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UNIVALENCE CONDITIONS FOR AN INTEGRAL OPERATOR DEFINED OUTSIDE THE UNIT DISK

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Abstract. In this paper we obtain the univalence conditions for an integral operator defined outside the unit disk.

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1. Introduction

We consider the unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and outside the unit disk $\mathcal{U}^- = \{z \in \mathbb{C} : |z| > 1\}$.

Let $\mathcal{A}$ be the class of functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

normalized by $f(0) = f'(0) - 1 = 0$, which are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$.

We denote Σ_0 the class of functions $g(\xi) = \xi + \frac{b_1}{\xi} + \frac{b_2}{\xi^2} + ..$, which are regular outside the unit disk $\mathcal{U}^-$.

In this paper we use the following lemmas.

Lemma 1. *(Pascu [1]). Let α be a complex number, $Re\ \alpha > 0$ and $f \in \mathcal{A}$. If*

$$\frac{1 - |z|^{2Re\ \alpha}}{Re\ \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1, \quad \forall z \in \mathcal{U}, \tag{1}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} \tag{2}$$

is regular and univalent in $\mathcal{U}$.

Lemma 2. *(Pescar [2]). Let* α *be complex number, Re* $\alpha > 0$*,* c *be a complex number,* $|c| \leq 1$*,* $c \neq -1$ *and* $f \in \mathcal{A}$*,* $f(z) = z + a_2 z^2 + \dots$*.*

If

$$\left| c|z|^{2\alpha} + (1 - |z|^{2\alpha}) \frac{zf''(z)}{\alpha f'(z)} \right| \leq 1, \quad \forall z \in \mathcal{U}, \tag{3}$$

then the function

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha - 1} f'(u) du \right]^{\frac{1}{\alpha}} \tag{4}$$

is regular and univalent in $\mathcal{U}$*.*

2. Main results

Theorem 3. *Let* α *be a complex numbers, Re* $\alpha > 0$ *and* $h \in \Sigma_0$*.*

If

$$\frac{|\xi|^{2Re\,\alpha} - 1}{|\xi|^{2Re\,\alpha}} \left| \left(2 + \xi \frac{h''(\xi)}{h'(\xi)} - 2\xi \frac{h'(\xi)}{h(\xi)} \right) \frac{1}{Re\,\alpha} \right| \leq 1, \tag{1}$$

for all $\xi \in \mathcal{U}^-$*, then the function*

$$H_\alpha(\xi) = \left[\alpha \int_0^{\frac{1}{\xi}} u^{\alpha - 3} \frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})} du \right]^{-\frac{1}{\alpha}} \tag{2}$$

is regular and univalent in $\mathcal{U}^-$*.*

Proof. Since $h \in \Sigma_0$, it results that $h(\frac{1}{z})$ is regular in $\mathcal{U} - \{0\}$ and

$$f_(z) = \frac{1}{h(\frac{1}{z})} \tag{3}$$

in $\mathcal{U}$.

The function h has a simple pole at $\xi = \infty$ and hence it results that the function f is regular in $z = 0$, where it has a simple zero, such that the function f has the form

$$f_(z) = \frac{1}{h(\frac{1}{z})} = z + \cdots \tag{4}$$

From (4) we obtain

$$\frac{zf''(z)}{f'(z)} = -\left(2 + \frac{1}{z}\frac{h''(\frac{1}{z})}{h'(\frac{1}{z})} - \frac{2}{z}\frac{h'(\frac{1}{z})}{h(\frac{1}{z})}\right). \tag{5}$$

Then

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| = \frac{|\xi|^{2Re\,\alpha}-1}{|\xi|^{2Re\,\alpha}}\left|\left(2+\xi\frac{h''(\xi)}{h'(\xi)} - 2\xi\frac{h'(\xi)}{h(\xi)}\right)\frac{1}{Re\,\alpha}\right|, \tag{6}$$

where $\xi = \frac{1}{z}$.

From (6) and (1) we obtain

$$\frac{1-|z|^{2Re\,\alpha}}{Re\,\alpha}\left|\frac{zf''(z)}{f'(z)}\right| \le 1, \tag{7}$$

for all $z \in \mathcal{U}$.

From Lemma 1 it results that the function F_α defined by (2) is regular and univalent in $\mathcal{U}$, hence

$$F_\alpha(z) = \left[\alpha\int_0^z u^{\alpha-1}f'(u)du\right]^{\frac{1}{\alpha}} = z + a_2z^2 + \cdots \tag{8}$$

Replacing in (4) $z = \frac{1}{\xi}$ we obtain

$$F_\alpha(\frac{1}{\xi}) = \left[\alpha\int_0^{\frac{1}{\xi}} u^{\alpha-1}\frac{1}{u^2}\frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})}du\right]^{\frac{1}{\alpha}}, \tag{9}$$

which is regular and univalent in $\mathcal{U}^-$. The function $F_\alpha(z)$ is regular, univalent in $\mathcal{U}$ and $F_\alpha(0) = 0$. Hence it results that $F_\alpha(z) \neq 0$ for all $z \in \mathcal{U} - \{0\}$, and the function

$$H_\alpha(\xi) = \frac{1}{F_\alpha(\frac{1}{\xi})} = \left[\alpha\int_0^{\frac{1}{\xi}} u^{\alpha-3}\frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})}du\right]^{-\frac{1}{\alpha}} \tag{10}$$

is regular and univalent in $\mathcal{U}^-$.

Corollary 4. *Let be the function $h \in \Sigma_0$.*

If

$$(|\xi|^2 - 1)\left|\left[2+\xi\frac{h''(\xi)}{h'(\xi)} - 2\xi\frac{h'(\xi)}{h(\xi)}\right]\right| \le 1, \tag{11}$$

for all $\xi \in \mathcal{U}^-$, then the function h is regular and univalent in $\mathcal{U}^-$.

Proof. Substituing $\alpha = 1$ in the relation (9) we obtain

$$F_1\left(\frac{1}{\xi}\right) = \int_0^{\frac{1}{\xi}} \frac{1}{u^2}\frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})}du = \left.\frac{1}{h(\frac{1}{u})}\right|_0^{\frac{1}{\xi}} = \frac{1}{h(\xi)}. \tag{12}$$

If $h \in \Sigma_0$, then $\frac{1}{h(\infty)} = 0$, such that the function
$H_1(\xi) = h(\xi)$, where $H_1(\xi) = \frac{1}{F_1(\frac{1}{\xi})}$.

From Theorem 3 we obtain that the function h is univalent in $\mathcal{U}^-$.

Theorem 5. *Let α be a complex number, Re $\alpha > 0$, c be a complex number, $|c| \leq 1$, $c \neq -1$ and $h \in \Sigma_0$.*

If

$$\frac{1}{||\xi|^{2\alpha}|}\left|\left[c + \left(|\xi|^{2\alpha} - 1\right)\left(2 + \xi\frac{h''(\xi)}{h'(\xi)} - 2\xi\frac{h'(\xi)}{h(\xi)}\right)\frac{1}{\alpha}\right]\right| \leq 1, \tag{13}$$

for all $\xi \in \mathcal{U}^-$, then the function H_α defined by

$$H_\alpha(\xi) = \left[\alpha\int_0^{\frac{1}{\xi}} u^{\alpha-3}\frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})}du\right]^{\frac{1}{\alpha}} \tag{14}$$

is regular and univalent in $\mathcal{U}^-$.

Proof. Because $h \in \Sigma_0$ we have $h(\frac{1}{z})$ is regular in $\mathcal{U} - \{0\}$ and

$$f(z) = \frac{1}{h(\frac{1}{z})}$$

in $\mathcal{U}$.

The function h has a simple pole at $\xi = \infty$ and hence we obtain that the function f is regular in $z = 0$, where it has a simple zero, such that the function f has the form

$$f_(z) = \frac{1}{h(\frac{1}{z})} = z + \cdots \tag{15}$$

From (4) we have

$$\frac{zf''(z)}{\alpha f'(z)} = \left[-2 - \frac{1}{z}\frac{h''(\frac{1}{z})}{h'(\frac{1}{z})} + \frac{2}{z}\frac{h'(\frac{1}{z})}{h(\frac{1}{z})}\right]\frac{1}{\alpha}. \tag{16}$$

Then

$$\left| c|z|^{2\alpha} + (1-|z|^{2\alpha})\frac{zf''(z)}{\alpha f'(z)} \right| =$$

$$= \frac{1}{||\xi|^{2\alpha}|} \left| \left\{ c + \left(|\xi|^{2\alpha} - 1\right) \frac{1}{\alpha} \left[2 + \xi\frac{h''(\xi)}{h'(\xi)} - 2\xi\frac{h'(\xi)}{h(\xi)} \right] \right\} \right| \leq$$

$$\leq \frac{1}{||\xi|^{2\alpha}|} \cdot \left||\xi|^{2\alpha}\right| = 1. \tag{17}$$

for all $z \in \mathcal{U}$. From Lemma 2 it follows that the function $F_\alpha(z)$ defined by (4) is regular and univalent in $\mathcal{U}$.

$$F_\alpha(z) = \left[\alpha \int_0^z u^{\alpha-1} f'(u) du \right]^{\frac{1}{\alpha}} = z + a_2 z^2 + \cdots$$

Replacing in (2) $z = \frac{1}{\xi}$ and $f(z) = \frac{1}{h(\frac{1}{z})}$, we have

$$F_\alpha(\frac{1}{\xi}) = \left[\alpha \int_0^{\frac{1}{\xi}} u^{\alpha-1} \frac{1}{u^2} \frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})} du \right]^{\frac{1}{\alpha}}, \tag{18}$$

which is regular and univalent in $\mathcal{U}^-$.

The function $F_{\alpha(z)}$ is regular, univalent in $\mathcal{U}$ and $F_\alpha(0) = 0$. Hence, it results that $F_\alpha(z) \neq 0$ for all $z \in \mathcal{U} - \{0\}$ and the function

$$H_\alpha(\xi) = \frac{1}{F_\alpha(\frac{1}{\xi})} = \left[\alpha \int_0^{\frac{1}{\xi}} u^{\alpha-3} \frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})} du \right]^{-\frac{1}{\alpha}} \tag{19}$$

is regular and univalent in $\mathcal{U}^-$.

Corollary 6. *Let c be a complex number, $|c| \leq 1$, $c \neq -1$ and the function $h \in \Sigma_0$. If*

$$\left| c + \left(|\xi|^2 - 1\right) \left[2 + \xi\frac{h''(\xi)}{h'(\xi)} - 2\xi\frac{h'(\xi)}{h(\xi)} \right] \right| \leq |\xi|^2, \tag{20}$$

for all $\xi \in \mathcal{U}^-$, then the function h is univalent in $\mathcal{U}^-$.

Proof. Substituing $\alpha = 1$ in the relation (19) we obtain

$$F_1\left(\frac{1}{\xi}\right) = \int_0^{\frac{1}{\xi}} \frac{1}{u^2} \frac{h'(\frac{1}{u})}{h^2(\frac{1}{u})} du = \frac{1}{h(\frac{1}{u})}\Bigg|_0^{\frac{1}{\xi}} = \frac{1}{h(\xi)}. \tag{21}$$

(if $h \in \Sigma_0$, then $\frac{1}{h(\infty)} = 0$), such that the function
$H_1(\xi) = h(\xi)$, where $H_1(\xi) = \frac{1}{F_1(\frac{1}{\xi})}$.

From Theorem 5 we obtain that the function h is univalent in $\mathcal{U}^-$.

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