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SOME SANDWICH-TYPE RESULTS FOR ϕ-LIKE FUNCTIONS

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Abstract. Using the technique of differential subordination, we here obtain certain results for ϕ-like, starlike and close-to-convex functions.

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1. Introduction

Let $\mathcal{H}$ be the class of functions analytic in $\mathbb{E} = \{z : |z| < 1\}$ and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions f, analytic in the open unit disk $\mathbb{E} = \{z : |z| < 1\}$ and normalized by the conditions $f(0) = f'(0) - 1 = 0$. A function $f \in \mathcal{A}$ is said to be starlike of order β, $0 \le \beta < 1$, if and only if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in \mathbb{E}.$$

The class of such functions is denoted by $\mathcal{S}^*(\beta)$. Note that $\mathcal{S}^*(0) = \mathcal{S}^*$, the class of univalent starlike functions.
A function $f \in \mathcal{A}$ is said to be close-to-convex in $\mathbb{E}$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \mathbb{E},\ \text{for } g \in \mathcal{S}^*.$$

The class of close-to-convex functions is denoted by $\mathcal{C}$. Noshiro [2] and Warchawski [6] independently proved in 1934-35 that f is close-to-convex if

$$\Re(f'(z)) > 0.$$

Let $\Phi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be an analytic function, p be an analytic function in $\mathbb{E}$ with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$ and h be univalent in $\mathbb{E}$. Then the function p is said to satisfy first order differential subordination if

$$\Phi(p(z), zp'(z); z) \prec h(z), \quad \Phi(p(0), 0; 0) = h(0). \tag{1}$$

A univalent function q is called a dominant of the differential subordination (1) if $p(0) = q(0)$ and $p \prec q$ for all p satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q} \prec q$ for all dominants q of (1) is said to be the best dominant of (1).
Let $\Psi : \mathbb{C}^2 \times \mathbb{E} \to \mathbb{C}$ be analytic and univalent in domain $\mathbb{C}^2 \times \mathbb{E}$, h be analytic in $\mathbb{E}$, p be analytic and univalent in $\mathbb{E}$, with $(p(z), zp'(z); z) \in \mathbb{C}^2 \times \mathbb{E}$ for all $z \in \mathbb{E}$. Then p is called a solution of the first order differential superordination if

$$h(z) \prec \Psi(p(z), zp'(z); z), \; h(0) = \Psi(p(0), 0; 0). \tag{2}$$

An analytic function q is called a subordinant of the differential superordination (2), if $q \prec p$ for all p satisfying (2). A univalent subordinant $\tilde{q}$ that satisfies $q \prec \tilde{q}$ for all subordinants q of (2) is said to be the best subordinant of (2).
The function $f \in \mathcal{A}$ is called $\phi-$like in the open unit disk $\mathbb{E}$, if

$$\Re\left(\frac{zf'(z)}{\phi(f(z))}\right) > 0, \; z \in \mathbb{E},$$

where ϕ is analytic in a domain containing $f(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. This concept was first introduced by Brickman [1] and he established that a function $f \in \mathcal{A}$ is univalent if and only if f is ϕ-like for some ϕ.
Using the concept of differential subordination Ruscheweyh [9] introduced and studied the following more general class of $\phi-$like functions:
Let ϕ be analytic function in the domain containing $f(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in f(\mathbb{E}) \setminus \{0\}$. Then $f \in \mathbb{A}$ is called ϕ-like w.r.t. a univalent function $q(z)$ if $\dfrac{zf'(z)}{\phi(f(z))} \prec q(z), \; z \in \mathbb{E}$.
In 2005, Ravichandran et al.[10] proved the following result for ϕ-like functions:
Let $\alpha \neq 0$ be a complex number and $q(z)$ be a convex univalent function in $\mathbb{E}$. Suppose $h(z) = \alpha q^2(z) + (1 - \alpha)q(z) + \alpha zq'(z)$ and

$$\Re\left\{\frac{1-\alpha}{\alpha} + 2q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0, \; z \in \mathbb{E}.$$

If $f \in \mathcal{A}$ satisfies

$$\frac{zf'(z)}{\phi(f(z)}\left(1 + \frac{\alpha zf''(z)}{f'(z)} + \frac{\alpha(f'(z) - (\phi(f(z)))'}{\phi(f(z))}\right) \prec h(z)$$

then

$$\frac{zf'(z)}{\phi(f(z))} \prec q(z),\ z \in \mathbb{E}$$

and $q(z)$ is best dominant.
Recently, Shanmugam et al. [5] and Ibrahim [3] also obtained the results for ϕ-like functions parallel to the results of Ravichandran [10] stated above.
In the present paper, we investigate the differential operator

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right),$$

where $f,\ g \in \mathcal{A}$ and ϕ is an analytic function in a domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0)-1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E})\setminus\{0\}$, for real numbers a and $b(\neq 0)$. We, here, obtain some sufficient conditions for ϕ-like, starlike and close-to-convex functions.

2. Preliminaries

We shall need the following definition and Lemmas to prove our main results.

Definition 1. *[7, Def. 2.2b, p.21]. We denote by Q the set of functions p that are analytic and injective in $\overline{\mathbb{E}} \setminus \mathbb{B}(p)$, where*

$$\mathbb{B}(p) = \left\{\zeta \in \partial\mathbb{E} : \lim_{z\to\zeta} p(z) = \infty\right\},$$

are such that $p'(\zeta) \neq 0$ for $\zeta \in \partial\mathbb{E} \setminus \mathbb{B}(p)$.

Lemma 1. *[7, Theorem 3.4h, p.132]. Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$, with $\varphi(w) \neq 0$, when $w \in q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that either*
(i) h is convex, or
(ii) Q_1 is starlike.
In addition, assume that
(iii) $\Re\left(\frac{zh'(z)}{Q_1(z)}\right) > 0$.
If p is analytic in $\mathbb{E}$, with $p(0)= q(0)$, $p(\mathbb{E}) \subset \mathbb{D}$ and

$$\theta[p(z)] + zp'(z)\varphi[p(z)] \prec \theta[q(z)] + zq'(z)\varphi[q(z)],$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Lemma 2. *[4]. Let q be univalent in $\mathbb{E}$ and let θ and φ be analytic in a domain $\mathbb{D}$ containing $q(\mathbb{E})$. Set $Q_1(z) = zq'(z)\varphi[q(z)],\ h(z) = \theta[q(z)] + Q_1(z)$ and suppose that*
(i) Q_1 is starlike in $\mathbb{E}$ and
(ii) $\Re\left[\dfrac{\theta'(q(z))}{\varphi(q(z))}\right] > 0,\ z \in \mathbb{E}$.
If $p \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{E}) \subset \mathbb{D}$ and $\theta[p(z)] + zp'(z)\varphi[p(z)]$ is univalent in $\mathbb{E}$ and

$$\theta[q(z)] + zq'(z)\varphi[q(z)] \prec \theta[p(z)] + zp'(z)\varphi[p(z)],\ z \in \mathbb{E},$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant.

3. Main results

Theorem 3. *Let q, $q(z) \neq 0$ be a univalent function in $\mathbb{E}$ and satisfies the condition*

$$\Re\left(1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}\right) > \max\left\{0,\ \ -\frac{a}{b}\Re(q(z))\right\}, \tag{3}$$

where a and $b(\neq 0)$ are real numbers. Let ϕ be analytic function in a domain containing $g(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right) \prec aq(z) + b\frac{zq'(z)}{q(z)}, \tag{4}$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec q(z),\ z \in \mathbb{E},$$

and q(z) is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{zf'(z)}{\phi(g(z))}.$$

Therefore

$$\frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}$$

and (4) reduces to

$$ap(z) + b\frac{zp'(z)}{p(z)} \prec aq(z) + b\frac{zq'(z)}{q(z)}.$$

Define θ and φ as $\theta(w) = aw$ & $\varphi(w) = \dfrac{b}{w}$. Both θ and φ are analytic in $\mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0,\ w \in \mathbb{C} \setminus \{0\}$. Therefore $Q_1(z) = zq'(z)\varphi(q(z)) = b\dfrac{zq'(z)}{q(z)}$ and

$$h(z) = \theta(q(z)) + Q_1(z) = aq(z) + b\frac{zq'(z)}{q(z)}.$$

A little calculation yields

$$\frac{zQ_1(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

and

$$\frac{zh'(z)}{Q_1(z)} = \frac{a}{b}q(z) + 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}.$$

In view of Condition 3, we have $Q_1(z)$ is starlike in $\mathbb{E}$ and $\Re\left(\dfrac{zh'(z)}{Q_1(z)}\right) > 0$.
The proof, now, follows from the Lemma 1.

On taking $\phi(z) = z$ in Theorem 3, we have the following result:

Theorem 4. *Let q, $q(z) \neq 0$, be a univalent function in $\mathbb{E}$, satisfying the Condition 3 of Theorem 3 for real numbers $a,\ b(\neq 0)$. If $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{g(z)} \neq 0,\ z \in \mathbb{E}$, satisfy the differential subordination*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec aq(z) + b\frac{zq'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{g(z)} \prec q(z),\ z \in \mathbb{E},$$

and q(z) is the best dominant.

On taking $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 3, we have the following result:

Theorem 5. *Let q, $q(z) \neq 0$ be a univalent function in $\mathbb{E}$ and satisfies the Condition 3 of Theorem 3 for real numbers a and $b(\neq 0)$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec aq(z) + b\frac{zq'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z),\ z \in \mathbb{E},$$

and q(z) is the best dominant.

On selecting $a = 1$ and $b = \alpha$ in the Theorem 5, we get the following result for the class of α-convex functions.

Theorem 6. *Let α be a non zero real number and let q, $q(z) \neq 0$ be a univalent function in $\mathbb{E}$ satisfying the Condition 3 of Theorem 3. If $f \in \mathcal{A},\ z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec q(z) + \alpha\frac{zq'(z)}{q(z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec q(z),\ z \in \mathbb{E},$$

and $q(z)$ is the best dominant.

By defining $\phi(z) = g(z) = z$ in Theorem 3, we obtain the following result:

Theorem 7. *Let q, $q(z) \neq 0$ be a univalent function in $\mathbb{E}$ and satisfying the Condition 3 of Theorem 3 for real numbers $a,\ b(\neq 0)$. If $f \in \mathcal{A},\ f'(z) \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec aq(z) + b\frac{zq'(z)}{q(z)},$$

then

$$f'(z) \prec q(z),\ z \in \mathbb{E},$$

and q(z) is the best dominant.

Remark 1. *It is easy to verify that dominant $q(z) = \left(\frac{1+z}{1-z}\right)^{\delta}$, $0 < \delta \leq 1$, satisfies the Condition 3 of Theorem 3, for real numbers a and $b(\neq 0)$. Consequently, we get:*

Theorem 8. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, and for real numbers a and $b(\neq 0)$, satisfy*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right) \prec a\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2b\delta z}{1-z^2},$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E},\ 0 < \delta \leq 1.$$

On taking $\phi(z) = z$ in above theorem, we obtain:

Corollary 9. *Let a and $b(\neq 0)$ are real numbers and $0 < \delta \leq 1$. If $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{g(z)} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec a\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2b\delta z}{1-z^2},$$

then

$$\frac{zf'(z)}{g(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E}.$$

For $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 8, we obtain the following result:

Corollary 10. *Let a and $b(\neq 0)$ are real numbers and $0 < \delta \leq 1$. If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2b\delta z}{1-z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

Selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result for the class of α-convex functions:

Corollary 11. *Let α be a non-zero real number. If $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2b\delta z}{1-z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec \left(\frac{1+z}{1-z}\right)^{\delta},\ z \in \mathbb{E},\ 0 < \delta \leq 1.$$

Hence $f(z)$ is strongly starlike.

On taking $\phi(z) = g(z) = z$ in Theorem 8, we have:

Corollary 12. *Let a and $b(\neq 0)$ are real numbers. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec a\left(\frac{1+z}{1-z}\right)^{\delta} + \frac{2b\delta z}{1-z^2},$$

then

$$f'(z) \prec \left(\frac{1+z}{1-z}\right)^{\delta}, \ z \in \mathbb{E}, \ 0 < \delta \leq 1,$$

and hence $f(z)$ is close-to-convex.

Remark 2. *When we select the dominant $q(z) = e^z$, then this dominant satisfies the Condition 3 of Theorem 3 for non-zero real numbers a and b such that $\Re(e^z) > -\dfrac{b}{a}$. Consequently, we obtain the following result:*

Theorem 13. *Let a and b be non-zero real numbers such that $\Re(e^z) > -\dfrac{b}{a}$ and let ϕ be analytic function in a domain containing $g(\mathbb{E})$, $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f, \ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z)}\right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec e^z, \ z \in \mathbb{E}.$$

On choosing $\phi(z) = z$ in above theorem, we obtain:

Corollary 14. *Let a and b non-zero real numbers such that $\Re(e^z) > -\dfrac{b}{a}$. If $f, \ g \in \mathcal{A}$, $\dfrac{zf'(z)}{g(z)} \neq 0$, $z \in \mathbb{E}$, satisfy the differential subordination*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{g(z)} \prec e^z, \ z \in \mathbb{E}.$$

On selecting $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 13, we get:

Corollary 15. *Let a and b are non-zero real numbers such that $\Re(e^z) > -\dfrac{b}{a}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec ae^z + bz,$$

then

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

on choosing $a = 1$ and $b = \alpha$ in above corollary, we obtain:

Corollary 16. *Let α be a non-zero real number such that $\Re(e^z) > -\alpha$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec e^z + \alpha z,$$

$$\frac{zf'(z)}{f(z)} \prec e^z, \ z \in \mathbb{E}.$$

Therefore, $f \in \mathcal{S}^*$.

For $\phi(z) = g(z) = z$ in Theorem 13, we obtain the following result:

Corollary 17. *Let a and b are non-zero real numbers such that $\Re(e^z) > -\dfrac{b}{a}$. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec ae^z + bz,$$

then

$$f'(z) \prec e^z, \ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

Remark 3. *By selecting the dominant $q(z) = 1 + mz$, $0 < m \leq 1$, we observed that the Condition 3 of Theorem 3 holds for all real numbers a and $b(\neq 0)$ having same sign. Thus from Theorem 3, we have the following result:*

Theorem 18. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(z) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. Let real numbers a and $b(\neq 0)$ be such that $\frac{a}{b} > 0$. If $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{\phi(g(z))} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z)}\right) \prec a(1+mz) + \frac{bmz}{1+mz},$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + mz,\ \textit{where}\ 0 < m \leq 1,\ z \in \mathbb{E}.$$

Taking $\phi(z) = z$ in above theorem, we get the following result:

Corollary 19. *Let a and b are non-zero real numbers having same sign and $0 < m \leq 1$. If $f,\ g \in \mathcal{A},\ \frac{zf'(z)}{g(z)} \neq 0,\ z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec a(1+mz) + \frac{bmz}{1+mz},$$

then

$$\frac{zf'(z)}{g(z)} \prec 1 + mz,\ z \in \mathbb{E}.$$

From Theorem 18, for $\phi(z) = z$ and $g(z) = f(z)$, we obtain:

Corollary 20. *Let a and b be non-zero real numbers having same sign and $0 < m \leq 1$. If $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a(1+mz) + \frac{bmz}{1+mz},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + mz,\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

On selecting $a = 1$ and $b = \alpha$ in above corollary, we get the following result:

Corollary 21. *For $\alpha > 0$, if $f \in \mathcal{A},\ \frac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec (1+mz) + \frac{\alpha mz}{1+mz},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + mz, \quad 0 < m \leq 1,$$

and hence $f(z)$ is starlike.

Selecting $\phi(z) = g(z) = z$, in Theorem 18, we have:

Corollary 22. *Let a and $b(\neq 0)$ be real numbers having same sign. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec a(1 + mz) + \frac{bmz}{1 + mz},$$

then

$$f'(z) \prec 1 + mz, \quad 0 < m \leq 1, \ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

Remark 4. *Let $q(z) = \dfrac{\beta(1 - z)}{\beta - z}$, then*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) = \Re\left(\frac{\beta - z^2}{(\beta - z)(1 - z)}\right) > 0, \ for \ \beta > 1$$

and

$$\Re q(z) = \Re\left(\frac{\beta(1 - z)}{\beta - z}\right) > 0.$$

In view of the above calculations, the Conditon 3 of Theorem 3 is satisfied for real numbers a and $b(\neq 0)$ such that $\dfrac{a}{b} > 0$. Consequently, we obtain the following result:

Theorem 23. *Let ϕ be analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(z) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \neq 0$, $z \in \mathbb{E}$, for real numbers a, and $b(\neq 0)$ such that $\dfrac{a}{b} > 0$, satisfies*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z)}\right) \prec a\frac{\beta(1 - z)}{\beta - z} + b\frac{(1 - \beta)z}{(\beta - z)(1 - z)},$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec \frac{\beta(1 - z)}{\beta - z}, \ z \in \mathbb{E}, \ where \ \beta > 1.$$

Taking $\phi(z) = z$, we get the following result from above theorem:

Corollary 24. *If* $f,\ g \in \mathcal{A},\ \dfrac{zf'(z)}{g(z)} \neq 0,\ z \in \mathbb{E}$, *satisfy the differential subordination*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec \frac{a\beta(1-z)}{\beta - z} + \frac{b(1-\beta)z}{(\beta - z)(1-z)},$$

then

$$\frac{zf'(z)}{g(z)} \prec \frac{\beta(1-z)}{\beta - z},\ z \in \mathbb{E},$$

where $\beta > 1$ *and* $a,\ b(\neq 0)$ *are real numbers having same sign.*

On selecting $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 23, we obtain:

Corollary 25. *Let* a *and* $b(\neq 0)$ *be real numbers having same sign and* $\beta > 1$. *If* $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{a\beta(1-z)}{\beta - z} + \frac{b(1-\beta)z}{(\beta - z)(1-z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\beta(1-z)}{\beta - z},\ z \in \mathbb{E},$$

and hence $f(z)$ *is starlike.*

Choosing $a = 1$ and $b = \alpha$ in above corollary, we get:

Corollary 26. *For* $\alpha > 0$, *if* $f \in \mathcal{A},\ \dfrac{zf'(z)}{f(z)} \neq 0,\ z \in \mathbb{E}$, *satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \frac{\beta(1-z)}{\beta - z} + \frac{\alpha(1-\beta)z}{(\beta - z)(1-z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\beta(1-z)}{\beta - z},\quad \beta > 1,\ z \in \mathbb{E},$$

i.e. $f \in \mathcal{S}^*$.

Taking $\phi(z) = g(z) = z$ in Theorem 23, we have:

Corollary 27. *Let a, $b(\neq 0)$ be real numbers having same sign and $\beta > 1$. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec \frac{a\beta(1-z)}{\beta - z} + \frac{b(1-\beta)z}{(\beta - z)(1-z)},$$

then

$$f'(z) \prec \frac{\beta(1-z)}{\beta - z}, \ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

Remark 5. *On selecting the dominant $q(z) = 1 + \frac{2}{3}z^2$ in Theorem 3, it is easy to check that this dominant satisfies the Condition 3 of Theorem 3 for real numbers a and b of same sign, as*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right) = 2\Re\left(1 + \frac{2}{3}z^2\right)^{-1} > 0$$

and

$$\Re q(z) = \Re\left(1 + \frac{2}{3}z^2\right) > 0.$$

Consequently, we obtain the following result:

Theorem 28. *For real numbers a and $b(\neq 0)$ of same sign, if f, $g \in \mathcal{A}$, $\frac{zf'(z)}{\phi(g(z))} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z)}\right) \prec a\left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3+2z^2},$$

then

$$\frac{zf'(z)}{\phi(g(z))} \prec 1 + \frac{2}{3}z^2, \ z \in \mathbb{E}.$$

Here, ϕ is an analytic function in the domain containing $g(\mathbb{E})$, such that $\phi(0) = 0 = \phi'(z) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$.

By selecting $\phi(z) = z$ in above theorem, we obtain:

Corollary 29. *Let a and $b(\neq 0)$ be real numbers such that $\frac{a}{b} > 0$. If f, $g \in \mathcal{A}$, $\frac{zf'(z)}{g(z)} \neq 0$, $z \in \mathbb{E}$, satisfy*

$$a\frac{zf'(z)}{g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zg'(z)}{g(z)}\right) \prec a\left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3+2z^2},$$

then

$$\frac{zf'(z)}{g(z)} \prec 1 + \frac{2}{3},\ z^2\ z \in \mathbb{E}.$$

On taking $\phi(z) = z$ and $g(z) = f(z)$ in Theorem 28, we have:

Corollary 30. *Let a and $b(\neq 0)$ be real numbers such that $\frac{a}{b} > 0$. If $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies*

$$(a-b)\frac{zf'(z)}{f(z)} + b\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec a\left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3+2z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E},$$

and hence $f(z)$ is starlike.

If we take $a = 1$ and $b = \alpha$ in above corollary, we get:

Corollary 31. *For $\alpha > 0$, if $f \in \mathcal{A}$, $\frac{zf'(z)}{f(z)} \neq 0$, $z \in \mathbb{E}$, satisfies the differential subordination*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \left(1 + \frac{2}{3}z^2\right) + \frac{4\alpha z^2}{3+2z^2},$$

then

$$\frac{zf'(z)}{f(z)} \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E},$$

and hence $f \in \mathcal{S}^*$.

In Theorem 28, by selecting $\phi(z) = g(z) = z$, we obtain:

Corollary 32. *Let real numbers a and $b(\neq 0)$ be such that, $\frac{a}{b} > 0$. If $f \in \mathcal{A}$, $f'(z) \neq 0$, $z \in \mathbb{E}$, satisfies*

$$af'(z) + b\frac{zf''(z)}{f'(z)} \prec a\left(1 + \frac{2}{3}z^2\right) + \frac{4bz^2}{3+2z^2},$$

then

$$f'(z) \prec 1 + \frac{2}{3}z^2,\ z \in \mathbb{E},$$

and hence $f(z)$ is close-to-convex.

4. Sandwich Type Results

Theorem 33. *Let a and $b(\neq 0)$ be real numbers such that $\frac{a}{b} > 0$. Let $q,\ q(z) \neq 0$ be univalent function in the unit disk $\mathbb{E}$, with $q(0) = 1$ such that $\frac{zq'(z)}{q(z)}$ is starlike univalent in $\mathbb{E}$ and $\Re q(z) > 0$. Let ϕ be analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. If $f,\ g \in \mathcal{A}$, $\frac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[q(0), 1] \cap Q$ with $\frac{zf'(z)}{\phi(g(z))}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right)$ is univalent in $\mathbb{E}$, satisfy*

$$aq(z) + b\frac{zq'(z)}{q(z)} \prec a\frac{zf'(z)}{\phi(g(z)} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z)}\right), \tag{5}$$

then

$$q(z) \prec \frac{zf'(z)}{\phi(g(z))},\ z \in \mathbb{E},$$

and $q(z)$ is the best subordinant.

Proof: Write $p(z) = \frac{zf'(z)}{\phi(g(z))}$, then (5) becomes

$$aq(z) + b\frac{zq'(z)}{q(z)} \prec ap(z) + b\frac{zp'(z)}{p(z)}$$

By defining θ and φ as $\theta(w) = aw$ and $\varphi(w) = \frac{b}{w}$, where θ and φ are analytic in $\mathbb{C} \setminus \{0\}$ and $\varphi(w) \neq 0,\ w \in \mathbb{C} \setminus \{0\}$. Therefore,

$$Q_1(z) = zq'(z)\varphi(q(z)) = b\frac{zq'(z)}{q(z)}.$$

A little calculation yields

$$\frac{zQ_1(z)}{Q_1(z)} = 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)}$$

and

$$\frac{\theta'(q(z))}{\varphi(q(z))} = \frac{aq(z)}{b}.$$

In view of the given conditions, $Q_1(z)$ is starlike and $\Re\left[\frac{\theta'(q(z))}{\varphi(q(z))}\right] > 0,\ z \in \mathbb{E}$. Hence the proof, now, follows from Lemma 2.

Theorem 34. *Let $q_1(z) \neq 0$ and $q_2(z) \neq 0$ be univalent in $\mathbb{E}$ such that $q_1(z)$ satisfies the condition of Theorem 33 whereas $q_2(z)$ satisfies the Condition 3 of Theorem 3. Let $\phi(z)$ be analytic function in the domain containing $g(\mathbb{E})$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. Let $f,\ g \in \mathcal{A}$, $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,1] \cap Q$ and $a\dfrac{zf'(z)}{\phi(g(z))} + b\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{z(\phi(g(z))'}{\phi(g(z))}\right)$ be univalent in $\mathbb{E}$, where a and $b(\neq 0)$ are real numbers. Further, if*

$$aq_1(z) + b\frac{zq_1'(z)}{q_1(z)} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right) \prec aq_2(z) + b\frac{zq_2'(z)}{q_2(z)},$$

then

$$q_1(z) \prec \frac{zf'(z)}{\phi(g(z))} \prec q_2(z),\ z \in \mathbb{E}.$$

Moreover, $q_1(z)$ and $q_2(z)$ are the best subordinant and the best dominant respectively.

Taking $q_1(z) = 1 + mz$ and $q_2(z) = 1 + nz,\ 0 < m < n \leq 1$, in Theorem 33, we have the following result:

Corollary 35. *Let $\phi(z)$ be a analytic function in the domain containing $g \in \mathbb{E}$ such that $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. Let $a,\ b(\neq 0)$ be real numbers such that $\dfrac{a}{b} > 0$. If $f,\ g \in \mathcal{A}$ be such that $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,1] \cap Q$ with $a\dfrac{zf'(z)}{\phi(g(z))} + b\left(1 + \dfrac{zf''(z)}{f'(z)} - \dfrac{z(\phi(g(z))'}{\phi(g(z))}\right)$ is univalent in $\mathbb{E}$ and satisfy*

$$a(1+mz) + \frac{bmz}{1+mz} \prec a\frac{zf'(z)}{\phi(g(z))} + b\left(1 + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))}\right) \prec a(1+nz) + \frac{bnz}{1+nz}$$

then

$$1 + mz \prec \frac{zf'(z)}{\phi(g(z))} \prec 1 + nz,\ z \in \mathbb{E},$$

where m and n are real numbers, such that $0 < m < n \leq 1$.

On selecting $m = 1/4$, $n = 1/2$ and $a = 1 = b$ in above corollary, we obtain:

Example 1. *Let $\phi(z)$ be a analytic function in the domain containing $g(\mathbb{E})$, where $\phi(0) = 0 = \phi'(0) - 1$ and $\phi(w) \neq 0$ for $w \in g(\mathbb{E}) \setminus \{0\}$. Let $f,\ g \in \mathcal{A}$ be such that $\dfrac{zf'(z)}{\phi(g(z))} \in \mathcal{H}[1,1] \cap Q$ with $1 + \dfrac{zf'(z)}{\phi(g(z))} + \dfrac{zf''(z)}{f'(z)} - \dfrac{z(\phi(g(z))'}{\phi(g(z))}$ is univalent in $\mathbb{E}$, and satisfy*

$$\frac{z}{4} + \frac{z}{4+z} \prec \frac{zf'(z)}{\phi(g(z))} + \frac{zf''(z)}{f'(z)} - \frac{z(\phi(g(z))'}{\phi(g(z))} \prec \frac{z}{2} + \frac{z}{2+z}, \tag{6}$$

then

$$1+\frac{z}{4}\prec\frac{zf'(z)}{\phi(g(z))}\prec 1+\frac{z}{2},\ z\in\mathbb{E}. \tag{7}$$

In Example 1, on taking $\phi(z)=z$, we get:

Example 2. *Let $f,\ g\in\mathcal{A}$ be such that $\frac{zf'(z)}{g(z)}\in\mathcal{H}[1,1]\cap Q$ with $1+\frac{zf'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}$ is univalent in $\mathbb{E}$ and satisfy*

$$\frac{z}{4}+\frac{z}{4+z}\prec\frac{zf'(z)}{g(z)}+\frac{zf''(z)}{f'(z)}-\frac{zg'(z)}{g(z)}\prec\frac{z}{2}+\frac{z}{2+z}$$

then

$$1+\frac{z}{4}\prec\frac{zf'(z)}{g(z)}\prec 1+\frac{z}{2},\ z\in\mathbb{E}.$$

On selecting $\phi(z)=z$ and $g(z)=f(z)$ in Example 1, we get:

Example 3. *Suppose $f\in\mathcal{A}$ is such that $\frac{zf'(z)}{f(z)}\in\mathcal{H}[1,1]\cap Q$ with $1+\frac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies*

$$\frac{z}{4}+\frac{z}{4+z}\prec\frac{zf''(z)}{f'(z)}\prec\frac{z}{2}+\frac{z}{2+z}$$

then

$$1+\frac{z}{4}\prec\frac{zf'(z)}{f(z)}\prec 1+\frac{z}{2},\ z\in\mathbb{E}.$$

On taking $\phi(z)=g(z)=z$ in Example 1, we have:

Example 4. *Suppose $f\in\mathcal{A}$ is such that $f'(z)\in\mathcal{H}[1,1]\cap Q$ with $f'(z)+\frac{zf''(z)}{f'(z)}$ is univalent in $\mathbb{E}$ and satisfies*

$$1+\frac{z}{4}+\frac{z}{4+z}\prec f'(z)+\frac{zf''(z)}{f'(z)}\prec 1+\frac{z}{2}+\frac{z}{2+z},$$

then

$$1+\frac{z}{4}\prec f'(z)\prec 1+\frac{z}{2},\ z\in\mathbb{E}.$$

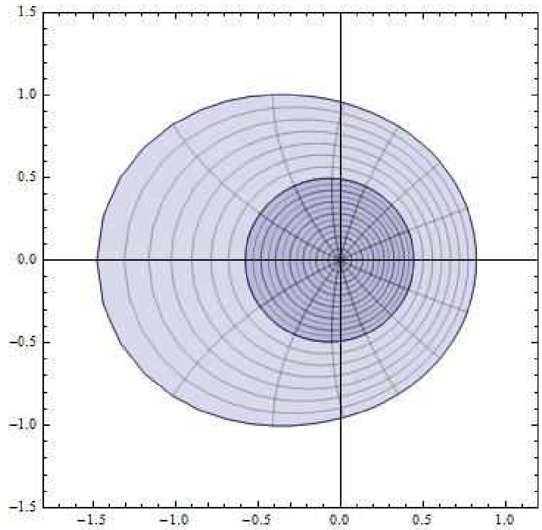

Figure 1

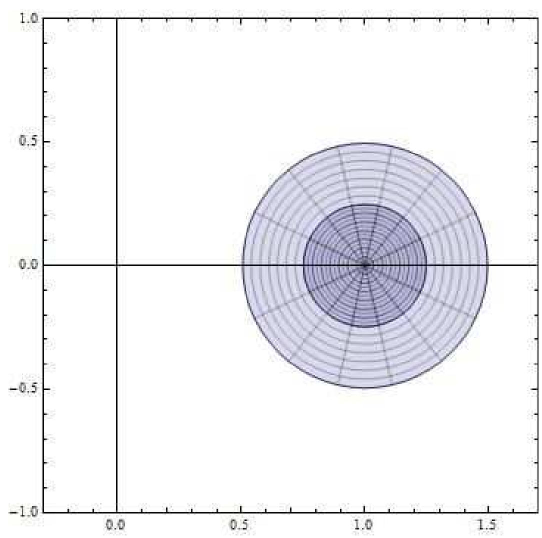

Figure 2

Using Mathematica 10.0, we plot the images of the unit disk under the functions $\dfrac{z}{4}+\dfrac{z}{4+z}$ and $\dfrac{z}{2}+\dfrac{z}{2+z}$ of (6) in Figure 1 and $1+\dfrac{z}{4}$ and $1+\dfrac{z}{2}$ of (7) in Figure 2. It follows that if $\dfrac{zf'(z)}{\phi(g(z))}+\dfrac{zf''(z)}{f'(z)}-\dfrac{z(\phi(g(z))'}{\phi(g(z))}$ takes values in the light shaded portion of Figure 1, then $\dfrac{zf'(z)}{\phi(g(z))}$ will take values in the light shaded portion of Figure 2. Consequently, in view of Example 3 and Example 4, f is starlike and close to convex respectively.

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