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MULTIPLE SOLUTIONS FOR A CLASS OF $(P_1(X), P_2(X))$-LAPLACIAN PROBLEMS WITH NEUMANN BOUNDARY CONDITIONS

N. Thanh Chung

Abstract. In this paper, we study the existence of solutions for a class of nonlinear Neumann problems with variable exponents of the form

$$\begin{cases} -\text{div}\left((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\right) + |u|^{p_{\max}(x)-2}u \\ \qquad\qquad = \lambda f(x,u) + \mu g(x,u) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ, μ are positive parameters, $p_i \in C_+(\overline{\Omega})$, $\inf_{x\in\overline{\Omega}} p_{\max}(x) > N$, $p_{\max}(x) = \max\{p_1(x), p_2(x)\}$ for all $x \in \overline{\Omega}$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions. Our proofs are essentially based on the three critical points theorem due to Ricceri [18].

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1. Introduction

In this paper, we are concerned with a class of nonlinear Neumann problems with variable exponents of the form

$$\begin{cases} -\text{div}\left((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\right) + |u|^{p_{\max}(x)-2}u \\ \qquad\qquad = \lambda f(x,u) + \mu g(x,u) \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 3$ is a smooth bounded domain, ν is the outward unit normal to $\partial\Omega$, λ, μ are positive parameters, $p_i \in C_+(\overline{\Omega})$, $\inf_{x\in\overline{\Omega}} p_{\max}(x) > N$, $p_{\max}(x) = \max\{p_1(x), p_2(x)\}$ for all $x \in \overline{\Omega}$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are two Carathéodory functions.

If $p_1(.)$ and $p_2(.)$ are two constants then problem (1) has been studied in some papers, we refer the readers to [6, 12, 13]. Problems of this type has evoked notable interest in the lastest years as they arise in several fields of physics and around, such as biophysics [10], plasma physics [21] and chemical reaction design [3]. In these applications, the problem is modeled as a general reaction-diffusion system

$$-u_t = \operatorname{div}\left((|\nabla u|^{p_1-2} + |\nabla u|^{p_2-2})\nabla u\right) + r(x, u), \tag{2}$$

where u describes a concentration, the first term on the right-hand side corresponds to diffusion with a diffusion coefficient $H(u) = |\nabla u|^{p_1-2} + |\nabla u|^{p_2-2}$, while r represents reaction and is related to processes of source and loss; typically in chemical and biological applications r has a polynomial form with respect to u. Boundary conditions are usually taken as zero flux i.e. the boundary of the domain is assumed impermeable to chemical species.

It should be noticed that in the case when $p_1(x) = p_2(x) = p(x)$ is a continuous function for all $x \in \overline{\Omega}$, problem (1) becomes the usual $p(x)$-Laplacian problem with Neumann boundary condition. In recent years, the study of differential equations and variational problems involving variable exponent conditions has been an interesting topic. The interest in studying such problems was stimulated by their applications in elastic mechanics, fluid dynamics and the mathematical models of stationary thermo-rheological viscous flows of non- Newtonian fluids. For more information on modeling physical phenomena by equations involving $p(x)$-growth condition we refer to [1]. $p(x)$-Laplacian problems have intensively studied in many papers, we refer to some interesting papers [5, 8, 11, 19, 22] in which the nonlinear terms $f(x, t)$ and $g(x, t)$ are subcritical and sublinear (or superlinear) at infinity with respect to the second variable $t \in \mathbb{R}$.

In [15], Mihăilescu firstly studied the existence and multiplicity of weak solutions for a class of nonlinear problems with Dirichlet boundary condition of the form

$$\begin{cases} -\operatorname{div}\left((|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2})\nabla u\right) = f(x, u) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

where the nonlinearity is given by

$$f(x, u) = \pm\left(-\lambda |u|^{p_{\max}(x)-2}u + |u|^{q(x)-2}u\right), \quad p_{\max}(x) < q(x) < \frac{Np_{\max}(x)}{N - p_{\max}(x)}$$

for any $x \in \overline{\Omega}$ and $\lambda > 0$ is a parameter. His proofs are essentially based on the minimum principle and the $\mathbb{Z}_2$ version for even functionals of the mountain pass theorem. By the presence of two variable exponents $p_1(x)$ and $p_2(x)$, problem (3) are more complicated. Some extensions of [15] can be found in [7, 16, 20, 23].

There, the authors studied the existence and multiplicity of solutions for Dirichlet problem (3) in the special cases involving indefinite weights. In [4], Avci et al. considered problem (3) with general nonlinearities satisfying Ambrosetti-Rabinowitz type conditions, that is, there exists $\mu > 0$ such that

$$\mu F(x,t) := \mu \int_0^t f(x,s)\, ds \leq f(x,t)t, \ \forall x \in \Omega, \ t \in \mathbb{R}\backslash\{0\}. \tag{4}$$

Very recently, Allaoui et al. [2] have studied a class of nonlocal problems involving this type of operators. Motivated by the papers mentioned above, we study the existence of mutiple solutions for Neumann nonlinear problem (1) by using variational methods. We do not assume the Ambrosetti-Rabinowitz type conditions as in [4], see condition (F_0). To the best of our knowledge, there has been no paper concering problem (1). Our proofs are essentially based on a variational principle due to Ricceri [18] involving the existence of at least three critical points.

In order to state the main result of this paper let us assume that the following conditions hold:

(F_0) There exist $C > 0$ and a function $q \in C_+(\overline{\Omega})$, $q^+ < p_{\max}^-$ such that
$$|f(x,t)| \leq C(1 + |t|^{q(x)-1}), \quad \forall (x,t) \in \Omega \times \mathbb{R};$$

(F_1) There exist $t_0 > 1$ and $R > 0$ such that $f(x,t) < 0$ when $|t| \in (0,1)$ and $f(x,t) \geq R$ when $|t| \in (t_0, +\infty)$.

(G_0) $g : \Omega \times \mathbb{R} \to \mathbb{R}$ satisfies
$$\sup_{|t| \leq k} |g(x,t)| \leq h_k(x),$$
for all $k > 0$ and almost every $x \in \Omega$ and $G(.,0) \in L^1(\Omega)$, where $h_k \in L^1(\Omega)$ and $G(x,t) = \int_0^t g(x,s)\, ds$.

The condition (F_0) means that $f(x,t)$ is sublinear at infinity with respect to $t \in \mathbb{R}$. There are many functions f satisfying the conditions (F_0) and (F_1), for example,
$$f(x,t) = |t|^{\alpha(x)-2}t - |t|^{\beta(x)-2}t, \quad t \in \mathbb{R},$$
where $\alpha, \beta \in C_+(\overline{\Omega})$ satisfy $\beta^- \leq \beta^+ < \alpha^- \leq \alpha^+ < p_{\max}^-$. It is worth mentioning that the nonlinear term in this paper may change sign in Ω.

Definition 1. *We say that $u \in X = W^{1,p_{\max}(x)}(\Omega)$ is a weak solution of problem (1) if*

$$\int_\Omega \left(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2}\right) (\nabla u, \nabla v)_{\mathcal{R}^N}\, dx + \int_\Omega |u|^{p_{\max}(x)-2} uv\, dx$$
$$-\lambda \int_\Omega f(x,u)v\, dx - \mu \int_\Omega g(x,u)v\, dx = 0$$

for all $v \in X$.

The result of this paper is formulated in the following theorem.

Theorem 1. *Assume that* $p_i^- = \inf_{x\in\overline{\Omega}} p_i(x) \geq 2$, $i = 1, 2$, $p_{\max}^- = \inf_{x\in\overline{\Omega}} p_{\max}(x) > N$ *and the conditions* (F_0)-(F_1) *hold. Then there exist an open interval* $\Lambda \subset (0, \infty)$ *and a positive real number* $\delta > 0$ *such that, for each* $\lambda \in \Lambda$ *and every Carathéodory function* $g : \Omega \times \mathbb{R} \to \mathbb{R}$ *satisfying the condition* (G_0)*, there exists a positive constant* $\mu^* > 0$ *such that for each* $\mu \in [0, \mu^*]$*, problem (1) has at least three solutions whose norms are less than* δ.

2. Preliminaries

We recall in what follows some definitions and basic properties of the generalized Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ where Ω is an open subset of $\mathbb{R}^N$. In that context, we refer to the book of Musielak [17] and the papers of Fan et al. [9], Mihăilescu and Rădulescu [16]. Set

$$C_+(\overline{\Omega}) := \{h : \quad h \in C(\overline{\Omega}), \quad h(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For any $h \in C_+(\overline{\Omega})$ we define

$$h^+ = \sup_{x\in\Omega} h(x) \text{ and } h^- = \inf_{x\in\Omega} h(x).$$

For any $p(x) \in C_+(\overline{\Omega})$, we define the variable exponent Lebesgue space

$$L^{p(x)}(\Omega) = \left\{u : \text{ a measurable real-valued function such that } \int_\Omega |u(x)|^{p(x)}\, dx < \infty\right\}.$$

We recall the following so-called *Luxemburg norm* on this space defined by the formula

$$|u|_{p(x)} = \inf\left\{\mu > 0 : \quad \int_\Omega \left|\frac{u(x)}{\mu}\right|^{p(x)} dx \leq 1\right\}.$$

Variable exponent Lebesgue spaces resemble classical Lebesgue spaces in many respects: they are Banach spaces, the Hölder inequality holds, they are reflexive if and only if $1 < p^- \leq p^+ < \infty$ and continuous functions are dense if $p^+ < \infty$. The inclusion between Lebesgue spaces also generalizes naturally: if $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents so that $p_1(x) \leq p_2(x)$ a.e. $x \in \Omega$ then there exists a continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$ the Hölder inequality

$$\left| \int_\Omega uv\, dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |u|_{p(x)} |v|_{p'(x)}$$

holds true.

An important role in manipulating the generalized Lebesgue-Sobolev spaces is played by the *modular* of the $L^{p(x)}(\Omega)$ space, which is the mapping $\rho_{p(x)} : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(u) = \int_\Omega |u|^{p(x)}\, dx.$$

Proposition 1 (see [9]). *If $u \in L^{p(x)}(\Omega)$ and $p^+ < \infty$ then the following relations hold*

$$|u|_{p(x)}^{p^-} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^+} \tag{5}$$

provided $|u|_{p(x)} > 1$ while

$$|u|_{p(x)}^{p^+} \le \rho_{p(x)}(u) \le |u|_{p(x)}^{p^-} \tag{6}$$

provided $|u|_{p(x)} < 1$ and

$$|u_n - u|_{p(x)} \to 0 \ \Leftrightarrow \ \rho_{p(x)}(u_n - u) \to 0. \tag{7}$$

Next, we define the Sobolev space with variable exponent

$$W^{1,p(x)}(\Omega) := \left\{ u \in L^{p(x)}(\Omega) : \ |\nabla u| \in L^{p(x)}(\Omega) \right\}$$

with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)},$$

which is a separable and reflexive Banach space. It has the following equivalent norm

$$\|u\|_{p(x)} = \inf \left\{ \mu > 0 : \ \int_\Omega \left(\left| \frac{u(x)}{\mu} \right|^{p(x)} + \left| \frac{\nabla u(x)}{\mu} \right|^{p(x)} \right) dx \le 1 \right\}.$$

Let

$$I_{p(x)}(u) = \int_\Omega \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx,$$

then there are the following relations

$$\|u\|_{p(x)} < 1 (= 1, > 1) \Leftrightarrow I_{p(x)}(u) < 1 (= 1, > 1), \tag{8}$$

$$\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq I_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}, \tag{9}$$

$$\|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq I_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}. \tag{10}$$

Remark 1. *If $N < p^- \leq p_{\max}(x)$ for any $x \in \overline{\Omega}$, by Theorem 2.2 in [9], we deduce that $W^{1,p_{\max}(x)}(\Omega)$ is continuously embedded in $W^{1,p^-_{\max}}(\Omega)$. Since $N < p^-_{\max}$ it follows that $W^{1,p^-_{\max}}(\Omega)$ is compactly embedded into $C(\overline{\Omega})$. Thus, we deduce that $W^{1,p_{\max}(x)}(\Omega)$ is compactly embedded in $C(\overline{\Omega})$. Defining $|u|_\infty = \sup_{x\in\overline{\Omega}} |u(x)|$, we find that there exists a positive constant $c > 0$ such that*

$$|u|_\infty \leq c\|u\|_{p_{\max}(x)}, \quad \forall u \in W^{1,p_{\max}(x)}(\Omega). \tag{11}$$

Since $p_{\max}(x) = \max\{p_1(x), p_2(x)\}$ for any $x \in \overline{\Omega}$, the space $W^{1,p_{\max}(x)}(\Omega)$ is continuously embedded into $W^{1,p_1(x)}(\Omega)$ and $W^{1,p_2(x)}(\Omega)$.

Finally, for proving our result in the next section, we introduce the following proposition.

Proposition 2. *Let $(X, \|.\|)$ be a separable and reflexive real Banach space; $\Phi : X \to \mathbb{R}$ a continuously Gâteaux differentiable and sequentially weakly lower semicontinuous fucntional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact. Assume that*

(i) $\lim_{\|u\|\to+\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty$ *for all $\lambda > 0$;*

(ii) There are $r \in \mathbb{R}$ and $u_0, u_1 \in X$ such that $\Phi(u_0) < r < \Phi(u_1)$;

(iii) $\inf_{u\in\Phi^{-1}((-\infty,r])} \Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)}$.

Then there exist an open interval $\Lambda \subset (0,\infty)$ and a positive real number δ such that each $\lambda \in \Lambda$, and every continuously Gâteaux differentiable functional $J : X \to \mathbb{R}$ with compact derivative, there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, the equation

$$\Phi'(u) + \lambda\Psi'(u) + \mu J'(u) = 0$$

has at least three solutions in X whose norms are less than δ.

3. Proof of the main result

In this section, we will prove Theorem 1 in details by using Proposition 2. We use the letter c_i to denote a general positive constant whose value may change from line to line.

Let us define the functionals $\Phi, \Psi : X := W^{1,p_{\max}(x)}(\Omega) \to \mathbb{R}$ by

$$\begin{aligned} \Phi(u) &= \int_\Omega \left(\frac{1}{p_1(x)}|\nabla u|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u|^{p_2(x)} \right) dx + \int_\Omega \frac{1}{p_{\max}(x)}|u|^{p_{\max}(x)}\,dx, \\ \Psi(u) &= -\int_\Omega F(x,u)\,dx, \end{aligned} \tag{12}$$

where $F(x,t) = \int_0^t f(x,s)\,ds$. It is easy to see that $\Phi, \Psi \in C^1(X, \mathbb{R})$ with the derivatives given by

$$\Phi'(u)(v) = \int_\Omega \left(|\nabla u|^{p_1(x)-2} + |\nabla u|^{p_2(x)-2} \right) (\nabla u, \nabla v)_{\mathbb{R}^N}\,dx + \int_\Omega |u|^{p_{\max}(x)-2}uv\,dx$$

and

$$\Psi'(u)(v) = -\int_\Omega f(x,u)v\,dx$$

for any $u, v \in X$.

Lemma 2. *The functional Φ is sequentially weakly lower semicontinuous, bounded on each bounded subset of X. Moreover, Φ' admits a continuous inverse on the dual space X^* of X.*

Proof. We first prove that Φ is convex. Indeed, since the function $t \in [0, +\infty) \mapsto t^\theta$ for any $\theta > 1$, we deduce that for each $x \in \Omega$ fixed the following inequalities hold

$$\left| \frac{\xi_1 + \xi_2}{2} \right|^{p_i(x)} \leq \left| \frac{|\xi_1| + |\xi_2|}{2} \right|^{p_i(x)} \leq \frac{1}{2}|\xi_1|^{p_i(x)} + \frac{1}{2}|\xi_2|^{p_i(x)}, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N, \quad i = 1, 2.$$

Using the above inequality we deduce that

$$\left| \frac{\nabla u + \nabla v}{2} \right|^{p_i(x)} \leq \frac{1}{2}|\nabla u|^{p_i(x)} + \frac{1}{2}|\nabla v|^{p_i(x)} \tag{13}$$

and

$$\left| \frac{u + v}{2} \right|^{p_{\max}(x)} \leq \frac{1}{2}|u|^{p_{\max}(x)} + \frac{1}{2}|v|^{p_{\max}(x)}, \quad \forall u, v \in X, \quad \forall x \in \Omega, \quad i = 1, 2. \tag{14}$$

From (13) and (14) we can obtain the following inequality

$$\Phi\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\Phi(u) + \frac{1}{2}\Phi(v), \quad \forall u, v \in X,$$

which means that Φ is convex. From this, by Corollary III.8 of [24], in order to show the weak lower semicontinuity of Φ, it is enough to show that Φ is strongly lower semicontinuous on X. For this purpose, let us fix $u \in E$ and $\epsilon > 0$. Let $v \in X$ be arbitrary. Since Φ is convex, applying the Hölder inequality, we deduce that

$$\begin{aligned}
\Phi(v) &\geq \Phi(u) + \Phi'(u)(v-u) \\
&= \Phi(u) - \int_\Omega |\nabla u|^{p_1(x)-1}|\nabla v - \nabla u|\,dx - \int_\Omega |\nabla u|^{p_2(x)-1}|\nabla v - \nabla u|\,dx \\
&\quad - \int_\Omega |u|^{p_{\max}(x)-1}|v-u|\,dx \\
&\geq \Phi(u) - 2\left||\nabla u|^{p_1(x)-1}\right|_{\frac{p_1(x)}{p_1(x)-1}} |\nabla v - \nabla u|_{p_1(x)} - 2\left||\nabla u|^{p_2(x)-1}\right|_{\frac{p_2(x)}{p_2(x)-1}} |\nabla v - \nabla u|_{p_2(x)} \\
&\quad - 2\left||u|^{p_{\max}(x)-1}\right|_{\frac{p_{\max}(x)}{p_{\max}(x)-1}} |v-u|_{p_{\max}(x)} \\
&\geq \Phi(u) - 2c_1\left(\left||\nabla u|^{p_1(x)-1}\right|_{\frac{p_1(x)}{p_1(x)-1}} + \left||\nabla u|^{p_2(x)-1}\right|_{\frac{p_2(x)}{p_2(x)-1}}\right)|\nabla v - \nabla u|_{p_{\max}(x)} \\
&\quad - 2\left||u|^{p_{\max}(x)-1}\right|_{\frac{p_{\max}(x)}{p_{\max}(x)-1}} |v-u|_{p_{\max}(x)} \\
&\geq \Phi(u) - c_2\|v-u\|_{p_{\max}(x)},
\end{aligned} \tag{15}$$

where c_1, c_2 are positive constants. This implies that

$$\Phi(v) \geq \Phi(u) - \epsilon, \quad \forall v \in X \text{ with } \|v-u\|_{p_{\max}(x)} < \delta = \frac{\epsilon}{c_2} \tag{16}$$

we thus have that Φ is strongly lower semicontinuous. Since Φ is convex, it follows that Φ is sequentially weakly lower semicontinuous on X.

Let us proceed for the boundedness of Φ on each bounded subset of X. Let X_0 be a bounded subset of X. By (5) and (6), and the continuous embeddings

$X \hookrightarrow W^{1,p_1(x)}(\Omega)$, $X \hookrightarrow W^{1,p_2(x)}(\Omega)$ we deduce that for any $u \in X$,

$$\begin{aligned}\int_\Omega \frac{1}{p_1(x)}|\nabla u|^{p_1(x)}\,dx &\leq \frac{1}{p_1^-}\left(|\nabla u|_{p_1(x)}^{p_1^+} + |\nabla u|_{p_1(x)}^{p_1^-}\right)\\ &\leq \frac{1}{p_1^-}\left(\|u\|_{p_1(x)}^{p_1^+} + \|u\|_{p_1(x)}^{p_1^-}\right)\\ &\leq \frac{c_3}{p_1^-}\left(\|u\|_{p_{\max}(x)}^{p_1^+} + \|u\|_{p_{\max}(x)}^{p_1^-}\right)\end{aligned} \tag{17}$$

and

$$\begin{aligned}\int_\Omega \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\,dx &\leq \frac{1}{p_2^-}\left(|\nabla u|_{p_2(x)}^{p_2^+} + |\nabla u|_{p_2(x)}^{p_2^-}\right)\\ &\leq \frac{1}{p_2^-}\left(\|u\|_{p_2(x)}^{p_2^+} + \|u\|_{p_2(x)}^{p_2^-}\right)\\ &\leq \frac{c_4}{p_2^-}\left(\|u\|_{p_{\max}(x)}^{p_2^+} + \|u\|_{p_{\max}(x)}^{p_2^-}\right).\end{aligned} \tag{18}$$

From (5) and (6) and the continuous embedding $X \hookrightarrow L^{p_{\max}}(\Omega)$ we also have

$$\begin{aligned}\int_\Omega \frac{1}{p_{\max}(x)}|u|^{p_{\max}(x)}\,dx &\leq \frac{1}{p_{\max}^-}\left(|u|_{p_{\max}(x)}^{p_{\max}^+} + |u|_{p_{\max}(x)}^{p_{\max}^-}\right)\\ &\leq \frac{c_5}{p_{\max}^-}\left(\|u\|_{p_{\max}(x)}^{p_{\max}^+} + \|u\|_{p_{\max}(x)}^{p_{\max}^-}\right).\end{aligned} \tag{19}$$

From (17)-(19), we obtain

$$\begin{aligned}\Phi(u) &= \int_\Omega \left(\frac{1}{p_1(x)}|\nabla u|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\right)dx + \int_\Omega \frac{1}{p_{\max}(x)}|u|^{p_{\max}(x)}\,dx\\ &\leq \frac{c_3}{p_1^-}\left(\|u\|_{p_{\max}(x)}^{p_1^+} + \|u\|_{p_{\max}(x)}^{p_1^-}\right) + \frac{c_4}{p_2^-}\left(\|u\|_{p_{\max}(x)}^{p_2^+} + \|u\|_{p_{\max}(x)}^{p_2^-}\right)\\ &\quad + \frac{c_5}{p_{\max}^-}\left(\|u\|_{p_{\max}(x)}^{p_{\max}^+} + \|u\|_{p_{\max}(x)}^{p_{\max}^-}\right).\end{aligned} \tag{20}$$

From (20), Φ is bounded on each bounded subset of X.

We continue to show the existence of the inverse function $(\Phi')^{-1} : X^* \to X$. To this end, let us show the strict monotonicity of Φ'. For all $u_1, u_2 \in X$, we have

$$\begin{aligned}
&(\Phi'(u_1) - \Phi'(u_2))(u_1 - u_2) \\
&= \int_\Omega \left(|\nabla u|^{p_1(x)-2}\nabla u_1 - |\nabla u_2|^{p_1(x)-2}\nabla u_2\right)(\nabla u_1 - \nabla u_2)\,dx \\
&\qquad + \int_\Omega \left(|\nabla u|^{p_2(x)-2}\nabla u_1 - |\nabla u_2|^{p_2(x)-2}\nabla u_2\right)(\nabla u_1 - \nabla u_2)\,dx \\
&\qquad + \int_\Omega \left(|u|^{p_{\max}(x)-2}\nabla u_1 - |u_2|^{p_{\max}(x)-2}u_2\right)(u_1 - u_2)\,dx.
\end{aligned} \tag{21}$$

It is known that

$$(|\xi_1|^{r-2}\xi_1 - |\xi_2|^{r-2}\xi_2) \geq \frac{1}{2^r}|\xi_1 - \xi_2|^r, \quad r \geq 2, \quad \forall \xi_1, \xi_2 \in \mathbb{R}^N. \tag{22}$$

From (21) and (22) we have

$$\begin{aligned}
&(\Phi'(u_1) - \Phi'(u_2))(u_1 - u_2) \\
&\geq c_6 \int_\Omega |\nabla u_1 - \nabla u_2|^{p_1(x)}\,dx + c_7 \int_\Omega |\nabla u_1 - \nabla u_2|^{p_2(x)}\,dx \\
&\qquad + c_8 \int_\Omega |u_1 - u_2|^{p_{\max}(x)}\,dx \\
&\geq \min\{c_6, c_7\} \int_\Omega \left(|\nabla u_1 - \nabla u_2|^{p_1(x)} + |\nabla u_1 - \nabla u_2|^{p_2(x)}\right)\,dx \\
&\qquad + c_8 \int_\Omega |u_1 - u_2|^{p_{\max}(x)}\,dx \\
&\geq \min\{c_6, c_7\} \int_\Omega |\nabla u_1 - \nabla u_2|^{p_{\max}(x)}\,dx + c_8 \int_\Omega |u_1 - u_2|^{p_{\max}(x)}\,dx
\end{aligned} \tag{23}$$

From (23), Φ' is strictly monotone.

For any $u \in X$ with $\|u\|_{p_{\max}(x)} > 1$, we have

$$\begin{aligned}
\frac{\Phi'(u)(u)}{\|u\|_{p_{\max}(x)}} &= \frac{1}{\|u\|_{p_{\max}(x)}}\left(\int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right)\,dx + \int_\Omega |u|^{p_{\max}(x)}\,dx\right) \\
&\geq \frac{1}{\|u\|_{p_{\max}(x)}}\left(\int_\Omega |\nabla u|^{p_{\max}(x)} + |u|^{p_{\max}(x)}\,dx\right) \\
&\geq \|u\|_{p_{\max}(x)}^{p_{\max}^- - 1},
\end{aligned} \tag{24}$$

from which we have the coercivity of Φ'. Standard arguments ensure that Φ' is hemicontinuous. Thus, in view of Theorem 26.A(d) of [24] there exists $\Phi'^{-1} : X^* \to$

X and it is bounded. Let us prove that Φ'^{-1} is continuous by showing that it is sequentially continuous. Let $\{w_m\} \subset X^*$ be a sequence strongly converging to $w \in X^*$ and let $u_m = \Phi'^{-1}(w_m)$, $m = 1, 2, ...$, and $u = \Phi'^{-1}(w)$. Then, $\{u_m\}$ is bounded in X and without loss of generality, we can assume that it converges weakly to a certain $u_0 \in X$. Since $\{w_m\}$ converges strongly to w, it is easy to see that

$$\lim_{m\to\infty} \Phi'(u_m)(u_m - u_0) = \lim_{m\to\infty} w_m(u_m - u_0) = 0$$

or

$$\begin{aligned}&\int_\Omega |\nabla u_m|^{p_1(x)-2}\nabla u_m(\nabla u_m - \nabla u_0)\,dx + \int_\Omega |\nabla u_m|^{p_2(x)-2}\nabla u_m(\nabla u_m - \nabla u_0)\,dx\\ &+ \int_\Omega |u_m|^{p_{\max}(x)-2}u_m(u_m - u_0)\,dx = 0_m(1).\end{aligned} \tag{25}$$

On the other hand, since $\{u_n\}$ converges weakly to a certain u_0 in X, and X is continuously embedded into $W^{1,p_i(x)}(\Omega)$ and $L^{p_{\max}(x)}(\Omega)$ we deduce that $\{u_n\}$ converges weakly to a certain u_0 in $W^{1,p_i(x)}(\Omega)$, $L^{p_i(x)}(\Omega)$ and $L^{p_{\max}(x)}(\Omega)$, so we have

$$\begin{aligned}&\int_\Omega |\nabla u|^{p_1(x)-2}\nabla u(\nabla u_m - \nabla u_0)\,dx + \int_\Omega |\nabla u|^{p_2(x)-2}\nabla u(\nabla u_m - \nabla u_0)\,dx\\ &+ \int_\Omega |u|^{p_{\max}(x)-2}u(u_m - u_0)\,dx = 0_m(1).\end{aligned} \tag{26}$$

From (25), (26) we can use (22) in order to get $\{u_m\}$ converges strongly to u_0 in X. The continuity and injectivity of Φ' imply that $\{u_m\}$ converges strongly to u, so Φ'^{-1} is continuous. The proof of Lemma 2 is completed.

Proof of Theorem 1. By Lemma 2, Φ is sequentially weakly lower semicontinuous, bounded on each bounded subset of X, and Φ' admits a continuous inverse on the dual space X^* of X. Moreover, by the hypothesis (F_0), Ψ' is compact.

Next, we will verify that the condition (i) of Proposition 2 is fulfilled. In fact, by relation (9), we have

$$\begin{aligned}\Phi(u) &= \int_\Omega \left(\frac{1}{p_1(x)}|\nabla u|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\right)dx + \int_\Omega \frac{1}{p_{\max}(x)}|u|^{p_{\max}(x)}\,dx\\ &\geq \frac{1}{p_{\max}^+}\int_\Omega \left(|\nabla u|^{p_1(x)} + |\nabla u|^{p_2(x)}\right)dx + \frac{1}{p_{\max}^+}\int_\Omega |u|^{p_{\max}(x)}\,dx\\ &\geq \frac{1}{p_{\max}^+}\int_\Omega \left(|\nabla u|^{p_{\max}(x)} + |u|^{p_{\max}(x)}\right)dx\\ &\geq \frac{1}{p_{\max}^+}\|u\|_{p_{\max}(x)}^{p_{\max}^-}\end{aligned} \tag{27}$$

for all $u \in X$ with $\|u\|_{p_{\max}(x)} > 1$.

On the other hand,

$$\Psi(u) = -\int_\Omega F(x,u)\,dx = \int_\Omega -F(x,u)\,dx$$

and due to the assumption (F_0),

$$|F(x,t)| \leq C\left(|t| + \frac{1}{q(x)}|t|^{q(x)}\right), \quad \forall (x,t) \in \Omega \times \mathbb{R}.$$

Therefore,

$$\begin{aligned}
\Psi(u) &\geq -C\int_\Omega |u|\,dx - C\int_\Omega \frac{1}{q(x)}|u|^{q(x)}\,dx \\
&\geq -c_9\|u\|_{p_{\max}(x)} - \frac{C}{q^-}\int_\Omega (|u|^{q^+} + |u|^{q^-})\,dx \\
&= -c_9\|u\|_{p_{\max}(x)} - \frac{C}{q^-}\left(|u|_{q^+}^{q^+} + |u|_{q^-}^{q^-}\right).
\end{aligned} \tag{28}$$

We know that X is continuously embedded into $L^{q^\pm}(\Omega)$. Furthermore, we can find two positive constants $c_{10} > 0$ such that

$$|u|_{q^+} \leq c_{10}\|u\|_{p_{\max}(x)}, \quad |u|_{q^-} \leq c_{10}\|u\|_{p_{\max}(x)}, \quad \forall u \in X. \tag{29}$$

From (28) and (29), we have

$$\Psi(u) \geq -c_9\|u\|_{p_{\max}(x)} - c_{10}\|u\|_{p_{\max}(x)}^{q^+} - c_{10}\|u\|_{p_{\max}(x)}^{q^-}. \tag{30}$$

Combining (27) and (30), it follows that for all $u \in X$ with $\|u\|_{p_{\max}(x)} > 1$,

$$\begin{aligned}
\Phi(u) + \lambda\Psi(u) = &\int_\Omega \left(\frac{1}{p_1(x)}|\nabla u|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u|^{p_2(x)}\right)\,dx \\
&+ \int_\Omega \frac{1}{p_{\max}(x)}|u|^{p_{\max}(x)}\,dx - \lambda\int_\Omega F(x.u)\,dx \\
\geq &\frac{1}{p_{\max}^+}\|u\|_{p_{\max}(x)}^{p_{\max}^-} - \lambda\left(c_9\|u\|_{p_{\max}(x)} + c_{10}\|u\|_{p_{\max}(x)}^{q^+} + c_{10}\|u\|_{p_{\max}(x)}^{q^-}\right).
\end{aligned} \tag{31}$$

Since $1 < q^- \leq q^+ < p_{\max}^-$, for any $\lambda > 0$ we have $\lim_{\|u\|_{p_{\max}(x)} \to +\infty}(\Phi(u) + \lambda\Psi(u)) = +\infty$ and (i) is verified.

In the sequel, we will verify the conditions (ii) and (iii) in Proposition 2. Indeed, it follows from the assumptions (F_0) and (F_1) that $F(x,t)$ is increasing for $t \in (1,+\infty)$ and decreasing for $t \in (0,1)$, uniformly with respect to $x \in \Omega$.

It is clear that $F(x,0) = 0$ and $F(x,t) \to \infty$ when $t \to \infty$, because of the definition of $F(x,t)$ and the assumption (F_1). Then there exists a real number $\delta > t_0$ such that

$$F(x,t) \geq 0 = F(x,0) \geq F(x,\tau), \quad \forall x \in X, \quad t > \delta, \quad \tau \in (0,1).$$

Let a, b be two real numbers such that

$$0 < a < \min\{1, c\}, \tag{32}$$

where c is given in (11) and $b > \delta$ satisfies

$$b^{p_{\max}^-}|\Omega| > 1. \tag{33}$$

From (F_1) we have $F(x,t) \leq F(x,0)$ for all $t \in [0,a]$, which implies that

$$\int_\Omega \sup_{0 \leq t \leq a} F(x,t)\, dx \leq \int_\Omega F(x,0)\, dx = 0.$$

Furthermore, since $b > \delta$ we get $\int_\Omega F(x,b)\, dx > 0$ and thus,

$$\int_\Omega \sup_{0 \leq t \leq a} F(x,t)\, dx \leq 0 < r.\frac{\int_\Omega F(x,b)\, dx}{\int_\Omega \frac{1}{p_{\max}(x)} b^{p_{\max}(x)}\, dx}. \tag{34}$$

Consider $u_0, u_1 \in X$, $u_0(x) = 0$ and $u_1(x) = b$ for any $x \in \Omega$, we define

$$r = \frac{1}{p_{\max}^+}\left(\frac{a}{c}\right)^{p_{\max}^+}. \tag{35}$$

From (32), we have $r \in (0,1)$. A simple computation implies $\Phi(u_0) = \Psi(u_0) = 0$ and

$$\begin{aligned}
\Phi(u_1) &= \int_\Omega \left(\frac{1}{p_1(x)}|\nabla u_1|^{p_1(x)} + \frac{1}{p_2(x)}|\nabla u_1|^{p_2(x)}\right) dx + \int_\Omega \frac{1}{p_{\max}(x)}|u_1|^{p_{\max}(x)}\, dx \\
&\geq \frac{1}{p_{\max}^+} b^{p_{\max}^-}|\Omega| \\
&> \frac{1}{p_{\max}^+}.1 \\
&> \frac{1}{p_{\max}^+}\left(\frac{a}{c}\right)^{p_{\max}^+} \\
&= r
\end{aligned} \tag{36}$$

and

$$\Psi(u_1) = -\int_\Omega F(x, u_1(x))\,dx = -\int_\Omega F(x, b)\,dx. \tag{37}$$

Thus, we obtain

$$\Phi(u_0) < r < \Phi(u_1)$$

and the condition (ii) in Proposition 2 is verified.

On the other hand, we have

$$\begin{aligned} -\frac{(\Phi(u_1) - r)\Psi(u_0) + (r - \Phi(u_0))\Psi(u_1)}{\Phi(u_1) - \Phi(u_0)} &= -r.\frac{\Psi(u_1)}{\Phi(u_1)} \\ &= r.\frac{\int_\Omega F(x, b)\,dx}{\int_\Omega \frac{1}{p_{\max}(x)} b^{p_{\max}(x)}\,dx} > 0. \end{aligned} \tag{38}$$

Next, we consider the case $u \in X$ with $\Phi(u) \le r < 1$. Since

$$r \ge \Phi(u) \ge \frac{1}{p^+_{\max}} I_{p_{\max}(x)}(u)$$

we obtain

$$I_{p_{\max}(x)}(u) \le r.p^+_{\max} = \left(\frac{a}{c}\right)^{p^+_{\max}} < 1,$$

which shows that $\|u\|_{p_{\max}(x)} < 1$ by (8). Furthermore, by (10), it is clear that

$$\begin{aligned} \frac{1}{p^+_{\max}} \|u\|^{p^+_{\max}}_{p_{\max}(x)} &\le \frac{1}{p^+_{\max}} I_{p_{\max}(x)}(u) \\ &\le \Phi(u) \\ &\le r. \end{aligned}$$

Thus, using Remark 1, for all $u \in X$ with $\Phi(u) \le r$, we have

$$\begin{aligned} |u(x)| &\le c\|u\|_{p_{\max}(x)} \\ &\le c\left(r.p^+_{\max}\right)^{\frac{1}{p^+_{\max}}} \\ &= a, \quad \forall x \in \Omega. \end{aligned} \tag{39}$$

The above inequality shows that

$$\begin{aligned} -\inf_{u \in \Phi^{-1}(-\infty, r]} \Psi(u) &= \sup_{u \in \Phi^{-1}(-\infty, r]} -\Psi(u) \\ &\le \int_\Omega \sup_{0 \le t \le a} F(x, t)\,dx \\ &\le 0. \end{aligned} \tag{40}$$

It follows from (38) and (40) that

$$-\inf_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) < r.\frac{\int_\Omega F(x,b)\,dx}{\int_\Omega \frac{1}{p_{\max}(x)} b^{p_{\max}(x)}\,dx}.$$

That is,

$$\inf_{u\in\Phi^{-1}(-\infty,r]}\Psi(u) > \frac{(\Phi(u_1)-r)\Psi(u_0)+(r-\Phi(u_0))\Psi(u_1)}{\Phi(u_1)-\Phi(u_0)},$$

which means that the condition (iii) in Proposition 2 is verified.

Since the function $g : \Omega\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function satisfying the condition (G_0), the functional

$$J(u) = -\int_\Omega G(x,u)\,dx$$

is well defined and continuously Gâteaux differentiable on X, with compact derivative, and one has

$$J'(u)(v) = -\int_\Omega g(x,u)v\,dx \text{ for all } u,v\in X.$$

So, according to Proposition 2, there exist an open interval $\Lambda\subset(0,\infty)$ and a positive real number δ such that for each $\lambda\in\Lambda$, and every continuously Gâteaux differentiable functional $J : X\to\mathbb{R}$ with compact derivative, there exists $\mu^*>0$ such that for each $\mu\in[0,\mu^*]$, the equation

$$\Phi'(u)+\lambda\Psi'(u)+\mu J'(u) = 0$$

has at least three solutions in X whose norms are less than δ. It follows that Theorem 1 holds.

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Nguyen Thanh Chung
Department of Mathematics,
Quang Binh University,
312 Ly Thuong Kiet, Dong Hoi, Quang Binh, Vietnam
email: *ntchung82@yahoo.com*