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PERIODIC SOLUTIONS FOR A KIND OF LIÉNARD-TYPE $p(t)$-LAPLACIAN EQUATION

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Abstract. Based on Manásevich-Mawhin's continuation theorem, the existence of periodic solutions for the Liénard-type $p(t)$-Laplacian equation with a deviating argument

$$\left(|x'(t)|^{p(t)-2}\,x'(t)\right)' + x'(t) + g\left(t, x\left(t-\tau(t)\right)\right) = e(t).$$

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1. Introduction

The speciality of this paper is that we discuss the existence of T-periodic solutions for Liénard-type $p(t)$-Laplacian equation (such an equation that can be derived from many fields, such as fluid mechanics and nonlinear elastic mechanics) of the from

$$\left(|x'(t)|^{p(t)-2}\,x'(t)\right)' + x'(t) + g\left(t, x\left(t-\tau(t)\right)\right) = e(t), \tag{1.1}$$

where $p, \tau, e : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions, p, τ, e are T-periodic, g is T-periodic in its first argument and $T > 0$.

In recent years, the problem of the existence of periodic solutions of Duffing type, Liénard-type and Rayleigh type p-Laplacian equations with deviating argument has received much attention. We refer the reader to (see, for example, [10, 11, 12, 13]) and the references there in. These papers were studied when $p > 1$ constant number. However, when $p = p(t)$ is a function, i.e. p is not a constant, the situation is very crucial, and therefore, the equation like (1.1) become a very challenging problem. As a consequence, one needs additional and different approaches to deal such problems.

Motivated by the ideas in [2], we will make some a priori estimates and use topological degree theory to obtain the existence of positive periodic solutions for (1.1).

This work is organized as follows. In Section 1, we introduce some necessary and preliminary results. In Section 2, we give the statement of our main result and its proof.

Definition 1. *We denote* $C_T = \{x : x \in C(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$ *with the norm*

$$\|x\|_0 = \max_{t\in[0,T]} |x(t)|,$$

and $C_T^1 = \{x : x \in C^1(\mathbb{R}, \mathbb{R}), x(t+T) \equiv x(t)\}$ *with the norm*

$$\|x\|_1 = \|x\|_0 + \|x'\|_0 .$$

Clearly, $(C_T, \|.\|_0)$ *and* $\left(C_T^1, \|.\|_1\right)$ *are two Banach spaces.*

Let $p \in C([0,T], \mathbb{R})$ with

$$1 < \min_{t\in[0,T]} p(t) = p^- \leq p(t) \leq p^+ = \max_{t\in[0,T]} p(t) < \infty.$$

Define

$$L^{p(t)}([0,T], \mathbb{R}) = \left\{ x \in L^1([0,T], \mathbb{R}) : \int_0^T |x(t)|^{p(t)} dt < \infty \right\}.$$

The variable exponent space $L_T^{p(t)}([0,T], \mathbb{R})$ denotes the Banach space of T-periodic functions on $[0,T]$ with values in $\mathbb{R}$ under the norm

$$\|x\|_{L^{p(t)}} := \|x\|_{p(t)} = \inf\left\{ \delta > 0 : \int_0^T \left| \frac{x(t)}{\delta} \right|^{p(t)} dt \leq 1 \right\}.$$

Define

$$W^{1,p(t)}([0,T], \mathbb{R}) = \left\{ x \in L^{p(t)}([0,T], \mathbb{R}) : x' \in L^{p(t)}([0,T], \mathbb{R}) \right\}$$

with the norm

$$\|x\|_{W^{1,p(t)}} := \|x\|_{1,p(t)} = \|x\|_{p(t)} + \|x'\|_{p(t)} .$$

The space $W_0^{1,p(t)}([0,T], \mathbb{R})$ is denoted by the closure of $C_0^\infty([0,T], \mathbb{R})$ in $W_T^{1,p(t)}([0,T], \mathbb{R})$, where

$$W_T^{1,p(t)}([0,T], \mathbb{R}) := \left\{ x : x \in W^{1,p(t)}([0,T], \mathbb{R}), x(t+T) \equiv x(t) \right\}$$

for all $x \in W^{1,p(t)}([0,T],\mathbb{R})$. We will use $\|x\|_{1,p(t)}$ and $\|x'\|_{p(t)}$ for $x \in W_0^{1,p(t)}([0,T],\mathbb{R})$ in the following discussions. Moreover, if $1 < p^- \leq p(t) \leq p^+ < \infty$ the spaces $L^{p(t)}([0,T],\mathbb{R}), W^{1,p(t)}([0,T],\mathbb{R})$ and $W_0^{1,p(t)}([0,T],\mathbb{R})$ are separable and reflexive Banach spaces (see [1],[3]).

Proposition 1. *[1, 3, 5] If we denote $\rho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$, $\forall u \in L^{p(x)}(\Omega)$, then*

i) $\|u\|_{p(x)} < 1 (= 1; > 1) \Leftrightarrow \rho_{p(x)}(u) < 1 (= 1; > 1)$;

ii) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|_{p(x)}^{p^-} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^+}, \|u\|_{p(x)} < 1 \Rightarrow \|u\|_{p(x)}^{p^+} \leq \rho_{p(x)}(u) \leq \|u\|_{p(x)}^{p^-}$;

iii) $\rho_{p(x)}(u) \leq \max\left\{\|u\|_{p(x)}^{p^-}, \|u\|_{p(x)}^{p^+}\right\} \leq \|u\|_{p(x)}^{p^-} + \|u\|_{p(x)}^{p^+}$.

Proposition 2. *[6] Assume that $r \in L_+^\infty(\Omega)$ and $p \in C_+(\Omega)$. If $|u|^{r(x)} \in L^{p(x)}(\Omega)$, then we have*

$$\min\left\{\|u\|_{r(x)p(x)}^{r^+}, \|u\|_{r(x)p(x)}^{r^-}\right\} \leq \left\||u|^{r(x)}\right\|_{p(x)} \leq \max\left\{\|u\|_{r(x)p(x)}^{r^+}, \|u\|_{r(x)p(x)}^{r^-}\right\}.$$

Proposition 3. *[1, 4] If p and $q \in L^1(\Omega)$ satisfy $1 < p(x) \leq q(x) < p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)}, p(x) < N; \\ +\infty, p(x) \geq N, \end{cases}$ which satisfies for all $x \in \Omega$, then there is a continuous embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ and then there exists a positive constant C such that*

$$\|u\|_{q(x)} \leq C \|u\|_{1,p(x)}.$$

Theorem 1. *[4] If G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, then G is continuous and bounded, and there is a constant $b \geq 0$ and a non-negative function $a \in L^{p_2(x)}(\Omega)$ such that*

$$|f(x,u)| \leq a(x) + b\,|u|^{\frac{p_1(x)}{p_2(x)}}, \tag{1.2}$$

for $x \in \Omega$ and $u \in \mathbb{R}$. On the other hand, if f satisfies (1.2)*, then G maps $L^{p_1(x)}(\Omega)$ into $L^{p_2(x)}(\Omega)$, and thus G is continuous and bounded.*

Remark 1. *$N_f(u,\upsilon) \in L^{q'(t)}([0,T],\mathbb{R})$ if and only if* *i)* $a \in L^{q'(t)}([0,T],\mathbb{R})$, $a(t) \geq 0$ *and there are constants* $c_1, c_2 \geq 0$ *such that*

$$|f(t,u,\upsilon)| \leq a(t) + c_1 |u|^{\frac{q(t)}{q'(t)}} + c_2 |\upsilon|^{\frac{s(t)}{q'(t)}}$$

for $\forall u \in L^{q(t)}([0,T],\mathbb{R})$ *and* $\forall \upsilon \in L^{s(t)}([0,T],\mathbb{R})$.

ii) if i) is satisfied, N_f *continuous and bounded from* $L^{q(t)}([0,T],\mathbb{R}) \times L^{s(t)}([0,T],\mathbb{R})$ *into* $L^{q'(t)}([0,T],\mathbb{R})$.

Lemma 2. *[2] Assume that* Ω *is an open bounded set in* C_T^1 *such that the following conditions hold.*

(1) *For each* $\lambda \in (0,1)$ *the problem*

$$\left(\Phi_p(x')\right)' = \lambda \widetilde{f}(t,x,x'), x(0) = x(T),\ x'(0) = x'(T) \tag{1.3}$$

has no solution on $\partial\Omega$*, where* $\widetilde{f}(t,x,x')$ *is a continuous function and* T*-periodic in the first variable.*

(2) *The equation*

$$F(a) = \frac{1}{T}\int_0^T \widetilde{f}(t,a,0)dt = 0.$$

has no solution on $\Omega \cap \mathbb{R}^N$.

(3) *The Brouwer degree*

$$\deg\left(F, \Omega \cap \mathbb{R}^N, 0\right) \neq 0.$$

Then the perodic boundary value problem (1.3) has at least one T*-periodic solution in* $\overline{\Omega}$.

2. Main result and its proof

We denote

$$f(t,x,x') := e(t) - x'(t) - g\left(t, x\left(t - \tau(t)\right)\right).$$

Assume that f and g satisfies the following conditions:

(F1) $f : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $a \in L^{q'(t)}(\mathbb{R}, \mathbb{R})$, $a(t) \geq 0$ and $s, q : \mathbb{R} \to \mathbb{R}$ are continuous functions with $s(t), q(t) < p(t) < p^*(t)$ and there are constants $c_1, c_2 \geq 0$ such that

$$|f(t, u, v)| \leq a(t) + c_1 |u|^{\frac{q(t)}{q'(t)}} + c_2 |v|^{\frac{s(t)}{q'(t)}}.$$

(F2) There exist a constant $M > 0$ for any $|D| \geq M$ such that

$$g(t, D) > e(t)$$

or

$$g(t, D) < e(t)$$

for all $t \in \mathbb{R}$.

(F3) There exist a constant $L > 0$ such that

$$|g(t, x_1) - g(t, x_2)| \leq L |x_1 - x_2|^2$$

for all $t, x_1, x_2 \in \mathbb{R}$.

Consequently, we obtain the auxiliary result.

For simplicity of natation, we write

$$\begin{aligned} X &= W_0^{1,p(t)}([0,T], \mathbb{R}), X^* = W_0^{-1,p'(t)}([0,T], \mathbb{R}), \\ Y_{q(t),s(t)} &= L^{q(t)}([0,T], \mathbb{R}) \times L^{s(t)}([0,T], \mathbb{R}), Y^*_{q(t)} = L^{q'(t)}([0,T], \mathbb{R}). \end{aligned}$$

Let us recall some results borrowed from [3, 7, 9] about $p(t)$-Laplacian and Nemytskii operator N_f. Firstly, since $s(t), q(t) < p(t) < p^*(t)$ for all $t \in [0, T]$, X is compactly embedded in $Y_{q(t),s(t)}$ (see [3, 7, 9]). Denote by i the compact injection of X in $Y_{q(t),s(t)}$ and by $i^* : Y^*_{q(t)} \to X^*$, $i^*\varphi = \varphi \circ i$ for all $\varphi \in Y^*_{q(t)}$, its adjoint. Since the Caratheodory function f satisfies (CAR), the Nemytskii operator N_f generated by f, $N_f(x, x')(t) = f(t, x(t), x'(t))$, is well defined from $Y_{q(t),s(t)}$ into $Y^*_{q(t)}$, continuous, and bounded (by Remark 1). To prove that Eq. (1.1) has a solution is sufficient to prove that the equation

$$\left(\Phi_{p(t)}(x')\right)' = \left(\left|x'(t)\right|^{p(t)-2} x'(t)\right)' = (i^* N_f i)(x, x') \tag{2.1}$$

has a solution in X.

If $x \in X$ satisfies (2.1) then, for all $\varphi \in X$, one has

$$\left\langle \left(\Phi_{p(t)}(x')\right)', \varphi \right\rangle_{X,X^*} = \left\langle (i^* N_f i)(x, x'), \varphi \right\rangle_{X,X^*} = \left\langle N_f\left((ix, ix')\right), i\varphi \right\rangle_{Y_{(t),s(t)}, Y^*_{q(t)}}.$$

Since $-\Phi_{p(t)}$ is a homeomorphism of X onto X^*, (2.1) may be equivalently written as

$$x = \left(-\Phi_{p(t)}\right)^{-1}\left[\left(i^* N_f i\right)(x, x')\right]. \tag{2.2}$$

Thus, compact operator

$$\Lambda = \left(-\Phi_{p(t)}\right)^{-1}\left(i^* N_f i\right) : X \to X$$

has a fixed point. According to the classical result of Leray, Schauder and Schaefer, a sufficient condition for Λ to have a fixed point is that a constant $R > 0$ exists such that

$$S = \left\{x \in X : x = \lambda\Lambda(x, x') \text{ for some } \lambda \in [0, 1]\right\} \subset B\left(0, R\right).$$

Since, for $\lambda = 0$, the only solution of equation $x = \lambda\Lambda(x, x')$ is $x = 0$, it is enough to show that there exists a constant $R > 0$ such that, any $x \in X$ which satisfies

$$x = \lambda\left(-\Phi_{p(t)}\right)^{-1}\left[\left(i^* N_f i\right)(x, x')\right] \tag{2.3}$$

for some $\lambda \in (0, 1]$, belongs to $B(0, R)$. So

$$\begin{aligned}
\left\langle -\Phi_{p(t)}\left(\frac{x'}{\lambda}\right), \left(\frac{x'}{\lambda}\right)\right\rangle_{X,X^*} &= \int_0^T \left|\frac{x'(t)}{\lambda}\right|^{p(t)} dt \\
&\geq \frac{1}{\lambda^{p^-}}\int_0^T \left|x'(t)\right|^{p(t)} dt \\
&= \frac{1}{\lambda^{p^-}}\rho_{p(t)}\left(x'(t)\right).
\end{aligned}$$

Then, we obtain

$$\int_0^T \left|x'(t)\right|^{p(x)} dt \leq \lambda^{p^- - 1}\left\langle\left(i^* N_f i\right)\left(x, x'\right), x\right\rangle_{X,X^*}$$

$$\leq \|i^*\|_{Y^*_{q(t)} \to X^*}\left\|N_f(ix, ix')\right\|_{Y^*_{q(t)}}\|x\|_X. \tag{2.4}$$

Consider the homotopic equation of Eq. (1.1) as follows:

$$\left(\Phi_{p(t)}(x'(t))\right)' = \lambda e(t) - \lambda x'(t) - \lambda g\left(t, x\left(t - \tau(t)\right)\right), \lambda \in (0, 1). \tag{2.5$_\lambda$}$$

The main result of the present paper is:

Theorem 3. *If the assumptions $(F1)$, $(F2)$, $(F3)$ hold and $p^- > 2$, then the Eq.(1.1) admits at least one positive periodic solution.*

First, let's give some lemmas and proofs that are necessary to prove the Theorem 3. We will prove the following lemma based on Shiping Lu and Weigao Ge's article [8].

Lemma 4. *Let $p^- > 2$ and $0 \leq \alpha \leq T$ be a constant, $s \in C_T^1$ with $\|s\|_0 \leq \alpha$. Then for $\forall x \in C_T^1$, we have*

$$\int_0^T |x(t-s(t)) - x(t)|^2 \, dt \leq \frac{4\alpha^2}{p^-} \int_0^T |x'(t)|^{p(t)} \, dt + \frac{2\alpha^2 T(p^+ - 2)}{p^-}.$$

Proof. By using the Lemma 1 given in [8], and the Young's inequality, we obtain the inequality as follows:

$$\int_0^T |x(t-s(t)) - x(t)|^2 \, dt \leq 2\alpha^2 \int_0^T |x'(t)|^2 \, dt$$
$$\leq \frac{4\alpha^2}{p^-} \int_0^T |x'(t)|^{p(t)} \, dt + \frac{2\alpha^2 T(p^+ - 2)}{p^-}. \tag{2.6}$$

This completes the proof of Lemma 4. ■

Next, we will show that the $x(t)$ must be uniform bounded in C_T^1 for any $\lambda \in [0,1]$. In order to make this we use $(F1)$, (2.4), Propositions 1, 2 and 3, we have

$$\begin{aligned}
&\int_0^T |x'(t)|^{p(t)} \, dt \\
\leq\ & \|i^*\|_{Y^*_{q(t)} \to X^*} \left\| N_f\left((ix, ix')\right)\right\|_{Y^*_{q(t)}} \|x\|_X \\
\leq\ & k_1 \left\| a(t) + c_1 |ix|^{\frac{q(t)}{q'(t)}} + c_2 |ix'|^{\frac{s(t)}{q'(t)}} \right\|_{Y^*_{q(t)}} \|x\|_X \\
\leq\ & k_1 \left(\|a(t)\|_{Y^*_{q(t)}} + c_1 \left\| |ix|^{\frac{q(t)}{q'(t)}} \right\|_{Y^*_{q(t)}} + c_2 \left\| |ix'|^{\frac{s(t)}{q'(t)}} \right\|_{Y^*_{q(t)}} \right) \|x\|_X \\
\leq\ & \left(k_1 \|a(t)\|_{Y^*_{q(t)}} + c_1 k_1 \max\left\{ \|ix\|_{Y_{q(t)}}^{q^+-1}, \|ix\|_{Y_{q(t)}}^{q^--1} \right\} \right) \|x\|_X \\
& + c_2 k_1 \left\{ \|ix'\|_{Y_{q(t)}}^{s^+-1}, \|ix'\|_{Y_{q(t)}}^{s^--1} \right\} \|x\|_X
\end{aligned}$$

$$\begin{aligned}
&\leq \left(k_1M_1 + c_1k_1k_2C_1 \max\left\{\|x\|_X^{q^+-1}, \|x\|_X^{q^--1}\right\}\right)\|x\|_X \\
&\quad + c_2k_1k_3C_2 \max\left\{\|x\|_X^{s^+-1}, \|x\|_X^{s^--1}\right\}\|x\|_X \\
&= k_1M_1\|x\|_X + c_1k_1k_2C_1 \max\left\{\|x\|_X^{q^+}, \|x\|_X^{q^-}\right\} \\
&\qquad + c_2k_1k_3C_2 \max\left\{\|x\|_X^{s^+}, \|x\|_X^{s^-}\right\}, \qquad (2.7)
\end{aligned}$$

where $k_1 = \|i^*\|_{Y^*_{q(t)} \to X^*}$, $k_2 = \max\left\{\|i\|_{Y_{q(t)}}^{q^+-1}, \|i\|_{Y_{q(t)}}^{q^--1}\right\}$, $k_3 = \max\left\{\|i\|_{Y_{q(t)}}^{s^+-1}, \|i\|_{Y_{q(t)}}^{s^--1}\right\}$ and $M_1 = \|a\|_{Y^*_{q(t)}}$. Since s^+,$q^+ < p^-$ in particular, if $x \in X$ satisfies (2.3) for some $\lambda \in (0,1]$, we derive from (2.7) the set S. Consequently, for $x \in X$ we obtain

$$\int_0^T \left|x'(t)\right|^{p(t)} dt \leq K. \qquad (2.8)$$

Let $x(t)$ be a T-periodic solution of Eq.(2.5$_\lambda$). Then, integrating Eq.(2.5$_\lambda$) from 0 to T, and noticing that $x'(0) = x'(T) = 0$, we have

$$\int_0^T |g(t, x(t - \tau(t))) - e(t)|\, dt = 0. \qquad (2.9)$$

This implies that there exists $\xi \in [0,T]$ such that

$$g(\xi, x(\xi - \tau(\xi)) - e(\xi) = 0.$$

Thus, taking this together with (2.9), we have

$$|x(\xi - \tau(\xi))| < M.$$

Since $x(t)$ is a T-periodic function, then there is an integer k and a constant $\xi^* \in [0,T]$ such that $\xi - \tau(\xi) = kT + \xi^*$. According to the Young's inequality, we have

$$\begin{aligned}
\|x\|_0 &\leq |x(\xi^*)| + \int_0^T \left|x'(t)\right| dt \leq M + \int_0^T \left|x'(t)\right| dt \\
&\leq \frac{1}{p^-}\int_0^T \left|x'(t)\right|^{p(t)} dt + \frac{p^+-1}{p^-}T + M.
\end{aligned}$$

If we take into account the inequality of (2.8), then we get

$$\|x\|_0 \leq \frac{K}{p^-} + \frac{p^+-1}{p^-}T + M := K_1. \qquad (2.10)$$

By the boundary condition $x(0) = x(T)$, it is easy to see that there exists $t^* \in [0,T]$, such that $x'(t^*) = 0$. Integrating (2.5_λ) from t_i^* to t, and from $(F3)$, Lemma 2 and Young's inequality, we obtain

$$\begin{aligned}
&\left|\int_{t^*}^{t}\left(x'(t)^{p(t)-2}x'(t)\right)' dt\right| = \left|x'(t)^{p(t)-2}x'(t)\right| = \left|x'(t)^{p(t)-1}\right| \\
\leq\ &\left|\int_{t^*}^{t}\left(e(t) - x'(t) - g\left(t, x\left(t-\tau(t)\right)\right)\right) dt\right| \\
\leq\ &\int_{t^*}^{t}\left|x'(t)\right| dt + \int_{t^*}^{t}\left|g\left(t, x\left(t-\tau(t)\right)\right)\right| dt + \int_{t^*}^{t}\left|e(t)\right| dt \\
\leq\ &\frac{1}{p^-}\int_0^T \left|x'(t)\right|^{p(t)} dt + \frac{p^+-1}{p^-}T \\
&+\int_0^T \left|g\left(t, x\left(t-\tau(t)\right)\right) - g(t,x)\right| dt + \int_0^T \left|g(t,x)\right| dt + \int_0^T \left|e(t)\right| dt \\
\leq\ &\frac{1}{p^-}\int_0^T \left|x'(t)\right|^{p(t)} dt + \frac{p^+-1}{p^-}T + L\int_0^T \left|x\left(t-\tau(t)\right) - x(t)\right|^2 dt \\
&+Tg_d + T\left\|e\right\|_0 \\
\leq\ &\frac{1}{p^-}\int_0^T \left|x'(t)\right|^{p(t)} dt + \frac{p^+-1}{p^-}T + \frac{4\alpha^2 L}{p^-}\int_0^T \left|x'(t)\right|^{p(t)} dt + \frac{2\alpha^2 TL\left(p^+-2\right)}{p^-} \\
&+Tg_d + T\left\|e\right\|_0 \\
\leq\ &\frac{\left(1+4\alpha^2 L\right)K}{p^-} + \left(\left\|e\right\|_0 + g_d + \frac{2\alpha^2 L\left(p^+-2\right) + p^+ - 1}{p^-}\right)T := K_2, \qquad (2.11)
\end{aligned}$$

where $g_d = \max\left\{|g(s,x)| : s \in [0,T], x \in [-M,M]\right\}$. So from (2.11), we obtain

$$|x'(t)| \leq \max\left\{K_2^{\frac{1}{p^- -1}}, K_2^{\frac{1}{p^+ -1}}\right\} := K_3.$$

That is,

$$\left\|x'\right\|_0 \leq K_3.$$

Combining this inequality with (2.10), we obtain

$$\|x\|_1 \leq K_1 + K_3 := K_4.$$

Thus, we obtain that $x(t)$ is uniform bounded in C_T^1 for any $\lambda \in [0,1]$.

Let $\Omega = \left\{x \in W_0^{1,p(t)}\left([0,T], \mathbb{R}\right) : x(0) = x(T), x'(0) = x'(T)\right\}$ and let us define the set

$$\Omega_1 = \{x \in \Omega \cap \mathbb{R} : F(x) = 0, F(x) = \int_0^T \left|g\left(t, x\left(t-\tau(t)\right)\right) - e(t)\right| dt\}.$$

We will prove that for any $D \in \Omega_1$, it holds $|D| \leq M$.

For any $D \in \Omega_1$, by $F(D) = 0$, we have

$$\int_0^T |g(t, D) - e(t)| \, dt = 0. \tag{2.12}$$

We can confirm that $|D| \leq M$. Otherwise, by $(F2)$

$$g(t, D) - e(t) > 0 \tag{2.13}$$

or

$$g(t, D) - e(t) < 0 \tag{2.14}$$

which implies that $F(D) < 0$ or $F(D) > 0$. That is a condradiction to (2.12).

Finally we will prove that the condition (3) of Lemma 2 is also satisfied.

For any $x \in \Omega \cap \mathbb{R}$, from (F2) we know that if $|D| > M$, such that $g(t, D) - e(t) > 0$ or $g(t, D) - e(t) < 0$. In the following, we assume (2.13). As for the case of (2.14), the proof is similar, so we omit it. Let us define

$$\Omega_2 = \{x \in \Omega \cap \mathbb{R} : \lambda(D - \zeta) + (1 - \lambda)F(D) = 0, \ \lambda \in [0, 1]\},$$

where $\zeta \in \mathbb{R}$ with $0 < |\zeta| < M$. Our aim is to show that Ω_2 is bounded by M. Argue by contrary; there exists $\zeta \in \Omega_2$ with $|D| > M$. Moreover, we have $\lambda(D - \zeta) + (1 - \lambda)F(D) > 0$, which contradicts the definition of Ω_2. That is for any $D \in \Omega_2$, we always have $|D| \leq M$. From the definition of F, it is easy to see that $F : \mathbb{R} \to \mathbb{R}$ is completely continuous. Let

$$h_\lambda(D) = \lambda(D - \zeta) + (1 - \lambda)F(D),$$

and define $\Omega^* = \{x \in \Omega : \|x\|_1 < K_4 + M\}$. Then clearly $h_\lambda(\partial\Omega^* \cap \mathbb{R}) \neq 0$ for any $\lambda \in [0, 1]$. By virtue of the invariance property of homotopy, we obtain

$$\begin{aligned} \deg(F, \partial\Omega^* \cap \mathbb{R}, 0) &= \deg(h_0, \partial\Omega^* \cap \mathbb{R}, 0) \\ &= \deg(h_1, \partial\Omega^* \cap \mathbb{R}, 0) = \deg([0, T], \partial\Omega^* \cap \mathbb{R}, \zeta) = 1. \end{aligned}$$

Hitherto, we have proved that the conditions (1)-(3) in Lemma 2 are all satisfied. Therefore, by Lemma 2, the Eq. (1.1) admits a solution in Ω^*. This completes the proof of Theorem 3. ■

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