$$

By the inequality (6) the first inequality follows. Since the coefficients of p_1^2 is positive when $\omega \leq 1$,the first inequality of $|a_3|$ follows from Lemma 2.
Now when $\omega \geq 1$, consider

$$a_3 = \frac{1}{3(1+2\omega)}[p_2 - \frac{\mu p_1^2}{2} + \frac{(1+3\omega)p_1c_1}{(1+\omega)^2} + \frac{1}{2}(c_2 + \frac{(1+3\omega)c_1^2}{(1+\omega)^2})]$$

where $\mu = \frac{\omega(\omega-1)}{(1+\omega)^2}$. Now using Lemma 3, the second inequality for $|a_3|$ follows.
In deriving the inequality for a_4, note that the coefficients of p_1^3 is positive and p_1^2 and p_1p_2 are negative, when $\omega \geq 1$.

$$\begin{aligned}
a_4 &= \frac{1}{4(1+3\omega)}[(p_3 - 2\beta p_1p_2 + \alpha p_1^3) + \frac{1+5\omega}{(1+\omega)(1+2\omega)}c_1(p_2 - \frac{\mu p_1^2}{2}) \\
&\quad + \frac{(1+5\omega)p_1}{2(1+\omega)(1+2\omega)}(c_2 + \frac{(17\omega^2+6\omega+1)c_1^2}{(1+\omega)^2(1+5\omega)}) \\
&\quad + \frac{1}{3}(c_3 + \frac{3(1+5\omega)c_1c_2}{2(1+\omega)(1+2\omega)} + \frac{(17\omega^2+6\omega+1)c_1^3}{2(1+\omega)^3(1+2\omega)})]
\end{aligned}$$

where $\beta = \frac{\omega(\omega-1)}{(1+\omega)(1+2\omega)}$, $\alpha = \frac{\omega(4\omega^3-\omega^2+2\omega-5)}{6(1+\omega)^3(1+2\omega)}$ and $\mu = \frac{\omega(5\omega^2-6\omega+1)}{(1+\omega)^2(1+5\omega)}$.
Here $\alpha - \beta \leq 0$ and $2\beta(\beta-1) \leq \alpha \leq \beta$, for all $\omega \geq 1$.
$\Rightarrow |p_3 - 2\beta p_1p_2 + p_1^3| \leq 2$, for all $\omega \geq 1$. Upon using Lemma 2, 3 and 4 third

inequality follows. To estimate $|a_4|$, corresponding to $\omega \in [.2, 1]$, we first express p_2 and p_3 in terms of p_1 using Lemma 1. Then normalize p_1 so that $p_1 = p$ and $0 \leq p \leq 2$. After a simple calculation and using Lemma 3 we arrive at

$$\begin{aligned} |a_4| \leq \; & \frac{(3+17\omega+43\omega^2+7\omega^3+2\omega^4)p^3}{48(1+\omega)^3(1+2\omega)(1+3\omega)} + \frac{(1+5\omega)p|y|(4-p^2)}{8(1+\omega)(1+2\omega)(1+3\omega)} \\ & + \frac{(4-p^2)(1-|y|^2)}{8(1+3\omega)} + \frac{(2\omega^4+37\omega^3+313\omega^2+131\omega+21)}{6(1+\omega)^3(1+2\omega)(1+3\omega)} - \frac{p(4-p^2)|y|^2}{16(1+3\omega)} \end{aligned}$$

Let

$$\begin{aligned} \phi(p,|y|) = \; & \frac{(3+17\omega+43\omega^2+7\omega^3+2\omega^4)p^3}{48(1+\omega)^3(1+2\omega)(1+3\omega)} + \frac{(4-p^2)(1-|y|^2)}{8(1+3\omega)} - \frac{p(4-p^2)|y|^2}{16(1+3\omega)} \\ & + \frac{(1+5\omega)p|y|(4-p^2)}{8(1+\omega)(1+2\omega)(1+3\omega)} \end{aligned}$$

The maximum and a saddle point of $\phi(p,|y|)$ are $p = |y| = 0$ and $p = 2$, $|y| = \frac{(1+5\omega)}{2(1+\omega)(1+2\omega)}$ respectively. When $p = |y| = 0$ we led to the second inequality for $|a_4|$. It is remaining to prove the inequality in $0 \leq \omega \leq .2$. The coefficients of p_1^2 and p_1^3 are negative when $0 \leq \omega \leq .2$. By using the same procedure used above, we arrive at,

$$\begin{aligned} a_4 \leq \; & \frac{(2\omega^4+7\omega^3+43\omega^2+17\omega+3)}{48(1+\omega)^3(1+2\omega)(1+3\omega)}p^3 + \frac{(1+5\omega)p|y|(4-p^2)}{8(1+\omega)(1+2\omega)(1+3\omega)} \\ & + \frac{(4-p^2)(1-|y|^2)}{8(1+3\omega)} + \frac{4\omega^4+134\omega^3+554\omega^2+274\omega+42}{12(1+\omega)^3(1+2\omega)(1+3\omega)} - \frac{p(4-p^2)|y|^2}{16(1+3\omega)} \end{aligned}$$

Let

$$\begin{aligned} \phi(p,|y|) = \; & \frac{(2\omega^4+7\omega^3+43\omega^2+17\omega+3)}{48(1+\omega)^3(1+2\omega)(1+3\omega)}p^3 + \frac{(4-p^2)(1-|y|^2)}{8(1+3\omega)} - \frac{p(4-p^2)|y|^2}{16(1+3\omega)} \\ & + \frac{(1+5\omega)p|y|(4-p^2)}{8(1+\omega)(1+2\omega)(1+3\omega)} \end{aligned}$$

Differentiating $\phi(p,|y|)$ with respect to p and $|y|$ we get the maximum point at $p = |y| = 0$ and the only saddle point is $p = 2$ and $|y| = \frac{(1+5\omega)}{2(1+\omega)(1+2\omega)}$. We arrive at third inequality when $p = |y| = 0$. Now by considering the boundary points in $[0,2] \times [0,1]$, again we obtain the third inequality.
The first inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ with respect to the convex function $g(z) = \frac{z}{1-z}$ and the second inequality is sharp for the function $p(z) = \frac{1+z^2}{1-z^2}$ with respect to the convex function $g(z) = \frac{z}{1-z}$.

Remark 1. *At this junction we remark that when $f(z) = g(z)$ and $\omega = 1$, Q^{ω} reduces to the well known class of convex functions and our result reduces to*

$$|a_n| \leq 1.$$

When $\omega = 0$, Q^{ω} reduces to C^* and our result reduces to

$$|a_n| \leq n,$$

the well known coefficient conjecture for class of close-to-convex functions. When $\omega = 1$, Q^{ω} reduces to the well known class of quasi-convex functions and our result reduces to

$$|a_n| \leq 1.$$

4. THE COEFFICIENTS OF $log\{\frac{f(z)}{z}\}$

For $f(z) \in S$, the logarthmic coefficients are derived from

$$\frac{1}{2}log(\frac{f(z)}{z}) = \sum_{n=1}^{\infty}\delta_n z^n. \tag{9}$$

These coefficients are very important in the study of Univalent(*Schlict*) functions. For $f(z) \in Q^{\omega}$, in the following theorem, we obtain similar results and these results are sharp for $|\delta_1|$.

Theorem 6. *Let $f \in Q^{\omega}$ for $\omega \geq 0$ and the coefficients of $log\frac{f(z)}{z}$ is given by (9). Then*

$$\begin{aligned} |\delta_1| &\leq \frac{1}{(1+\omega)} \\ |\delta_2| &\leq \frac{(2\omega^2+10\omega+3)}{4(1+\omega)^2(1+2\omega)} \\ |\delta_3| &\leq \begin{cases} \frac{\omega^4+8\omega^3+14\omega^2+8\omega+1}{2(1+\omega)^3(1+2\omega)(1+3\omega)} & , if\ \omega < \frac{1}{5} \\ \frac{2(\omega^4+8\omega^3+19\omega^2+7\omega+1)}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , if\ \omega \geq \frac{1}{5} \end{cases} \end{aligned}$$

Proof. Differentiating both sides of (9) and equating the coefficients we get,

$$\begin{aligned} \delta_1 &= \frac{1}{2}a_2 \\ \delta_2 &= \frac{1}{2}[a_3 - \frac{a_2^2}{2}] \\ \delta_3 &= \frac{1}{2}[a_4 - a_2a_3 + \frac{a_2^3}{3}] \end{aligned}$$

The inequality for δ_1 is trivial from Theorem 5. Now we have,

$$\begin{aligned}\delta_2 &= \frac{p_2}{6(1+2\omega)} - \frac{4\omega^2+2\omega+3}{48(1+\omega)^2(1+2\omega)}p_1^2 + \frac{1+6\omega}{48(1+\omega)^2(1+2\omega)}c_1^2 \\ &+ \frac{c_2}{12(1+2\omega)} + \frac{1+6\omega}{24(1+\omega)^2(1+2\omega)}p_1c_1\end{aligned}$$

The coefficients of p_1^2 is negative for all $\omega \geq 0$. Consider

$$\delta_2 = \frac{1}{6(1+2\omega)}[p_2 - \frac{\mu}{2}p_1^2 + \frac{c_2}{2} + \frac{1+6\omega}{8(1+\omega)^2}c_1^2 + \frac{1+6\omega}{4(1+\omega)^2}p_1c_1]$$

where $\mu = \frac{4\omega^2+2\omega+3}{4(1+\omega)^2}$. By using Lemma (3) we obtain the second inequality.

$$\begin{aligned}\delta_3 &= \frac{1}{8(1+3\omega)}(p_3 - \frac{2(1+3\omega^2)p_1p_2}{3(1+\omega)(1+2\omega)} + \frac{(4\omega^4+5\omega^3+4\omega^2-2\omega+1)}{6(1+\omega)^3(1+2\omega)}p_1^3) \\ &+ \frac{(1+9\omega)c_1}{24(1+\omega)(1+2\omega)(1+3\omega)}(p_2 - \frac{(9\omega^3+4\omega^2+14\omega+1)}{2(1+\omega)^2(1+9\omega)}p_1^2) + \frac{c_3}{24(1+3\omega)} \\ &+ \frac{(1+9\omega)c_1c_2}{48(1+\omega)(1+2\omega)(1+3\omega)} + \frac{\omega(5\omega-1)c_1^3}{48(1+\omega)^3(1+2\omega)(1+3\omega)} \\ &+ \frac{(1+9\omega)p_1}{48(1+\omega)(1+2\omega)(1+3\omega)}(c_2 - \frac{3\omega(1-5\omega)}{(1+\omega)^2(1+9\omega)}c_1^2)\end{aligned}$$

The coefficients of p_1p_2 and p_1^2 are always negative. The coefficient of p_1^3 is always positive. When $\omega \geq \frac{1}{5}$ the coefficients of c_1^2 and c_1^3 are positive. We have

$$\begin{aligned}\delta_3 &= \frac{1}{8(1+3\omega)}[p_3 - 2\beta p_1p_2 + \alpha p_1^3] + \frac{c_3}{24(1+3\omega)} + \frac{(1+9\omega)c_1c_2}{48(1+\omega)(1+2\omega)(1+3\omega)} \\ &+ \frac{\omega(5\omega-1)c_1^3}{48(1+\omega)^3(1+2\omega)(1+3\omega)} + \frac{(1+9\omega)}{24(1+\omega)(1+2\omega)(1+3\omega)}c_1[p_2 - \frac{\mu}{2}p_1^2] \\ &+ \frac{(1+9\omega)p_1c_2}{48(1+\omega)(1+2\omega)(1+3\omega)} + \frac{\omega(5\omega-1)c_1^2p_1}{16(1+\omega)^3(1+2\omega)(1+3\omega)}\end{aligned}$$

where $\beta = \frac{1+3\omega^2}{3(1+\omega)(1+2\omega)}$, $\alpha = \frac{4\omega^4+5\omega^3+4\omega^2-2\omega+1}{6(1+\omega)^3(1+2\omega)}$ and $\mu = \frac{9\omega^3+4\omega^2+14\omega+1}{(1+\omega)^2(1+9\omega)}$.
Note that $0 \leq \beta \leq 1$ and $\beta(2\beta-1) \leq \alpha \leq \beta$, when $\omega \geq \frac{1}{5}$ and now applying Lemma 3 and 4 we arrive at the second inequality for $|\delta_3|$. When $\omega \leq \frac{1}{5}$, the coefficients of

c_1^3 and c_1^2 are negative. So let

$$\begin{aligned}\delta_3 &= \frac{1}{8(1+3\omega)}[p_3 - 2\beta p_1 p_2 + \alpha p_1^3] + \frac{(1+9\omega)}{24(1+\omega)(1+2\omega)(1+3\omega)} c_1 [p_2 - \frac{\mu}{2} p_1^2] \\ &+ \frac{(1+9\omega)}{48(1+\omega)(1+2\omega)(1+3\omega)} p_1 [c_2 - \frac{\lambda}{2} c_1^2] + \frac{(1+9\omega) c_1 c_2}{48(1+\omega)(1+2\omega)(1+3\omega)} \\ &+ \frac{\omega(5\omega-1) c_1^3}{48(1+\omega)^3(1+2\omega)(1+3\omega)} + \frac{c_3}{24(1+3\omega)}\end{aligned}$$

where $\beta = \frac{1+3\omega^2}{3(1+\omega)(1+2\omega)}$, $\alpha = \frac{4\omega^4+5\omega^3+4\omega^2-2\omega+1}{6(1+\omega)^3(1+2\omega)}$, $\mu = \frac{9\omega^3+4\omega^2+14\omega+1}{(1+\omega)^2(1+9\omega)}$ and $\lambda = \frac{6\omega(1-5\omega)}{(1+\omega)^2(1+9\omega)}$. Here $0 \leq \beta \leq 1$ and $\beta(2\beta-1) \leq \alpha \leq \beta$, when $\omega < \frac{1}{5}$ and now we use the same normalization procedure used in finding a_4 to obtain the inequality,

$$\begin{aligned}|\delta_3| &\leq \frac{\omega^2+6\omega+1}{2(1+\omega)(1+2\omega)(1+3\omega)} + \frac{(\omega^4+8\omega^3+19\omega^2+7\omega+1)}{48(1+\omega)^3(1+2\omega)(1+3\omega)} c^3 \\ &+ \frac{(4\omega^2+6\omega+1)(4-c^2)c|y|}{96(1+\omega)(1+2\omega)(1+3\omega)} - \frac{(4-c^2)c|y|^2}{96(1+3\omega)} + \frac{(4-c^2)(1-|y|^2)}{48(1+3\omega)} \\ &:= \phi(c,|y|)\end{aligned}$$

Differentiating $\phi(c,|y|)$ with respect to c and $|y|$, using elementary calculus it is easily seen that the maximum point is $c = 0 = |y|$ and the saddle point is $c = 2$ and $|y| = \frac{16\omega^2+33\omega+9}{16(2\omega^2+3\omega+1)}$. We obtain maximum for the first inequality of $|\delta_3|$ at the end points of $[0,2] \times [0,1]$.
The first inequality is sharp for the function $p(z) = \frac{1+z}{1-z}$ with respect to the convex function $g(z) = \frac{z}{1-z}$.

5. THE COEFFICIENTS OF INVERSE FUNCTION

$Q^\omega \subset S$, inverse function f^{-1} exists and defined in some disk $|w| < r_0(f)$. Let

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + ...,$$

then $f(f^{-1}(w)) = w$. Then equating the coefficients we have,

$$\begin{aligned}A_2 &= -a_2 \\ A_3 &= 2a_2^2 - a_3 \\ A_4 &= -5a_2^3 + 5a_2 a_3 - a_4\end{aligned}$$

Theorem 7. *Let $f(z) \in Q^{\omega}$ for $\omega \geq 0$ and $f^{-1}(w)$ be the inverse of $f(z)$. Then*

$$|A_2| \leq \frac{2}{(1+\omega)}$$
$$|A_3| \leq \frac{3\omega^2+30\omega+19}{3(1+\omega)^2(1+2\omega)}$$

$$|A_4| \leq \begin{cases} \frac{(55+197\omega+171\omega^2+39\omega^3+2\omega^4)}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , 0 \leq \omega \leq \omega_0 \text{ and } \omega_1 \leq \omega \leq \omega_2 \\ \frac{(107+391\omega+367\omega^2+59\omega^3+4\omega^4)}{6(1+\omega)^3(1+2\omega)(1+3\omega)} & , \omega_0 \leq \omega < \omega_1 \\ \frac{(50+181\omega+163\omega^2+46\omega^3+4\omega^4)}{3(1+\omega)^3(1+2\omega)(1+3\omega)} & , \omega_2 \leq \omega \end{cases}$$

where $\omega_0 = 0.5855$, $\omega_1 = 1$ and $\omega_2 = 1.769$.

Proof. By Theorem 5 the inequality for $|A_2|$ is trivial . We have

$$\begin{aligned} A_3 &= -\frac{p_2}{3(1+2\omega)} + \frac{(3+5\omega+\omega^2)}{6(1+\omega)^2(1+2\omega)}p_1^2 + \frac{(2+3\omega)}{3(1+\omega)^2(1+2\omega)}c_1p_1 \\ &\quad -\frac{c_2}{6(1+\omega)} + \frac{(2+3\omega)c_1^2}{6(1+\omega)^2(1+2\omega)} \\ &= \frac{1}{3(1+2\omega)}[-\{(p_2 - \frac{\mu}{2}p_1^2) + \frac{1}{2}(c_2 - \frac{\lambda}{2}c_1^2)\} + \frac{(2+3\omega)}{(1+\omega)^2}c_1p_1] \end{aligned}$$

where $\mu = \frac{(3+5\omega+\omega^2)}{(1+\omega)^2}$ and $\lambda = \frac{2(2+3\omega)}{(1+\omega)^2}$. Using Lemma 4, we have

$$max\{2, 2|2\mu - 1|\} = \frac{2(\omega^2+8\omega+5)}{(1+\omega)^2}$$

and

$$max\{2, 2|2\lambda - 1|\} = 2.$$

Now using lemma 3 we obtain the inequality for $|A_3|$.

$$\begin{aligned} A_4 &= -\frac{1}{4(1+3\omega)}[p_3 - 2\beta_1p_1p_2 + \alpha_1p_1^3] + \frac{(7+15\omega)c_1}{12(1+\omega)(1+2\omega)(1+3\omega)}[p_2 - \frac{\mu}{2}p_1^2] \\ &\quad -\frac{1}{12(1+3\omega)}[c_3 - 2\beta_2c_1c_2 + \alpha_2c_1^3] + \frac{(7+15\omega)p_1}{24(1+\omega)(1+2\omega)(1+3\omega)}[c_2 - \frac{\lambda}{2}c_1^2] \end{aligned}$$

where $\beta_1 = \frac{3\omega^2+12\omega+5}{3(1+\omega)(1+2\omega)}$, $\alpha_1 = \frac{15+60\omega+72\omega^2+29\omega^3+4\omega^4}{6(1+\omega)^3(1+2\omega)}$, $\mu = \frac{25+92\omega+88\omega^2+15\omega^3}{(1+\omega)^2(7+15\omega)}$, $\lambda = \frac{2(6+21\omega^1+7\omega^2)}{(1+\omega)^2(7+15\omega)}$, $\beta_2 = \frac{7+15\omega}{4(1+\omega)(1+2\omega)}$ and $\alpha_2 = \frac{5+16\omega+6\omega^2+17\omega^3}{2(1+\omega)^3(1+2\omega)}$.

Here we see that $0 \leq \beta_1 \leq 1$ and $\beta_1(2\beta_1 - 1) \leq \alpha_1 \leq \beta_1$ for $\omega \geq \omega_2$ where $\omega_2 = 1.769...$, is the only positive root of $5 + 16\omega + 8\omega^2 - 7\omega^3 - 2\omega^4$. Whenever $\omega \leq \omega_2$, we have $\alpha_1 - \beta_1 \geq 0$. There we use Lemma 4 with $\alpha_1 = \beta_1$. Also $0 \leq \beta_2 \leq 1$ and $\beta_2(2\beta_2 - 1) \leq \alpha_2 \leq \beta_2$ for $\omega_0 \leq \omega \leq \omega_1$ where $\omega_0 = 0.5855$ and $\omega_1 = 1$ are the positive roots of $3 + 3\omega - 25\omega^2 + 19\omega^3$. Similarly whenever $\omega \notin [0.5855, 1]$, we have $\alpha_2 - \beta_2 \geq 0$. There we use Lemma 4 with $\alpha_2 = \beta_2$. Using these along with Lemma 2, we arrive at the required inequalities for $\left|A_4\right|$. The inequality for $|A_2|$ is sharp for the function $p(z) = \frac{1+z}{1-z}$ with respect to the convex function $g(z) = \frac{z}{1-z}$.

6. THE SECOND HANKEL DETERMINANT

In [10], Pommerenke defined qth Hankel determinant for a function $f(z)$. Here we find the second Hankel determinant $H_2(2) = |a_2 a_4 - a_3^2|$ for $f(z) \in Q^\omega$, when $0 \leq \omega \leq 1$.

Theorem 8. *$f(z) \in Q^\omega$ for $0 \leq \omega \leq 1$*

$$|H_2(2)| \leq \begin{cases} \frac{1}{8} & , if\ \omega = 0 \\ \frac{390+4090\omega+14835\omega^2+23065\omega^3+15425\omega^4+2655\omega^5+456\omega^6}{36(1+\omega)^4(1+2\omega)^2(1+3\omega)} & , if\ 0 < \omega \leq 0.0799 \\ \frac{489+3427\omega+8553\omega^2+11683\omega^3+14486\omega^4+12114\omega^5+4848\omega^6}{18(1+\omega)^4(1+2\omega)^2(1+3\omega)} & , if\ 0.0799 \leq \omega \leq 1 \end{cases} \tag{10}$$

Proof. When $\omega = 0$, $Q^\omega = K$, the corresponding inequality was proved by Duren in [1]. Now consider $\omega \neq 0$. Then we have,

$$H_2(2) = A + B$$

where,

$$\begin{aligned} A \;=\; & \frac{p_1}{8(1+\omega)(1+2\omega)}[p_3 + \frac{\omega(12\omega^4 + 26\omega^3 + 5\omega^2 - 28\omega - 15)p_1^3}{18(1+\omega)^3(1+2\omega)} \\ & - \frac{2\omega(\omega-1)p_1p_2}{(1+\omega)(1+2\omega)}] + \frac{(1+15\omega+18\omega^2)}{72(1+\omega)^2(1+2\omega)^2(1+3\omega)}c_1^2[p_2 \\ & - \frac{(18\omega^4 + 393\omega^3 + 436\omega^2 + 161\omega + 16)}{2(1+15\omega+18\omega^2)}p_1^2] + \frac{c_1}{8(1+\omega)(1+3\omega)}[p_3 \\ & - \frac{(36\omega^3 + 36\omega^2 + 15\omega + 7)p_1p_2}{9(1+\omega)(1+2\omega)^2} + \frac{\omega(12\omega^4 - 114\omega^3 + 60\omega^2 + 46\omega - 4)}{9(1+\omega)^3(1+2\omega)^2}p_1^3] \\ & - \frac{c_2}{9(1+2\omega)^2}[p_2 - \frac{\omega(\omega-1)}{2(1+\omega)^2}p_1^2] - \frac{p_2}{9(1+2\omega)^2}[p_2 - \frac{\omega(\omega-1)}{(1+\omega)^2}p_1^2] \end{aligned}$$

and

$$\begin{aligned} B &= -\frac{c_2^2}{36(1+2\omega)^2} + \frac{(1+5\omega)p_1^2}{16(1+\omega)^2(1+2\omega)(1+3\omega)}[c_2 + \frac{(17\omega^2+6\omega+1)c_1^2}{(1+\omega)^2(1+5\omega)}] \\ &+ \frac{p_1}{24(1+\omega)(1+3\omega)}[c_3 + \frac{(180\omega^3+252\omega^2+111\omega+11)c_1c_2}{6(1+\omega)(1+2\omega)^2} \\ &- \frac{(102\omega^4-39\omega^3-147\omega^2-69\omega-7)c_1^3}{6(1+\omega)^3(1+2\omega)^2}] + \frac{c_1}{24(1+\omega)(1+3\omega)}[c_3 \\ &+ \frac{(18\omega^2+15\omega+1)c_1c_2}{6(1+\omega)(1+2\omega)} - \frac{(102\omega^4-21\omega^3-84\omega^2-33\omega-4)c_1^3}{6(1+\omega)^3(1+2\omega)}]. \end{aligned}$$

The coefficient of p_1^4 in A is negative for all $0 < \omega \leq 1$, where 1 is the root of the equation $12\omega^4+26\omega^3+5\omega^2-28\omega-15$ and the coefficient of $c_1p_1^3$ in A is negative when $\omega \leq 0.0799$, where 0.0799 is the root of the equation $12\omega^4-114\omega^3+60\omega^2+46\omega-4$. Other coefficients are positive both in A and B for all the values of $\omega \in (0,1]$.
Now consider $0 < \omega \leq 0.0799$. Use Lemma 1 to express p_2 and p_3 in terms of p_1 in A. Then normalize p_1 so that $p_1 = p$ and $0 \leq p \leq 2$. After subsequent simplification using Lemma 2 and 3, we get

$$\begin{aligned} |H_2(2)| &\leq \frac{1296\omega^6+2878\omega^5+1819\omega^4+1069\omega^3+661\omega^2+481\omega+96}{9(1+\omega)^4(1+2\omega)^2(1+3\omega)}p^3 \\ &- \frac{(5+11\omega)(4-p^2)p_1|y|^2}{16(1+\omega)(1+2\omega)(1+3\omega)} + \frac{(11\omega+5)p(4-p^2)(1-|y|^2)}{8(1+\omega)(1+2\omega)} \\ &+ \frac{(27+189\omega+75\omega^2-62\omega^3-648\omega^4)(4-p^2)|y|p_1}{72(1+\omega)^2(1+2\omega)^2(1+3\omega)} \\ &+ \frac{78+740\omega+2227\omega^2+2386\omega^3+699\omega^4-168\omega^5}{9(1+\omega^3)(1+2\omega)^2(1+3\omega)}. \end{aligned}$$

Using elementary calculus we arrive at the second inequality. Now to prove the third inequality in $0.0799 < \omega \leq 1$. Using similar arguments we arrive at the third inequality.

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