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ORTHOGONAL WAVELET FRAMES GENERATED BY THE WALSH POLYNOMIALS

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Abstract. Continuing our investigation on wavelet frames associated with the Walsh polynomials, in this article we give an algorithm for constructing a pair of orthogonal wavelet frames generated by the Walsh polynomials using polyphase matrices. Moreover, a general form for all orthogonal tight wavelet frames generated by an appropriate Walsh polynomial is also described and we investigate their properties by means of the Walsh-Fourier transform.

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1. Introduction

In recent years, the Walsh analysis has been innovated and investigated to various fields of science and technology including signal processing, pattern recognition, transform spectroscopy, sampling theory, differential and integral equations, quantum mechanics and variational problems. Walsh functions were invented by J. L. Walsh in 1923 and are the only known functions with desirable features comparable to sine-cosine functions. These functions are defined on the interval $0 < x < 1$ and assume only the values $+1$ and -1. Similar to those of the Haar functions and trigonometric series, they form a complete orthogonal system in $L^2[0,1]$. There are two ways of considering these functions: either they may be defined on the positive half-line $\mathbb{R}^+$ or they may be identified with the characters of the locally compact Abelian group G_2 which is isomorphic to the Cantor dyadic group $\mathcal{C}$. We work in the former form.

The first construction of wavelets related to the Walsh functions was given by Lang [7] via scaling filters and these wavelets turn out to be certain lacunary Walsh series on the real line. Later, Farkov [4] introduced the notion of multiresolution analysis on a positive half-line $\mathbb{R}^+$ and pointed out a method for constructing compactly supported orthogonal p-wavelets related to the Walsh functions. He also proved a necessary and sufficient condition for scaling filters with p^n many terms $(p, n \geq 2)$ to generate a p-MRA in $L^2(\mathbb{R}^+)$. Recently, Shah [13] introduced the concept of p-frame multiresolution analysis associated with Walsh polynomials on positive half-line $\mathbb{R}^+$ and established a complete characterization of all p-wavelet frames related with p-FMRA using the shift-invariant space theory. Recent results in this direction can also be found in [1,3,5,9-15,18] and the references therein.

Along with the study of compactly supported wavelet bases, there had been a continuing research effort in the study of wavelet frames and their promising features in applications have attracted a great deal of interest in recent years to extensively study them. A wavelet frame is a generalization of an orthonormal wavelet basis by introducing redundancy into a wavelet system. Dyadic wavelet frames on the positive half-line $\mathbb{R}^+$ were constructed by Shah and Debnath [14] using the machinery of Walsh-Fourier transforms. An excellent construction of tight wavelet frames generated by the Walsh polynomials was first reported by the author in [12] by adapting the extension principles of Daubechies et al.[2]. To be more precise, we provide a sufficient condition for finite number of functions $\{\psi_1, \psi_2, \ldots, \psi_L\}$ to form a tight wavelet frame for $L^2(\mathbb{R}^+)$. These studies were continued by Shah and his colleagues in [16,17], where they have provided some excellent tools for constructing minimum-energy wavelet frames and periodic wavelet frames generated by the Walsh polynomials on $\mathbb{R}^+$.

In this paper, we shall introduce the notion of orthogonal wavelet frames generated by the Walsh polynomials on positive half-line $\mathbb{R}^+$ using extension principles. A general algorithm for the construction of orthogonal tight wavelet frames related to Walsh polynomials from a compactly supported scaling function is given. Moreover, we investigate their properties by means of the Walsh-Fourier transforms.

The rest of the paper is organized as follows. In Section 2, we introduce some notations and preliminaries related to the operations on positive half-line $\mathbb{R}^+$ including the definitions of Walsh-Fourier transform and MRA based wavelet frame generated by the Walsh polynomials. In Section 3, we construct a pair of orthogonal wavelet frames generated by the Walsh polynomials and establish more conditions for the existence of orthogonal wavelet frames in $L^2(\mathbb{R}^+)$.

2. Walsh-Fourier Analysis and MRA Based Wavelet Frames

We start this section with certain results on Walsh-Fourier analysis. We present a brief review of generalized Walsh functions, Walsh-Fourier transforms and its various properties.

As usual, let $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{Z}^+ = \{0, 1, 2, \dots\}$ and $\mathbb{N} = \mathbb{Z}^+ - \{0\}$. Denote by $[x]$ the integer part of x. Let p be a fixed natural number greater than 1. For $x \in \mathbb{R}^+$ and any positive integer j, we set

$$x_j = [p^j x] (\mathrm{mod}\, p), \qquad x_{-j} = [p^{1-j} x] (\mathrm{mod}\, p), \tag{1}$$

where $x_j, x_{-j} \in \{0, 1, \dots, p-1\}$. It is clear that for each $x \in \mathbb{R}^+$, there exist $k = k(x)$ in $\mathbb{N}$ such that $x_{-j} = 0,\ \forall\, j > k$.

Consider on $\mathbb{R}^+$ the addition defined as follows:

$$x \oplus y = \sum_{j<0} \zeta_j p^{-j-1} + \sum_{j>0} \zeta_j p^{-j},$$

with $\zeta_j = x_j + y_j (\mathrm{mod}\, p)$, $j \in \mathbb{Z} \setminus \{0\}$, where $\zeta_j \in \{0, 1, \dots, p-1\}$ and x_j, y_j are calculated by (1). As usual, we write $z = x \ominus y$ if $z \oplus y = x$, where $\ominus$ denotes subtraction modulo p in $\mathbb{R}^+$.

For $x \in [0, 1)$, let $r_0(x)$ is given by

$$r_0(x) = \begin{cases} 1, & \text{if } x \in [0, 1/p) \\ \varepsilon_p^{\ell}, & \text{if } x \in \left[\ell p^{-1}, (\ell+1) p^{-1}\right), \quad \ell = 1, 2, \dots, p-1, \end{cases}$$

where $\varepsilon_p = \exp(2\pi i/p)$. The extension of the function r_0 to $\mathbb{R}^+$ is given by the equality $r_0(x+1) = r_0(x)$, $x \in \mathbb{R}^+$. Then, the *generalized Walsh functions* $\{w_m(x) : m \in \mathbb{Z}^+\}$ are defined by

$$w_0(x) \equiv 1 \quad \text{and} \quad w_m(x) = \prod_{j=0}^{k} \left(r_0(p^j x)\right)^{\mu_j}$$

where $m = \sum_{j=0}^{k} \mu_j p^j$, $\mu_j \in \{0, 1, \dots, p-1\}$, $\mu_k \neq 0$. They have many properties similar to those of the Haar functions and trigonometric series, and form a complete orthogonal system. Further, by a Walsh polynomial we shall mean a finite linear combination of Walsh functions.

For $x, y \in \mathbb{R}^+$, let

$$\chi(x, y) = \exp\left(\frac{2\pi i}{p} \sum_{j=1}^{\infty} (x_j y_{-j} + x_{-j} y_j)\right), \tag{2}$$

where x_j, y_j are given by (1).

We observe that

$$\chi\left(x, \frac{m}{p^n}\right) = \chi\left(\frac{x}{p^n}, m\right) = w_m\left(\frac{x}{p^n}\right), \qquad \forall\, x \in [0, p^n),\ m, n \in \mathbb{Z}^+,$$

and
$$\chi(x \oplus y, z) = \chi(x, z)\,\chi(y, z), \quad \chi(x \ominus y, z) = \chi(x, z)\,\overline{\chi(y, z)},$$
where $x, y, z \in \mathbb{R}^+$ and $x \oplus y$ is p-adic irrational. It is well known that systems $\{\chi(\alpha, .)\}_{\alpha=0}^{\infty}$ and $\{\chi(\cdot, \alpha)\}_{\alpha=0}^{\infty}$ are orthonormal bases in $L^2[0,1]$ (See Golubov et al.[6]).

The *Walsh-Fourier transform* of a function $f \in L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+)$ is defined by
$$\hat{f}(\xi) = \int_{\mathbb{R}^+} f(x)\,\overline{\chi(x, \xi)}\,dx, \tag{3}$$
where $\chi(x, \xi)$ is given by (2). The Walsh-Fourier operator $\mathcal{F} : L^1(\mathbb{R}^+) \cap L^2(\mathbb{R}^+) \to L^2(\mathbb{R}^+)$, $\mathcal{F}f = \hat{f}$, extends uniquely to the whole space $L^2(\mathbb{R}^+)$. The properties of the Walsh-Fourier transform are quite similar to those of the classic Fourier transform (See [6]). In particular, if $f \in L^2(\mathbb{R}^+)$, then $\hat{f} \in L^2(\mathbb{R}^+)$ and
$$\left\|\hat{f}\right\|_{L^2(\mathbb{R}^+)} = \left\|f\right\|_{L^2(\mathbb{R}^+)}. \tag{4}$$

For given $\Psi := \{\psi_1, \ldots, \psi_L\} \subset L^2(\mathbb{R}^+)$, define the wavelet system
$$\mathcal{F}(\Psi) = \left\{\psi_{j,k}^{\ell}(x) = p^{j/2}\psi_\ell(p^j x \ominus k),\, j \in \mathbb{Z}, k \in \mathbb{Z}^+, 1 \le \ell \le L\right\}, \tag{5}$$

The wavelet system $\mathcal{F}(\Psi)$ is called a *wavelet frame*, if there exist positive constants A and B such that
$$A\big\|f\big\|_2^2 \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+} \big|\langle f, \psi_{\ell,j,k}\rangle\big|^2 \le B\big\|f\big\|_2^2. \tag{6}$$

holds for every $f \in L^2(\mathbb{R}^+)$, and we call the optimal constants A and B the lower frame bound and the upper frame bound, respectively. A *tight wavelet frame* refers to the case when $A = B$, and a Parseval wavelet frame refers to the case when $A = B = 1$. On the other hand if only the right hand side of the above double inequality holds, then we say $\mathcal{F}(\Psi)$ a *Bessel system*.

Corresponding to the system (5), we have the dual system as
$$\mathcal{F}(\Phi) = \left\{\phi_{j,k}^{\ell} := p^{j/2}\phi_\ell(p^j x \ominus k),\, j \in \mathbb{Z}, k \in \mathbb{Z}^+, 1 \le \ell \le L\right\}. \tag{7}$$

If both $\mathcal{F}(\Psi)$ and $\mathcal{F}(\Phi)$ are wavelet frames and for any $f \in L^2(\mathbb{R}^+)$, we have the reconstruction formula
$$f = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+} \langle f, \psi_{\ell,j,k}\rangle \phi_{\ell,j,k} \tag{8}$$

in the L^2-sense, then we say that $\mathcal{F}(\Phi)$ is a dual wavelet frame of $\mathcal{F}(\Psi)$ (and vice versa) or we simply say that $(\mathcal{F}(\Psi), \mathcal{F}(\Phi))$ is a *pair of dual wavelet frames.*

The most common method for constructing wavelet frames relies on unitary extension principles (UEP) introduced by Ron and Shen [8] and subsequently extended by Daubechies et al.[2] in the form of the oblique extension principle (OEP). Compared to other wavelet frame characterizations, the conditions indicated in the two extension principles are practically easy to check, which makes the construction of wavelet frames painless. Following the unitary extension principle, one often starts with a refinable function ϕ or even with a refinement mask to construct desired wavelet frames.

A compactly supported function $\varphi \in L^2(\mathbb{R}^+)$ is said to be p-refinable if it satisfies the following refinement equation

$$\varphi(x) = p \sum_{k=0}^{p^n-1} c_k \varphi(px \ominus k), \quad x \in \mathbb{R}^+ \tag{9}$$

where c_k are complex coefficients. In the Fourier domain, the above refinement equation can be written as

$$\hat{\varphi}(\xi) = h_0\left(\frac{\xi}{p}\right) \hat{\varphi}\left(\frac{\xi}{p}\right), \tag{10}$$

where

$$h_0(\xi) = \sum_{k=0}^{p^n-1} c_k \, \overline{w_k(\xi)}, \tag{11}$$

is a *generalized Walsh polynomial*, which is called the *mask* or *symbol* of the p-refinable function φ and is of course a p-adic step function. Observe that $w_k(0) = \hat{\varphi}(0) = 1$. By letting $\xi = 0$ in (10) and (11), we obtain $\sum_{k=0}^{p^n-1} c_k = 1$. Since φ is compactly supported and in fact $\operatorname{supp}\varphi \subset [0, p^{n-1})$, hence $\hat{\varphi}(\xi) = 1$ for all $\xi \in [0, p^{1-n})$ as $\hat{\varphi}(0) = 1$. Further, it is proved in [4] that a function $\varphi \in L^2(\mathbb{R}^+)$ generates a p-MRA in $L^2(\mathbb{R}^+)$ if and only if

$$\sum_{k\in\mathbb{Z}^+} \left|\hat{\varphi}\big(\xi \ominus k\big)\right|^2 = 1, \text{ for } a.e.\ \xi \in [0,1], \quad \lim_{j\to\infty} \left|\hat{\varphi}(p^{-j}\xi)\right| = 1, \text{ for } a.e.\ \xi \in \mathbb{R}^+. \tag{12}$$

Suppose $\Psi = \{\psi_1, \ldots, \psi_L\}$ is a set of p-MRA functions derived from

$$\hat{\psi}_\ell(\xi) = h_\ell\left(\frac{\xi}{p}\right) \hat{\varphi}\left(\frac{\xi}{p}\right), \tag{13}$$

where

$$h_\ell(\xi) = \sum_{k=0}^{p^n-1} d_k^\ell\, \overline{w_k(\xi)}, \quad \ell = 1, \dots, L \tag{14}$$

are the generalized Walsh polynomials, called the *wavelet masks* or high pass filters of wavelet frame. With $h_\ell(\xi)$, $\ell = 0, 1, \dots, L$, $L \geq p-1$ as the Walsh polynomials (wavelet masks), we formulate the matrix $\mathcal{M}(\xi)$ as:

$$\mathcal{H}(\xi) = \begin{pmatrix} h_0(\xi) & h_0(\xi \oplus 1/p) & \dots & h_0(\xi \oplus (p-1)/p) \\ h_1(\xi) & h_1(\xi \oplus 1/p) & \dots & h_1(\xi \oplus (p-1)/p) \\ \vdots & \vdots & \ddots & \vdots \\ h_L(\xi) & h_L(\xi \oplus 1/p) & \dots & h_L(\xi \oplus (p-1)/p) \end{pmatrix}. \tag{15}$$

The matrix $\mathcal{H}(\xi)$ is called the *modulation matrix.* The so-called unitary extension principle (UEP) provides a sufficient condition on $\Psi = \{\psi_1, \dots, \psi_L\}$ such that the wavelet system $\mathcal{F}(\Psi)$ given by (5) constitutes a tight frame for $L^2(\mathbb{R}^+)$. It is well known that in order to apply the UEP to derive wavelet tight frame from a given refinable function, the corresponding refinement mask must satisfy

$$\sum_{k=0}^{p-1} \left| h_0\big(\xi \oplus k/p\big)\right|^2 \leq 1, \quad \xi \in \mathbb{R}^+. \tag{16}$$

In [12], the author has given a general procedure for the construction of tight wavelet frames generated by the Walsh polynomials using unitary extension principles and established a complete characterization of such frames by virtue of the modulation matrix $\mathcal{H}(\xi)$. More precisely, we prove that the wavelet system $\mathcal{F}(\Psi)$ given by (5) forms a tight wavelet frame for $L^2(\mathbb{R}^+)$ if the modulation matrix $\mathcal{H}(\xi)$ given by (15) satisfy the UEP condition

$$\mathcal{H}(\xi)\mathcal{H}^*(\xi) = I_p, \quad \text{for } a.e.\ \xi \in \sigma(V_0), \tag{17}$$

where $\sigma(V_0) := \big\{\xi \in [0,1] : \sum_{k \in \mathbb{Z}^+} |\hat{\varphi}\big(\xi \oplus k\big)|^2 \neq 0\big\}$.

3. Orthogonal Wavelet Frames Related to Walsh Polynomials

In this section, we present an algorithm for constructing orthogonal wavelet frames generated by the Walsh polynomials based on polyphase representation of wavelet masks.

Polyphase matrix plays an important role in the design and efficient implementations of filter banks with the desired properties. The *polyphase representation* of the refinement mask $h_0(\xi)$ can be derived by using the properties of Walsh polynomials as

$$\begin{aligned} h_0(\xi) &= \sum_{k=0}^{p^n-1} c_k \overline{w_k(\xi)} \\ &= \sum_{k=0}^{p^n-1} \sum_{m=0}^{p-1} c_{pk+m} \overline{w_{pk+m}(\xi)} \\ &= \sum_{m=0}^{p-1} \overline{w_m(\xi)} \sum_{k=0}^{p^n-1} c_{pk+m} \overline{w_k(p\xi)} \\ &= \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} \mu_{0,m}(p\xi) \overline{w_m(\xi)}, \end{aligned}$$

where

$$\mu_{0,m}(\xi) = \sqrt{p} \sum_{k=0}^{p^n-1} c_{pk+m} \overline{w_k(\xi)}, \quad m = 0, 1, \ldots, p-1. \tag{18}$$

Similarly, the wavelet masks $h_\ell(\xi), 1 \le \ell \le L$, as defined in (14) can be splitted into polyphase components as

$$h_\ell(\xi) = \frac{1}{\sqrt{p}} \sum_{m=0}^{p-1} \mu_{\ell,m}(p\xi) \overline{w_m(\xi)}, \tag{19}$$

where

$$\mu_{\ell,m}(\xi) = \sqrt{p} \sum_{k=0}^{p^{n-1}-1} d^\ell_{pk+m} \overline{w_k(\xi)}, \quad m = 0, 1, \ldots, p-1. \tag{20}$$

With the polyphase components given by (18) and (20), we formulate the *polyphase matrix* $\Gamma(\xi)$ as:

$$\Gamma(\xi) = \begin{pmatrix} \mu_{0,0}(\xi) & \mu_{1,0}(\xi) & \ldots & \mu_{L,0}(\xi) \\ \mu_{0,1}(\xi) & \mu_{1,1}(\xi) & \ldots & \mu_{L,1}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{0,p-1}(\xi) & \mu_{1,p-1}(\xi) & \ldots & \mu_{L,p-1}(\xi) \end{pmatrix}. \tag{21}$$

The polyphase matrix $\Gamma(\xi)$ is called a *unitary matrix* if

$$\Gamma(p\xi)\Gamma^*(p\xi) = I_p, \quad a.e.\, \xi \in [0,1] \tag{22}$$

which is equivalent to

$$\sum_{\ell=0}^{L} \overline{\mu_{\ell,r}(p\xi)}\mu_{\ell,r'}(p\xi) = \delta_{r,r'} \Leftrightarrow \sum_{\ell=1}^{L} \overline{\mu_{\ell,r'}(p\xi)}\mu_{\ell,r}(p\xi) = \delta_{r,r'} - \overline{\mu_{0,r}(p\xi)}\mu_{0,r'}(p\xi),\ 0 \le r, r' \le p-1. \tag{23}$$

The following theorem is proved in [18] which shows that a unitary polyphase matrix leads to a tight wavelet frame generated by Walsh polynomial on $\mathbb{R}^+$.

Theorem 1. *Let $\varphi \in L^2(\mathbb{R}^+)$ be a compactly supported refinable function and every element of the framelet symbols, $h_0(\xi), h_\ell(\xi)$, $\ell = 1, 2, \dots, L$, in* (11) *and* (14) *is a Walsh polynomial. Moreover, if the polyphase matrix $\Gamma(\xi)$ given by* (21) *satisfy UEP condition* (22)*, then the wavelet system $\mathcal{F}(\Psi)$ given by* (5) *constitutes a tight frame for $L^2(\mathbb{R}^+)$.*

Given a collection of Walsh polynomials $\mathbf{H} = [h_0, h_1, \dots, h_L]$. Consider the following matrices

$$\mathcal{M}(\xi) = \begin{pmatrix} h_0(\xi) & h_0(\xi \oplus \delta_k) \\ h_1(\xi) & h_1(\xi \oplus \delta_k) \\ \vdots & \vdots \\ h_L(\xi) & h_L(\xi \oplus \delta_k) \end{pmatrix}, \qquad \mathcal{M}_0(\xi) = \begin{pmatrix} h_1(\xi) & h_1(\xi \oplus \delta_k) \\ h_2(\xi) & h_2(\xi \oplus \delta_k) \\ \vdots & \vdots \\ h_L(\xi) & h_L(\xi \oplus \delta_k) \end{pmatrix} \tag{24}$$

where $\delta_k = k/p$, for $k = 1, 2, \dots, p-1$. Let there be another wavelet frame whose Walsh polynomials are given by $\tilde{h}_0, \tilde{h}_1, \dots, \tilde{h}_L$. Denoting the matrices as in (24) for these wavelet masks by $\tilde{\mathcal{M}}(\xi)$ and $\tilde{\mathcal{M}}_0(\xi)$, respectively. With the above definitions, we present an algorithm for constructing arbitrarily many orthogonal wavelet frames generated by the Walsh polynomials on $\mathbb{R}^+$.

Theorem 2. Suppose that $\varphi, \tilde{\varphi} \in L^2(\mathbb{R}^+)$ are compactly supported refinable functions which satisfy the conditions of the unitary extension principle, and let $h_0(\xi), \tilde{h}_0(\xi)$ be the associated low-pass filter. Let the corresponding high-pass filters be $h_\ell, \tilde{h}_\ell$, $\ell = 1, 2, \dots, L$. Let the matrices $\mathcal{M}(\xi), \mathcal{M}_0(\xi), \tilde{\mathcal{M}}(\xi)$ and $\tilde{\mathcal{M}}_0(\xi)$ be as defined in (24). For all δ_k, $k = 1, 2, \dots, p-1$, suppose that the following matrix equations hold for a.e. $\xi \in \mathbb{R}^+$,

$$\mathcal{M}^*(\xi)\mathcal{M}(\xi) = I_2, \quad \tilde{\mathcal{M}}^*(\xi)\tilde{\mathcal{M}}(\xi) = I_2, \quad \textit{and} \quad \mathcal{M}_0(\xi)\tilde{\mathcal{M}}_0(\xi) = 0. \tag{25}$$

Let $\hat{\psi}_\ell(\xi) = h_\ell(p^{-1}\xi)\hat{\varphi}(p^{-1}\xi)$ and $\hat{\phi}_\ell(\xi) = h_\ell(p^{-1}\xi)\hat{\tilde{\varphi}}(p^{-1}\xi)$, $1 \le \ell \le L$. Then $\{\psi_1, \psi_2, \dots, \psi_L\}$ and $\{\phi_1, \phi_2, \dots, \phi_L\}$ generate orthogonal wavelet tight frame.

Proof. From the unitary extension principle, it follows that both $\{\psi_1, \psi_2, \dots, \psi_L\}$ and $\{\phi_1, \phi_2, \dots, \phi_L\}$ generate normalized tight wavelet frames for $L^2(\mathbb{R}^+)$. Thus, it only remains to prove the orthogonality of these wavelet frames. For each $\ell = 1, 2, \dots, L$, by Hölder's inequality and virtue of the fact that ψ_ℓ and ϕ_ℓ generate Bessel sequences, we have

$$\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_\ell(p^j\xi)\overline{\hat{\phi}_\ell(p^j\xi)}\right| \le \left(\sum_{j\in\mathbb{Z}} \left|\hat{\psi}_\ell(p^j\xi)\right|^2\right) \left(\sum_{j\in\mathbb{Z}} \left|\hat{\phi}_\ell(p^j\xi)\right|^2\right) < \infty. \tag{26}$$

Thus, the order of summation can be changed. With this, by equation (25), we have

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} \hat{\psi}_\ell(p^j\xi)\overline{\hat{\phi}_\ell(p^j\xi)} &= \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}} h_\ell(p^{j-1}\xi)\hat{\varphi}(p^{j-1}\xi)\overline{\tilde{\hat{h}}_\ell(p^{j-1}\xi)\hat{\tilde{\varphi}}(p^{j-1}\xi)} \\ &= \sum_{j\in\mathbb{Z}} \hat{\varphi}(p^{j-1}\xi)\overline{\hat{\tilde{\varphi}}(p^{j-1}\xi)} \sum_{\ell=1}^{L} h_\ell(p^{j-1}\xi)\overline{\tilde{\hat{h}}_\ell(p^{j-1}\xi)} \\ &= 0,\end{aligned}$$

holds for almost every $\xi \in \mathbb{R}^+$. Likewise, for $k \in \mathbb{Z}^+ \setminus p\mathbb{Z}^+$, again by (25), we obtain

$$\begin{aligned}\sum_{\ell=1}^{L}\sum_{j=0}^{\infty} \hat{\psi}_\ell(p^j\xi)\overline{\hat{\phi}_\ell\big(p^j(\xi\oplus k)\big)} &= \sum_{\ell=1}^{L}\sum_{j=0}^{\infty} h_\ell(p^{j-1}\xi)\hat{\varphi}(p^{j-1}\xi)\overline{\tilde{\hat{h}}_\ell\big(p^{j-1}(\xi\oplus k)\big)\hat{\tilde{\varphi}}\big(p^{j-1}(\xi\oplus k)\big)} \\ &= \sum_{j=0}^{\infty} \hat{\varphi}(p^{j-1}\xi)\overline{\hat{\tilde{\varphi}}\big(p^{j-1}(\xi\oplus k)\big)} \sum_{\ell=1}^{L} h_\ell(p^{j-1}\xi)\overline{\tilde{\hat{h}}_\ell\big(p^{j-1}(\xi\oplus k)\big)} \\ &= 0.\end{aligned}$$

This completes the proof of the theorem.

Next, we briefly describe how to obtain a pair of compactly supported orthogonal tight frames from a given compactly supported tight frame system $\mathcal{F}(\Psi)$ constructed via the UEP. More precisely, we construct a pair of orthogonal wavelet frames generated by the Walsh polynomials for the space $L^2(\mathbb{R}^+)$ with slightly different approach as described in Theorem 2.

Let A be a $2L \times 2L$ paraunitary matrix. Partition $A = (A_1 : A_2)$ where A_1 and A_2 are the first and last L columns of A. Let B and C be the matrices as

$$B = \begin{pmatrix} 1 & 0 \\ 0 & A_1 \end{pmatrix}, \qquad C = \begin{pmatrix} 1 & 0 \\ 0 & A_2 \end{pmatrix}.$$

With B and C in hand, we construct new polyphase matrices as $\Gamma_1 = B\Gamma$, $\Gamma_2 = C\Gamma$. The new polyphase matrix Γ_1 looks like

$$\Gamma_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{1,1} & \cdots & a_{1,L} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_{2L,1} & \cdots & a_{2L,L} \end{pmatrix} \begin{pmatrix} \mu_{0,0}(\xi) & \mu_{0,1}(\xi) & \dots & \mu_{0,p-1}(\xi) \\ \mu_{1,0}(\xi) & \mu_{1,1}(\xi) & \dots & \mu_{1,p-1}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \mu_{L,0}(\xi) & \mu_{L,1}(\xi) & \dots & \mu_{L,p-1}(\xi) \end{pmatrix}$$

$$= \begin{pmatrix} \mu_{0,0}(\xi) & \mu_{0,1}(\xi) & \dots & \mu_{0,p-1}(\xi) \\ \sum\limits_{\ell=1}^{L} a_{1,L}\, \mu_{\ell,0}(\xi) & \sum\limits_{\ell=1}^{L} a_{1,\ell}\, \mu_{\ell,1}(\xi) & \dots & \sum\limits_{\ell=1}^{L} a_{1,\ell}\, \mu_{\ell,p-1}(\xi) \\ \vdots & \vdots & \ddots & \vdots \\ \sum\limits_{\ell=1}^{L} a_{2L,\ell}\, \mu_{\ell,0}(\xi) & \sum\limits_{\ell=1}^{L} a_{2L,\ell}\, \mu_{\ell,1}(\xi) & \dots & \sum\limits_{\ell=1}^{L} a_{2L,\ell}\, \mu_{\ell,p-1}(\xi) \end{pmatrix}.$$

It is easy to verify that both the matrices Γ_1 and Γ_2 constructed above are unitary. Moreover, under this algorithm the scaling function does not change. Therefore, for $k = 1, 2, \dots, 2L$, the new wavelet masks $G_k(\xi)$ are given by

$$\begin{aligned} G_k(\xi) &= \sum_{m=0}^{p-1} \overline{w_m(\xi)} \sum_{\ell=1}^{L} a_{k,\ell}(p\xi)\, \mu_{m,\ell}(\xi) \\ &= \sum_{\ell=1}^{L} a_{k,\ell}(p\xi) \sum_{m=0}^{p-1} \overline{w_m(\xi)}\, \mu_{m,\ell}(\xi) \\ &= \sum_{\ell=1}^{L} a_{k,\ell}(p\xi) h_\ell(\xi). \end{aligned} \tag{27}$$

Likewise one obtains $\tilde{G}_k(\xi)$ as

$$\tilde{G}_k(\xi) = \sum_{\ell=L+1}^{2L} a_{k,\ell}(p\xi)\tilde{h}_\ell(\xi). \tag{28}$$

Let $\mathcal{M}(\xi)$ and $\tilde{\mathcal{M}}(\xi)$ be as in equation (24). Then, $\mathcal{M}^*(\xi)\mathcal{M}(\xi) = I_2$, $\tilde{\mathcal{M}}^*(\xi)\tilde{\mathcal{M}}(\xi) = I_2$, as both the matrices $\mathcal{M}$ and $\tilde{\mathcal{M}}$ consist of the columns of the modulation matrices. This satisfies one of the conditions of Theorem 2.

Lemma 3. Let $\mathcal{M}_0(\xi)$ and $\tilde{\mathcal{M}}_0(\xi)$ be the matrices of Walsh polynomials (wavelet masks) as in Theorem 2 Then

$$\mathcal{M}_0(\xi)^* \tilde{\mathcal{M}}_0(\xi) = 0. \tag{29}$$

Proof. Since the entries of matrix A are Walsh polynomials, so they are periodic in each components. Therefore, we have

$$A_{\ell,m}\big(p(\xi \oplus \delta_\ell)\big) = A_{\ell,m}\big(p\xi \oplus p\delta_\ell\big) = A_{\ell,m}\big(p\xi\big).$$

Hence, equations (27) and (28) can be expressed as:

$$G_k\big(\xi \oplus \delta_\ell\big) = \sum_{\ell=1}^{L} a_{k,\ell}(p\xi) h_\ell\big(\xi \oplus \delta_\ell\big), \quad \tilde{G}_k\big(\xi \oplus \delta_\ell\big) = \sum_{\ell=L+1}^{2L} a_{k,\ell}(p\xi) \tilde{h}_\ell\big(\xi \oplus \delta_\ell\big).$$

Thus, we have

$$\begin{aligned}
\mathcal{M}_0(\xi) &= \begin{pmatrix}
\sum_{\ell=1}^{L} a_{1,L}(p\xi) h_\ell(\xi) & \sum_{\ell=1}^{L} a_{1,\ell}(p\xi) h_\ell\big(\xi \oplus \delta_\ell\big) \\
\sum_{\ell=1}^{L} a_{2,L}(p\xi) h_\ell(\xi) & \sum_{\ell=1}^{L} a_{2,\ell}(p\xi) h_\ell\big(\xi \oplus \delta_\ell\big) \\
\vdots & \vdots \\
\sum_{\ell=1}^{L} a_{L,\ell}(p\xi) h_\ell(\xi) & \sum_{\ell=1}^{L} a_{L,\ell}(p\xi) h_\ell\big(\xi \oplus \delta_\ell\big)
\end{pmatrix} \\
&= \begin{pmatrix}
a_{1,1}(p\xi) & a_{1,2}(p\xi) & \cdots & a_{1,L}(p\xi) \\
a_{2,1}(p\xi) & a_{2,1}(p\xi) & \cdots & a_{2,L}(p\xi) \\
\vdots & \vdots & \ddots & \vdots \\
a_{L,1}(p\xi) & a_{L,2}(p\xi) & \cdots & a_{L,L}(p\xi)
\end{pmatrix}
\begin{pmatrix}
h_1(\xi) & h_1(\xi \oplus \delta_\ell) \\
h_2(\xi) & h_2(\xi \oplus \delta_\ell) \\
\vdots & \vdots \\
h_L(\xi) & h_L(\xi \oplus \delta_\ell)
\end{pmatrix}.
\end{aligned}$$

The corresponding dual matrix $\tilde{\mathcal{M}}_0(\xi)$ can be obtained similarly. Using the fact that the matrix A is paraunitary, it follows that $\mathcal{M}_0(\xi)^* \tilde{\mathcal{M}}_0(\xi) = 0$.

For $k = 1, 2, \ldots, 2L$, we consider the new wavelet system as

$$\hat{\psi}_k^*(p\xi) = G_k(\xi)\hat{\varphi}(\xi), \quad \hat{\phi}_k^*(p\xi) = \tilde{G}_k(\xi)\hat{\tilde{\varphi}}(\xi)$$

and let $\Psi^* = \{\psi_1^*, \psi_2^*, \ldots, \psi_{2L}^*\}$ and $\Phi^* = \{\phi_1^*, \phi_2^*, \ldots, \phi_{2L}^*\}$.

Theorem 4. The wavelet systems $\mathcal{F}(\Psi^*)$ and $\mathcal{F}(\Phi^*)$ generated by $\{\psi_1^*, \psi_2^*, \ldots, \psi_{2L}^*\}$ and $\{\phi_1^*, \phi_2^*, \ldots, \phi_{2L}^*\}$ form a pair of orthogonal wavelet frames for $L^2(\mathbb{R}^+)$.

Proof. The proof of the theorem follows immediately from Theorem 1, Lemma 3 and the fact that the matrices Γ_1 and Γ_2 are unitary.

The following result show the relationship between a pair of orthogonal MRA based wavelet frames generated by the Walsh polynomials.

Theorem 5. Suppose that $\mathcal{F}(\Psi)$ and $\mathcal{F}(\Phi)$ are a pair of orthogonal MRA wavelet frames generated by the Walsh polynomials. If $P(\Psi) = P(\Phi)$ and there exists functions $h, g \in L^2(\mathbb{R}^+)$ such that $\Psi^g := \{\psi_1^g, \psi_2^g, \ldots, \psi_L^g\}$ and $\Phi^h := \{\phi_1^h, \phi_2^h, \ldots, \phi_L^h\}$ are wavelet frames, where ψ_ℓ^g and ϕ_ℓ^h are defined by $\hat{\psi}_\ell^g(\xi) = \hat{\psi}_\ell(\xi)\hat{g}(\xi)$, $\hat{\phi}_\ell^h(\xi) = \hat{\phi}_\ell(\xi)\hat{h}(\xi)$, $1 \le \ell \le L$, respectively. Then, $\mathcal{F}(\Psi^g)$ and $\mathcal{F}(\Phi^h)$ are a pair of orthogonal wavelet frames for $L^2(\mathbb{R}^+)$.

Proof. Suppose that $\mathcal{F}(\Psi)$ and $\mathcal{F}(\Phi)$ are a pair of orthogonal wavelet frames in $L^2(\mathbb{R}^+)$ and $P(\Psi) = P(\Phi)$. Then, by the property of MRA based wavelet frames, for any $n \neq m \in \mathbb{Z}$, we have $P(p^m\Psi) \perp P(p^n\Phi)$. Therefore, for all $f_1 \in P(\Psi)$, we have

$$0 = Pf_1(x) = \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}^+}\Big\langle f_1(x), \psi_\ell\big(p^j x \ominus k\big)\Big\rangle \phi_\ell\big(p^j x \ominus k\big) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_1,k}^{\ell}\phi_\ell(x \ominus k), \tag{30}$$

where $C_{f_1,k}^{\ell} = \big\langle f_1(x), \psi_\ell(x \ominus k)\big\rangle$. For any $f \in L^2(\mathbb{R}^+)$, we define $f = f_1 + f_2$, where $f_1 \in P(\Psi)$, $f_2 \in L^2(\mathbb{R}^+) \setminus P(\Psi)$, then, $\langle f_1, f_2\rangle = 0$. With this, we get

$$Pf_2(x) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f_1(x), \psi_\ell(x \ominus k)\Big\rangle \phi_\ell(x \ominus k) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_2,k}^{\ell}\phi_\ell(x \ominus k) = 0. \tag{31}$$

Combining (30) and (31), we conclude that

$$Pf(x) = Pf_1(x) + Pf_2(x) = 0. \tag{32}$$

Since $\hat{\phi}_\ell^h(\xi)=\hat{\phi}_\ell(\xi)\hat{h}(\xi)$ and $Pf(x)=0$, we have

$$\begin{aligned}
0 &= \widehat{Pf(x)} \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f,k}^{\ell}\hat{\phi}_\ell(\xi)\,\overline{w_k(\xi)} \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_1,k}^{\ell}\hat{\phi}_\ell(\xi)\overline{w_k(\xi)} + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_2,k}^{\ell}\hat{\phi}_\ell(\xi)\,\overline{w_k(\xi)} \\
&= h(\xi)\left(\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_1,k}^{\ell}\hat{\phi}_\ell(\xi)\overline{w_k(\xi)} + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_2,k}^{\ell}\hat{\phi}_\ell(\xi)\,\overline{w_k(\xi)}\right) \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_1,k}^{\ell}\hat{\phi}_\ell^h(\xi)\overline{w_k(\xi)} + \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f_2,k}^{\ell}\hat{\phi}_\ell^h(\xi)\,\overline{w_k(\xi)} \\
&= \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f,k}^{\ell}\hat{\phi}_\ell^h(\xi)\,\overline{w_k(\xi)}.
\end{aligned} \tag{33}$$

Applying inverse Walsh-Fourier transform on equation (33), we obtain

$$0 = Pf(x) = \sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f,k}^{\ell}\phi_\ell^h(x\ominus k). \tag{34}$$

From the above equality, we deduce that

$$\begin{aligned}
\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f,k}^{\ell}\Big\langle f(x),\phi_\ell^h\big(x\ominus k\big)\Big\rangle &= \left\langle f(x),\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+} C_{f,k}^{\ell}\phi_\ell^h\big(x\ominus k\big)\right\rangle \\
&= \Big\langle f(x),0\Big\rangle \\
&= \left\langle f(x),\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f(x),\phi_\ell^h\big(x\ominus k\big)\Big\rangle\psi_\ell(x\ominus k)\right\rangle.
\end{aligned} \tag{35}$$

Thus, we have

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f(x),\phi_\ell^h\big(x\ominus k\big)\Big\rangle\psi_\ell(x\ominus k)=0. \tag{36}$$

In a similar manner, we can show that

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f(x),\phi_\ell^h\big(x\ominus k\big)\Big\rangle\psi_\ell^g(x\ominus k)=0. \tag{37}$$

Therefore, for any $j\in\mathbb{Z}$, we have

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f(x),\phi_\ell^h\big(p^jx\ominus k\big)\Big\rangle\psi_\ell^g(p^jx\ominus k)=p^{-j}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f\big(p^{-j}x\big),\phi_\ell^h\big(x\ominus k\big)\Big\rangle\psi_\ell^g\big(x\ominus k\big)=0. \tag{38}$$

Putting everything together, we conclude that

$$\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^+}\Big\langle f(x),\phi_\ell^h\big(p^jx\ominus k\big)\Big\rangle\psi_\ell^g(p^jx\ominus k)=0.$$

Hence, $\mathcal{F}(\Psi^g)$ and $\mathcal{F}(\Phi^h)$ constitutes a pair of orthogonal wavelet frames generated by Walsh polynomials for $L^2(\mathbb{R}^+)$.

The following theorem describes a general construction algorithm for orthogonal wavelet tight frames related to the Walsh polynomials.

Theorem 6. Suppose $A(\xi)$ is an $L\times L$ paraunitary matrix with integral periodic entries $a_{\ell,m}(\xi)$ and let $A_m(\xi)$ denotes the mth column. Let $h_0,h_1,\ldots,h_L$ be the Walsh polynomials (masks) given by (11) and (14) such that $\mathbf{H}^*(\xi)\mathbf{H}(\xi)=I_2$, where $\mathbf{H}=[h_0(\xi),h_1(\xi),\ldots,h_L(\xi)]$ is the combined mask of the wavelet masks, and let the wavelet system $\mathcal{F}(\Psi)$ forms a normalized wavelet frame for $L^2(\mathbb{R}^+)$. For $m=1,2,\ldots,L$, define new wavelet masks via

$$\begin{pmatrix}\eta_{1,1}^m(\xi)\\ \eta_{1,2}^m(\xi)\\ \vdots\\ \eta_{1,L}^m(\xi)\\ \eta_{2,1}^m(\xi)\\ \vdots\\ \eta_{2,L}^m(\xi)\\ \vdots\\ \eta_{N,1}^m(\xi)\\ \eta_{N,2}^m(\xi)\\ \vdots\\ \eta_{N,L}^m(\xi)\end{pmatrix}=\begin{pmatrix}A_m(\xi)h_1(\xi)\\ A_m(\xi)h_2(\xi)\\ \vdots\\ A_m(\xi)h_N(\xi)\end{pmatrix}. \tag{39}$$

Then, for $m = 1, 2, \ldots, L$, the wavelet systems generated by $\Psi^m = \{\psi^m_{n,\ell} : 1 \le n \le N,\ 1 \le \ell \le L\}$, obtained via

$$\hat{\psi}^m_{n,\ell}(p\xi) = \eta^m_{n,\ell}(\xi)\hat{\varphi}(\xi), \tag{40}$$

are tight wavelet frames and are pairwise orthogonal.

Proof. We first prove that the systems $\mathcal{F}(\Psi^m), 1 \le m \le L$ are tight wavelet frames for $L^2(\mathbb{R}^+)$. To do so, we first consider

$$\mathcal{M}_m = \Big[h_0(\xi), \eta^m_{1,1}(\xi), \eta^m_{1,2}(\xi), \ldots, \eta^m_{1,L}(\xi), \eta^m_{2,1}(\xi), \ldots, \eta^m_{2,L}(\xi), \ldots, \eta^m_{N,1}(\xi), \ldots, \eta^m_{N,L}(\xi)\Big].$$

Then, we define $\mathcal{M}_m(\xi)$ according to (24) as:

$$\mathcal{M}_m(\xi) = \begin{pmatrix} h_0(\xi) & h_0(\xi \oplus \delta_k) \\ \eta^m_{1,1}(\xi) & \eta^m_{1,1}(\xi \oplus \delta_k) \\ \eta^m_{1,2}(\xi) & \eta^m_{1,2}(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta^m_{1,L}(\xi) & \eta^m_{1,L}(\xi \oplus \delta_k) \\ \eta^m_{2,1}(\xi) & \eta^m_{2,1}(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta^m_{2,L}(\xi) & \eta^m_{2,L}(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta^m_{N,1}(\xi) & \eta^m_{N,1}(\xi \oplus \delta_k) \\ \eta^m_{N,2}(\xi) & \eta^m_{N,2}(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta^m_{N,L}(\xi) & \eta^m_{N,L}(\xi \oplus \delta_k) \end{pmatrix}, \tag{41}$$

where $\delta_k = k/p$, for $k = 1, 2, \ldots, p-1$. Clearly, $\mathcal{M}^*_m(\xi)\mathcal{M}_m(\xi)$ is a 2×2 matrix. Next, we examine the entries of $\mathcal{M}^*_m(\xi)\mathcal{M}_m(\xi)$ individually. Since the columns of $A(\xi)$ have length 1, it follows that

$$\begin{aligned}
\Big[\mathcal{M}^*_m(\xi)\mathcal{M}_m(\xi)\Big]_{1,1} &= \big|h_0(\xi)\big|^2 + \sum_{\ell=1}^{L}\sum_{n=1}^{N}\big|a_{\ell,m}(\xi)h_n(\xi)\big|^2 \\
&= \big|h_0(\xi)\big|^2 + \sum_{\ell=1}^{L}\big|a_{\ell,m}(\xi)\big|^2\sum_{n=1}^{N}\big|h_n(\xi)\big|^2 \\
&= \big|h_0(\xi)\big|^2 + \sum_{n=1}^{N}\big|h_n(\xi)\big|^2 \\
&= 1.
\end{aligned}$$

Similarly,

$$\begin{aligned}\Big[\mathcal{M}_m^*(\xi)\mathcal{M}_m(\xi)\Big]_{2,2} &= \big|h_0(\xi\oplus\delta_k)\big|^2 + \sum_{\ell=1}^{L}\sum_{n=1}^{N}\big|a_{\ell,m}(\xi\oplus\delta_k)h_n(\xi\oplus\delta_k)\big|^2\\ &= \big|h_0(\xi\oplus\delta_k)\big|^2 + \sum_{\ell=1}^{L}\sum_{n=1}^{N}\big|a_{\ell,m}(\xi\oplus\delta_k)\big|^2\big|h_n(\xi\oplus\delta_k)\big|^2\\ &= \big|h_0(\xi\oplus\delta_k)\big|^2 + \sum_{n=1}^{N}\big|h_n(\xi\oplus\delta_k)\big|^2\\ &= 1.\end{aligned}$$

Using the fact that $\mathcal{M}^*(\xi)\mathcal{M}(\xi) = I_2$ and all the entries of $A(\xi)$ are integral periodic, we obtain

$$\begin{aligned}\Big[\mathcal{M}_m^*(\xi)\mathcal{M}_m(\xi)\Big]_{1,2} &= h_0(\xi\oplus\delta_k)\overline{h_0(\xi)} + \sum_{\ell=1}^{L}\sum_{n=1}^{N}a_{\ell,m}(\xi\oplus\delta_k)h_n(\xi\oplus\delta_k)\,\overline{a_{\ell,m}(\xi)\,h_n(\xi)}\\ &= h_0(\xi\oplus\delta_k)\overline{h_0(\xi)} + \sum_{\ell=1}^{L}\sum_{n=1}^{N}\big|a_{\ell,m}(\xi)\big|^2\overline{h_n(\xi)}h_n(\xi\oplus\delta_k)\\ &= h_0(\xi\oplus\delta_k)\overline{h_0(\xi)} + \sum_{n=1}^{N}\overline{h_n(\xi)}h_n(\xi\oplus\delta_k)\\ &= 0.\end{aligned}$$

By the conjugate symmetry of $\mathcal{M}_m^*(\xi)\mathcal{M}_m(\xi)$, the entry (2,1) must be zero. Hence, we can say that

$$\mathcal{M}_m^*(\xi)\mathcal{M}_m(\xi) = I_2, \quad 1 \le m \le L. \tag{42}$$

Putting everything together, from Theorem (2), the wavelet systems $\mathcal{F}(\Psi^m)$ defined via (40) are tight wavelet frames for $L^2(\mathbb{R}^+)$. It only remains to prove the orthogonality. According to equation (24), for $1 \le m \le L$, we have

$$\mathcal{M}_m^0(\xi) = \begin{pmatrix} \eta_{1,1}^m(\xi) & \eta_{1,1}^m(\xi \oplus \delta_k) \\ \eta_{1,2}^m(\xi) & \eta_{1,2}^m(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta_{1,L}^m(\xi) & \eta_{1,L}^m(\xi \oplus \delta_k) \\ \eta_{2,1}^m(\xi) & \eta_{2,1}^m(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta_{2,L}^m(\xi) & \eta_{2,L}^m(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta_{N,1}^m(\xi) & \eta_{N,1}^m(\xi \oplus \delta_k) \\ \eta_{N,2}^m(\xi) & \eta_{N,2}^m(\xi \oplus \delta_k) \\ \vdots & \vdots \\ \eta_{N,L}^m(\xi) & \eta_{N,L}^m(\xi \oplus \delta_k) \end{pmatrix} = \begin{pmatrix} A_m(\xi)h_1(\xi) & A_m(\xi \oplus \delta_k)h_1(\xi \oplus \delta_k) \\ A_m(\xi)h_2(\xi) & A_m(\xi \oplus \delta_k)h_2(\xi \oplus \delta_k) \\ \vdots & \vdots \\ A_m(\xi)h_N(\xi) & A_m(\xi \oplus \delta_k)h_N(\xi \oplus \delta_k) \end{pmatrix}. \tag{43}$$

If $1 \le m \ne m' \le L$, then

$$\begin{aligned} &\mathcal{M}_m^0(\xi)^* \mathcal{M}_{m'}^0(\xi) \\ &= \begin{pmatrix} A_m(\xi)h_1(\xi) & A_m(\xi \oplus \delta_k)h_1(\xi \oplus \delta_k) \\ A_m(\xi)h_2(\xi) & A_m(\xi \oplus \delta_k)h_2(\xi \oplus \delta_k) \\ \vdots & \vdots \\ A_m(\xi)h_N(\xi) & A_m(\xi \oplus \delta_k)h_N(\xi \oplus \delta_k) \end{pmatrix}^* \begin{pmatrix} A_{m'}(\xi)h_1(\xi) & A_{m'}(\xi \oplus \delta_k)h_1(\xi \oplus \delta_k) \\ A_{m'}(\xi)h_2(\xi) & A_{m'}(\xi \oplus \delta_k)h_2(\xi \oplus \delta_k) \\ \vdots & \vdots \\ A_{m'}(\xi)h_N(\xi) & A_{m'}(\xi \oplus \delta_k)h_N(\xi \oplus \delta_k) \end{pmatrix} \\ &= \begin{pmatrix} A_m^*(\xi)A_{m'}(\xi)\sum_{n=1}^N |h_n(\xi)|^2 & A_m^*(\xi)A_{m'}(\xi \oplus \delta_k)\sum_{n=1}^N \overline{h_n(\xi)}h_n(\xi \oplus \delta_k) \\ A_m^*(\xi \oplus \delta_k)A_{m'}(\xi)\sum_{n=1}^N \overline{h_n(\xi \oplus \delta_k)}h_n(\xi) & A_m^*(\xi \oplus \delta_k)A_{m'}(\xi \oplus \delta_k)\sum_{n=1}^N \overline{h_n(\xi \oplus \delta_k)}h_n(\xi \oplus \delta_k) \end{pmatrix} \\ &= \begin{pmatrix} A_m^*(\xi)A_{m'}(\xi)\sum_{n=1}^N |h_n(\xi)|^2 & A_m^*(\xi)A_{m'}(\xi)\sum_{n=1}^N \overline{h_n(\xi)}h_n(\xi \oplus \delta_k) \\ A_m^*(\xi \oplus \delta_k)A_{m'}(\xi)\sum_{n=1}^N \overline{h_n(\xi \oplus \delta_k)}h_n(\xi) & A_m^*(\xi \oplus \delta_k)A_{m'}(\xi)\sum_{n=1}^N \overline{h_n(\xi \oplus \delta_k)}h_n(\xi \oplus \delta_k) \end{pmatrix} \\ &= 0, \end{aligned}$$

where we use the fact that the product of the two matrices $A_m^*(\xi)A_{m'}(\xi) = 0$ by the orthogonality of the columns of $A(\xi)$. Using Theorem 2, we get the desired result.

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