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SOME FAMILIES OF UNIFORMLY STARLIKE AND CONVEX FUNCTIONS WITH NEGATIVE COEFFICIENTS

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Abstract. We introduce a subclass of uniformly starlike and convex functions with negative coefficients defined by Sălăgean operator. In this paper, we obtain coefficient estimates, distortion theorems, closure theorems and radii of close-to-convexity, starlikeness, and convexity for functions belonging to this class. Several results for the modified Hadamard products of functions belonging to this class are obtained. Finally, distortion theorems for fractional calculus functions are also considered.

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1. Introduction

Let A_j denote the class of the functions of the from

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \qquad (j \in \mathbb{N} = \{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic in the open unit disc $U=\{z : |z| < 1\}$. We note that $A_1 = A$. For a function $f(z) \in A_j$,let

$$\begin{aligned} D^0 f(z) &= f(z), \\ D^1 f(z) &= Df(z) = zf'(z), \\ D^n f(z) &= D\left(D^{n-1} f(z)\right) \end{aligned}$$

$$= z + \sum_{k=j+1}^{\infty} k^n a_k z^k \ , \ n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}. \tag{1.2}$$

The differential operator D^n was introduced by Sălăgean [9]. With the help of the differential operator D^n, for $0 \leq \alpha < 1$, $0 \leq \lambda \leq 1$, $\beta \geq 0$, $n \in \mathbb{N}_0$ and $m \in \mathbb{N}$, let $S_j(n, m, \lambda, \alpha, \beta)$ denote the subclass of A_j consisting of functions $f(z)$ of the form (1.1) and satisfying condition

$$Re\left\{\frac{(1-\lambda)\, z\,(D^n f(z))' + \lambda z\,(D^{n+m} f(z))'}{(1-\lambda)\, D^n f(z) + \lambda D^{n+m} f(z)} - \alpha\right\} >$$

$$\beta\left|\frac{(1-\lambda)\, z\ (D^n f\ (z))' + \lambda z\,(D^{n+m} f(z))'}{(1-\lambda)\, D^n f(z) + \lambda D^{n+m} f(z)} - 1\right|, z \in \mathbb{U}. \tag{1.3}$$

The operator D^{n+m} was studied by Sekine [11], Hossen et al. [6] and Aouf and Sălăgean [4] .We denote by T_j the subclass of A_j consisting of the functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k\, z^k \qquad (a_k \geq 0, k \geq j+1; j \in \mathbb{N}). \tag{1.4}$$

Further, we define the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$ by

$$\mathbb{Q}_j(m, n, \lambda, \alpha, \beta) = S_j(m, n, \lambda, \alpha, \beta) \cap T_j.$$

We note that, specializing the parameters $\alpha, \beta, \lambda, n$ and m, we obtain the following subclasses studied by various authors:

(i) $\mathbb{Q}_1(1, n, \lambda, \alpha, \beta) = TS_\lambda(n, \alpha, \beta)$ (see Aouf et al. [2]);
(ii) $\mathbb{Q}_1(m, n, 0, \alpha, \beta) = T(n, \alpha, \beta)$ (see Aouf [1]);
(iii) $\mathbb{Q}_j(m, n, \lambda, \alpha, 0) = T_j(m, n, \lambda, \alpha)$ (see Aouf et al. [3]);
(iv) $\mathbb{Q}_j(1, n, \lambda, \alpha, 0) = P(j, n, \lambda, \alpha)$ (see Aouf and Srivastava [5]);
(v) $\mathbb{Q}_j(1, n, \lambda, \beta, k) = U_j(n, \lambda, \alpha, \beta)$ (see Shanmugam et al. [12]);
(vi) $\mathbb{Q}_j(m, 0, 0, \alpha, \beta) = TS(m, \alpha, \beta)$ (see Rosy and Murugusudaramoorthy [8]);

(vii) $\mathbb{Q}_j(1, 0, 0, \alpha, 0) = C_\alpha(j)$ (see Srivastava et al. [15]);
$(viii)$ $\mathbb{Q}_1(0, 0, 0, \alpha, 0) = T^*(\alpha)$ and $\mathbb{Q}_j(0, 1, 0, \alpha, 0) = C(\alpha)$ (see Silverman [13]).

2. Coefficient estimates

Theorem 1. *Let the function $f(z)$ be defined by (1.4). Then $f(z) \in \mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$ if and only if*

$$\sum_{k=j+1}^{\infty} k^n\,[k(1+\beta) - (\alpha+\beta)]\,[1 + (k^m - 1)\lambda]\, a_k \leq 1 - \alpha. \tag{2.1}$$

Proof. Assume that (2.1) holds.Then we must show that

$$\beta\left|\frac{(1-\lambda)\,z\ \left(D^{n}f\ \ (z)\right)'+\lambda z\left(D^{n+m}f\,(z)\right)'}{(1-\lambda)\,D^{n}f\,(z)+\lambda D^{n+m}f\,(z)}-1\right|-$$

$$Re\left\{\frac{(1-\lambda)\,z\ \left(D^{n}f\ \ (z)\right)'+\lambda z\left(D^{n+m}f\,(z)\right)'}{(1-\lambda)\,D^{n}f\,(z)+\lambda D^{n+m}f\,(z)}-\alpha\right\}\leq 1-\alpha.$$

We have

$$\beta\left|\frac{(1-\lambda)\,z\left(D^{n}f\,(z)\right)'+\lambda z\left(D^{n+m}f\,(z)\right)'}{(1-\lambda)\,D^{n}f\,(z)+\lambda D^{n+m}f\,(z)}-1\right|-$$

$$Re\left\{\frac{(1-\lambda)\,z\left(D^{n}f\ \ (z)\right)'+\lambda z\left(D^{n+m}f\,(z)\right)'}{(1-\lambda)\,D^{n}f\,(z)+\lambda D^{n+m}f\,(z)}-\alpha\right\}$$

$$\leq\frac{(1+\beta)\sum\limits_{k=j+1}^{\infty}k^{n}\,(k-1)\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}$$

$$\leq\frac{(1+\beta)\sum\limits_{k=j+1}^{\infty}k^{n}\,(k-1)\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}}{1-\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}}\leq 1-\alpha.$$

Hence, $f\,(z)\in\mathbb{Q}_{j}\,(m,n,\lambda,\alpha,\beta)\,.$

Conversely, let $f\,(z)\in\mathbb{Q}_{j}\,(m,n,\lambda,\alpha,\beta)$. Then we have

$$\frac{1-\sum\limits_{k=j+1}^{\infty}k^{n+1}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}-\alpha\geq$$

$$\beta\left|\frac{\sum\limits_{k=j+1}^{\infty}k^{n}\,(k-1)\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}k^{n}\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}z^{k-1}}\right|.$$

Letting $z\to 1^{-}$ along the real axis, we obtain the desired inequality

$$\sum_{k=j+1}^{\infty}k^{n}\left[k\,(1+\beta)-(\alpha+\beta)\right]\left[1+\left(k^{m}-1\right)\lambda\right]a_{k}\leq 1-\alpha. \tag{2.3}$$

This completes the proof of Theorem 1. ∎

Corollary 1. Let the function $f(z)$ be define by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$, Then

$$a_k \leq \frac{1-\alpha}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]} \quad (k \geq j+1). \tag{2.4}$$

The result is sharp for the function

$$f(z) = z - \frac{1-\alpha}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}z^k \quad (k \geq j+1). \tag{2.5}$$

3. Distortion theorems

Theorem 2. *Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then for $|z| = r < 1$, we have*

$$\left|D^i f(z)\right| \geq r - \frac{1-\alpha}{(j+1)^{n-i}\left[j(1+\beta)+(1-\alpha)\right]\left[1+\left[(1+j)^m-1\right]\lambda\right]}r^{j+1}, \tag{3.1}$$

$$\left|D^i f(z)\right| \leq r + \frac{1-\alpha}{(j+1)^{n-i}\left[j(1+\beta)+(1-\alpha)\right]\left[1+((1+j)^m-1)\lambda\right]}r^{j+1}, \tag{3.2}$$

for $z \in U$ and $0 \leq i \leq n$. The equalities in (3.1) and (3.2) are attained for the function $f(z)$ given by

$$f(z) = z - \frac{1-\alpha}{(j+1)^n\left[j(1+\beta)+(1-\alpha)\right]\left[1+\left[(1+j)^m-1\right]\lambda\right]}z^{j+1} \quad (z = \pm r). \tag{3.3}$$

Proof. Note that $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ if and only if $D^i f(z) \in \mathbb{Q}_j(m,n-i,\lambda,\alpha,\beta)$ and that

$$D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k. \tag{3.4}$$

By Theorem 1, we have

$$(j+1)^{n-i}\left[j(1+\beta)+(1-\alpha)\right]\left[1+((j+1)^m-1)\lambda\right]\sum_{k=j+1}^{\infty} k^i a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right] a_k \leq 1-\alpha,$$

that is, that

$$\sum_{k=j+1}^{\infty} k^i a_k \ \leq \frac{1-\alpha}{\left(j+1\right)^{n-i}\left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]}. \tag{3.5}$$

The assertions (3.1) and (3.2) of Theorem 2 would now follow readily form (3.4) and (3.5). Finally, we note that equalities in (3.1) and (3.2) are attained for the function $f(z)$ defined by

$$D^i f\left(z\right) = z - \frac{1-\alpha}{\left(j+1\right)^{n-i}\left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]} z^{j+1}. \tag{3.6}$$

This completes the proof of Theorem 2. ∎

Corollary 2. Let the function $f(z)$ defined by (1.4)be in the class $\mathbb{Q}_j(m, n,\lambda, \alpha, \beta$).Then for $|z| = r < 1$

$$\left|f\left(z\right)\right| \geq r - \frac{1-\alpha}{\left(j+1\right)^{n}\ \left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]} r^{j+1} \tag{3.7}$$

and.

$$\left|f\left(z\right)\right| \leq r + \frac{1-\alpha}{\left(j+1\right)^{n}\ \left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]} r^{j+1}. \tag{3.8}$$

The equalities in (3.7) and(3.8) are attained for the function $f(z)$ given by (3.3).

Proof. Taking $i = 0$ in Theorem 2, we immediately obtain (3.7) and (3.8). ∎

Corollary 3. Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m, n,\lambda, \alpha, \beta$). Then, for $|z| = r < 1$

$$\left|f'\left(z\right)\right| \geq 1 - \frac{1-\alpha}{\left(j+1\right)^{n-1}\ \left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]} r^{j} \tag{3.9}$$

and

$$\left|f'\left(z\right)\right| \leq 1 + \frac{1-\alpha}{\left(j+1\right)^{n-1}\ \left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]\left[1+\left[\left(1+j\right)^m-1\right]\lambda\right]} r^{j}. \tag{3.10}$$

The equalities in (3.9) and (3.10)are attained for the function $f(z)$ given by (3.3).

Proof. Setting $i = 1$ in Theorem 2, and making use of (1.3), we arrive at Corollary 3. ∎

4. Convex linear combination

In this section, we shall prove that the class $\mathbb{Q}_j(m;n,\lambda,\alpha,\beta)$ is closed under convex linear combination.

Theorem 3. . *$\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ is a convex set.*

Proof. Let the functions

$$f_\nu(z) = z - \sum_{k=j+1}^{\infty} a_{\nu,k}\, z^k \qquad (a_{\nu,k} \geq 0; \nu = 1,2) \tag{4.1}$$

be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. It is sufficient to show that the function $h(z)$ defined by

$$h(z) = \mu f_1(z) + (1-\mu) f_2(z) \qquad (0 \leq \mu \leq 1) \tag{4.2}$$

is also in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Since, for $0 \leq \mu \leq 1$,

$$h(z) = z - \sum_{k=j+1}^{\infty} \{\mu a_{k,1} + (1-\mu) a_{k,2}\} z^k, \tag{4.3}$$

with the aid of Theorem 1, we have

$$\sum_{k=j+1}^{\infty} k^n [k(1+\beta) - (\alpha+\beta)] [1 + (k^m - 1)\lambda] \{\mu a_{k,1} + (1-\mu) a_{k,2}\} \leq 1 - \alpha, \tag{4.4}$$

which implies that $h(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Hence $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$ is a convex set. ∎

Theorem 4. *Let $f_j(z) = z$ and*

$$f_k(z) = z - \frac{1-\alpha}{k^n [k(1+\beta) - (\alpha+\beta)] [1 + (k^m - 1)\lambda]} z^k \quad (k \geq j+1; n \in \mathbb{N}_0; m \in \mathbb{N}) \tag{4.5}$$

for $0 \leq \alpha < 1$, $\beta \geq 0$ and $0 \leq \lambda \leq 1$, then $f(z)$ is in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$, if and only if it can be expressed in the form

$$f(z) = \sum_{k=j}^{\infty} \upsilon_k\, f_k(z), \tag{4.6}$$

where $\upsilon_k \geq 0\, (k \geq j)$ *and* $\sum_{k=j}^{\infty} \upsilon_k = 1$.

Proof. Assume that

$$f(z) = \sum_{k=j}^{\infty} \upsilon_k \; f_k(z)$$

$$= z - \sum_{k=j+1}^{\infty} \frac{1-\alpha}{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]} \upsilon_k \; z^k \,. \tag{4.7}$$

Then it follows that

$$\sum_{k=j+1}^{\infty} \left\{ \frac{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha} \cdot \right.$$

$$\left. \cdot \frac{1-\alpha}{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]} \upsilon_k \right\}$$

$$= \sum_{k=j+1}^{\infty} \upsilon_k = 1 - \upsilon_j \leq 1. \tag{4.8}$$

So, by Theorem 1, $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta\ )$.

Conversely, assume that the function $f(z)$ defined by (1.4) belongs to the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then

$$a_k \leq \frac{1-\alpha}{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]} \qquad \left(k \geq j+1;\; n \in \mathbb{N}_0; m \in \mathbb{N}\right).$$

Setting

$$\upsilon_k = \frac{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha} a_k \quad \left(k \geq j+1;\; n \in \mathbb{N}_0; m \in \mathbb{N}\right) \tag{4.9}$$

and

$$\upsilon_j = 1 - \sum_{k=j+1}^{\infty} \upsilon_k$$

we can see that $f(z)$ can be expressed in the form (4.4) .This completes the proof of Theorem 4. ∎

5. Radii of close-to-convexity, starlikeness and convexity

Theorem 5. *Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then $f(z)$ is close-to-convex of order ρ $(0 \leq \rho < 1)$ in $|z| < r_1$, where*

$$r_1 = r_1(m,n,\lambda,\alpha,\beta,\rho)$$

$$\inf_k \left\{ \frac{(1-\rho)\, k^{n-1} \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha} \right\}^{\frac{1}{k-1}} (k \geq j+1). \quad (5.1)$$

The result is sharp, the extremal function $f(z)$ being given by (2.5).

Proof. We must show that

$$\left|f'(z)-1\right| \leq 1-\rho \qquad \text{for } |z| < r_1(m,n,\lambda,\alpha,\beta,\rho),$$

where $r_1(m,n,\lambda,\alpha,\beta,p)$ is given by (5.1). Indeed we find from (1.4), that

$$\left|f'(z)-1\right| \leq \sum_{k=j+1}^{\infty} k a_k \, |z|^{k-1}.$$

Thus

$$\left|f'(z)-1\right| \leq 1-\rho$$

if

$$\sum_{k=j+1}^{\infty} \left(\frac{k}{1-\rho}\right) a_k \, |z|^{k-1} \leq 1. \quad (5.2)$$

But, by Theorem 1, (5.2) will be true if

$$\left(\frac{k}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha},$$

that is, if

$$|z| \leq \left[\frac{(1-\rho)\, k^{n-1} \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha}\right]^{\frac{1}{k-1}}. \quad (5.3)$$

Theorem 5 follows easily from (5.3). ∎

Theorem 6. *Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then $f(z)$ is starlike of order ρ $(0 \leq \rho < 1)$ in $|z| < r_2$, where*

$$r_2 = r_2(m,n,\lambda,\alpha,\beta,\rho) =$$

$$\inf_k \left\{ \frac{(1-\rho)\, k^{n} \left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{(k-\rho)(1-\alpha)} \right\}^{\frac{1}{(k-1)}} (k \geq j+1). \quad (5.4)$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

Proof. It is sufficient to show that

$$\left|\frac{z\ f'(z)}{f(z)}-1\right| \leq\ 1-\rho \quad \text{for } |z|\ <\ r_2(m,n,\lambda,\alpha,\beta,\rho),$$

where $r_2(m,n,\lambda,\alpha,\beta,\rho)$ is given by (5.4) . Indeed we find, again from (1.4) , that

$$\left|\frac{z\ f'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{k=j+1}^{\infty}(k-1)\,a_k\ |z|^{k-1}}{1-\sum\limits_{k=j+1}^{\infty}a_k\ |z|^{k-1}}.$$

Thus

$$\left|\frac{z\ f'(z)}{f(z)}-1\right| \leq 1-\rho$$

if

$$\sum_{k=j+1}^{\infty}\left(\frac{k-\rho}{1-\rho}\right)a_k\ |z|^{k-1} \leq 1. \tag{5.5}$$

But, by Theorem 1, (5.5) will be true if

$$\left(\frac{k-\rho}{1-\rho}\right)|z|^{k-1} \leq \frac{k^n\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha},$$

that is ,if

$$|z| \leq \left[\frac{(1-\rho)\,k^n\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{(k-\rho)(1-\alpha)}\right]^{\frac{1}{(k-1)}} \quad (k \geq j+1)\,. \tag{5.6}$$

Theorem 6 follows easily from (5.6). ∎

Corollary 4. Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then $f(z)$ is convex of order ρ $(0\ \leq \rho\ < 1)$ in $|z| < r_3,$ where

$$r_3 = r_3(m,n,\lambda,\alpha,\beta,\rho) = \tag{5.7}$$

$$\inf_k\left\{\frac{(1-\rho)\,k^{n-1}\left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{(k-\rho)(1-\alpha)}\right\}^{\frac{1}{k-1}} (k \geq j+1)\,.$$

The result is sharp, with the extremal function $f(z)$ given by (2.5).

6. A family of integral operators

Theorem 7. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$*, let* c *be a real number such that* $c > -1$*. Then the function* $F(z)$ *defined by*

$$F(z) = \frac{c+1}{z^c}\int_0^z t^{c-1} f(t)\, dt \qquad (c > -1) \tag{6.1}$$

also belongs to the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$.

Proof. From the representation (6.1) of $F(z)$, it follows that

$$F(z) = z - \sum_{k=j+1}^{\infty} b_k\, z^k,$$

where

$$b_k = \left(\frac{c+1}{c+k}\right) a_k.$$

Therefore,we have

$$\sum_{k=j+1}^{\infty} k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right] b_k$$

$$= \sum_{k=j+1}^{\infty} k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]\left(\frac{c+1}{c+k}\right) a_k$$

$$\leq \sum_{k=j+1}^{\infty} k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right] a_k \leq 1-\alpha,$$

since $f(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Hence, by Theorem 1, $F(z) \in \mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. ∎

Theorem 8. *Let the function*

$$F(z) = z - \sum_{k=j+1}^{\infty} a_k\, z^k \qquad (a_k \geq 0)$$

be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$*. And let* c *be a real number such that* $c > -1$*.Then the function* $f(z)$ *given by* (6.1)*, is univalent in* $|z| < R^*$*, where*

$$R^* = \inf_k \left[\frac{k^{n-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right](c+1)}{(c+k)(1-\alpha)}\right]^{\frac{1}{(k-1)}} (k \geq j+1). \tag{6.2}$$

The result is sharp.

Proof. From (6.1), we have

$$f(z) = \frac{z^{1-c}\left\{z^{c}F(z)\right\}'}{c+1} = z - \sum_{k=j+1}^{\infty}\left(\frac{c+k}{c+1}\right)a_k\ z^k.$$

In order to obtain the required result, it suffices to show that

$$\left|f'(z)-1\right| \leq 1 \qquad \textit{whenever} \quad |z| < R^*,$$

where $R^*$is given by (6.2). Now

$$\left|f'(z)-1\right| \leq \sum_{k=j+1}^{\infty}\frac{k(c+k)}{c+1}a_k\ |z|^{k-1}.$$

Thus $|f'(z)-1| \leq 1$ if

$$\sum_{k=j+1}^{\infty}\frac{k(c+k)}{c+1}a_k\ |z|^{k-1} \leq 1. \tag{6.3}$$

But Theorem 1 confirms that

$$\frac{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha}a_k \leq 1. \tag{6.4}$$

Hence (6.3) will be satisfied if

$$\frac{k(c+k)}{c+1}\ |z|^{k-1} < \frac{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha},$$

that is, if

$$|z| < \left[\frac{k^{n-1}\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]\ (c+1)}{(c+k)(1-\alpha)}\right]^{\frac{1}{(k-1)}} (k \geq j+1). \tag{6.5}$$

Therefore, the function $f(z)$ given by (6.1) is univalent in $|z| < R^*$. Sharpness of the result follows if we take

$$f(z) = z - \frac{(c+k)(1-\alpha)}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]\ (c+1)}z^k\,(k \geq j+1).$$

∎

7. Modified Hadamard products

Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by (4.1). The modified Hadamard product (or convolution) of the functions $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z - \sum_{k=j+1}^{\infty} a_{k,1} a_{k,2} z^k. \tag{7.1}$$

Theorem 9. *Let the functions $f_\nu(z)(\nu = 1, 2)$ defined by (4.1) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$. Then $(f_1 * f_2)(z) \in \mathbb{Q}_j(m, n, \lambda, \delta(m, n, \lambda, \alpha, \beta), \beta)$, where*

$$\delta(m, n, \lambda, \alpha, \beta) =$$

$$1 - \frac{j(1+\beta)(1-\alpha)^2}{(j+1)^n [(j+1)(1+\beta) - (\alpha+\beta)]^2 [1 + ((j+1)^m - 1)\lambda] - (1-\alpha)^2}. \tag{7.2}$$

The result is sharp.

Proof. Employing the technique used earlier by Schild and Silverman [10], we need to find the largest $\delta = \delta(m, n, \lambda, \alpha, \beta)$ such that

$$\sum_{k=j+1}^{\infty} \frac{k^n [k(1+\beta) - (\delta+\beta)] [1 + (k^m - 1)\lambda]}{1-\delta} a_{k,1} a_{k,2} \leq 1. \tag{7.3}$$

Since

$$\sum_{k=j+1}^{\infty} \frac{k^n [k(1+\beta) - (\alpha+\beta)] [1 + (k^m - 1)\lambda]}{1-\alpha} a_{k,1} \leq 1 \tag{7.4}$$

and

$$\sum_{k=j+1}^{\infty} \frac{k^n [k(1+\beta) - (\alpha+\beta)] [1 + (k^m - 1)\lambda]}{1-\alpha} a_{k,2} \leq 1, \tag{7.5}$$

then by the Cauchy-Schwarz inequality, we have

$$\sum_{k=j+1}^{\infty} \frac{k^n [k(1+\beta) - (\alpha+\beta)] [1 + (k^m - 1)\lambda]}{1-\alpha} \sqrt{a_{k,1} a_{k,2}} \leq 1. \tag{7.6}$$

Thus it is sufficient to show that

$$\frac{k^n [k(1+\beta) - (\delta+\beta)] [1 + (k^m - 1)\lambda]}{1-\delta} a_{k,1} a_{k,2}$$

$$\leq \frac{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}{1-\alpha}\sqrt{a_{k,1}a_{k,2}}$$
$$(k \geq j+1), \tag{7.7}$$

that is, that

$$\frac{k^n\left[k(1+\beta)-(\delta+\beta)\right]}{1-\delta}a_{k,1}a_{k,2} \leq \frac{k^n\left[k(1+\beta)-(\alpha+\beta)\right]}{1-\alpha}\sqrt{a_{k,1}a_{k,2}}$$

or

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{\left[k(1+\beta)-(\alpha+\beta)\right](1-\delta)}{\left[k(1+\beta)-(\delta+\beta)\right](1-\alpha)}\,(k \geq j+1). \tag{7.8}$$

Note that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{1-\alpha}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]}\ \ (k \geq j+1). \tag{7.9}$$

Consequently, we need only to prove that

$$\frac{1-\alpha}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]\left[1+(k^m-1)\lambda\right]} \tag{7.10}$$
$$\leq \frac{\left[k(1+\beta)-(\alpha+\beta)\right](1-\delta)}{\left[k(1+\beta)-(\delta+\beta)\right](1-\alpha)}\ (\ k \geq j+1)$$

or, equivalently, that

$$\delta \leq 1-\frac{(k-1)(1+\beta)(1-\alpha)^2}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]^2\left[1+(k^m-1)\lambda\right]-(1-\alpha)^2}\quad (k \geq j+1). \tag{7.11}$$

Since

$$\phi(k) = 1-\frac{(k-1)(1+\beta)(1-\alpha)^2}{k^n\left[k(1+\beta)-(\alpha+\beta)\right]^2\left[1+(k^m-1)\lambda\right]-(1-\alpha)^2}\quad (k \geq j+1) \tag{7.12}$$

is an increasing function of k $(k \geq j+1)$ then letting $k = j+1$ we obtain

$$\delta \leq \phi(j+1) = 1-\frac{j(1+\beta)(1-\alpha)^2}{(j+1)^n\left[j(1+\beta)+(1-\alpha)\right]^2\left[1+((j+1)^m-1)\lambda\right]-(1-\alpha)^2} \tag{7.13}$$

which proves the main assertion of Theorem 9. Finally, by taking the functions

$$f_\nu(z) = z-\frac{1-\alpha}{(j+1)^n\left[j(1+\beta)+(1-\alpha)\right]\left[1+((j+1)^m-1)\lambda\right]}z^{j+1}\ \ (\nu = 1,2), \tag{7.14}$$

we can see that the result is sharp. ∎

Remark 1. Putting $m = 1$ in Theorem 9, we obtain the following corollary which corrects the result obtained by Shanmugam et al.[12, Theorem 5.1].

Corollary 5. Let the functions $f_\nu(z)\,(\nu = 1, 2)$ defined by (4.1) be in the class $U_j(n, \lambda, \alpha, \beta)$. Then $(f_1 * f_2)(z) \in U_j(n, \lambda, \gamma, \beta)$, where

$$\gamma = 1 - \frac{j(1+\beta)(1-\alpha)^2}{(j+1)^n[(j+1)(1+\beta) - (\alpha+\beta)]^2\,(1+j\lambda) - (1-\alpha)^2}. \tag{7.15}$$

The result is sharp.

Theorem 10. *Let* $f_1(z) \in \mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$ *and* $f_2(z) \in \mathbb{Q}_j(m, n, \lambda, \gamma, \beta)$. *Then* $(f_1 * f_2)(z) \in \mathbb{Q}_j(m, n, \lambda, \xi(j, m, n, \lambda, \gamma, \beta), \beta)$, *where*

$$\begin{aligned} &\xi(j, m, n, \lambda, \gamma, \beta) = 1 - \\ &[j(1+\beta)(1-\alpha)(1-\gamma)]\,. \\ &.\{(j+1)^n[j(1+\beta)+(1-\alpha)][j(1+\beta)+(1-\gamma)]\,. \\ &\cdot[1+((j+1)^m-1)\lambda] - (1-\alpha)(1-\gamma)\}^{-1} \end{aligned} \tag{7.16}$$

The result is the best possible for the functions

$$f_1(z) = z - \frac{1-\alpha}{(j+1)^n[j(1+\beta)+(1-\alpha)][1+((j+1)^m-1)\lambda]} z^{j+1} \tag{7.17}$$

and

$$f_2(z) = z - \frac{1-\gamma}{(j+1)^n[j(1+\beta)+(1-\gamma)][1+((j+1)^m-1)\lambda]} z^{j+1}\ . \tag{7.18}$$

Proof. Proceeding as in the proof of Theorem 9, we get

$$\begin{aligned} &\xi \le 1 - [j(1+\beta)(1-\alpha)(1-\gamma)]\,. \\ &.\{(j+1)^n[j(1+\beta)+(1-\alpha)][j(1+\beta)+(1-\gamma)] \\ &[1+((j+1)^m-1)\lambda] - (1-\alpha)(1-\gamma)\}^{-1} \\ &(k \ge j+1)\,. \end{aligned} \tag{7.19}$$

Since the right-hand side of (7.19) is an increasing function of k, setting $k = j+1$ in (7.19), we obtain (7.16). This completes the proof of Theorem 10. ∎

Corollary 6. Let the functions $f_\nu(z)$ $(\nu = 1, 2, 3)$ defined by (4.1), be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$. Then $(f_1 * f_2 * f_3)(z) \in Q_j(m, n, \lambda, \zeta(m, n, \lambda, \alpha, \beta), \beta)$, where

$$\zeta(m, n, \lambda, \alpha, \beta) = 1 - \{j(1+\beta)(1-\alpha)^3\}.$$

$$.\{(j+1)^{2n}[j(1+\beta)+(1-\alpha)]^3[1+((j+1)^m-1)\lambda]-(1-\alpha)^3\}^{-1}. \tag{7.20}$$

The result is the best possible for the functions $f_\nu(z)(\nu = 1, 2, 3)$ given by (7.14).

Proof. From Theorem 9, we have

$$(f_1 * f_2)(z) \in \mathbb{Q}_j(m, n, \lambda, \delta(m, n, \lambda, \alpha, \beta), \beta),$$

where δ is given by (7.2). Now, using Theorem 10, we get $(f_1 * f_2 * f_3)(z) \in \mathbb{Q}_j(m, n, \lambda, \zeta(m, n, \lambda, \alpha, \beta), \beta)$, where

$$\zeta(m, n, \lambda, \alpha, \beta) = 1 - \{j(1+\beta)(1-\alpha)^3\}.$$

$$.\{(j+1)^{2n}[j(1+\beta)(1-\alpha)]^3[1+((j+1)^m-1)\lambda]-(1-\alpha)^3\}^{-1}. \tag{7.21}$$

This completes the proof of corollary 6. ∎

Remark 2. Putting $m = 1$ in Corollary 6, we obtain similar result for the class $U_j(n, \lambda, \alpha, \beta)$.

Theorem 11. *Let the functions $f_\nu(z)$ $(\nu = 1, 2)$ be defined by (4.1) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$. Then the function*

$$h(z) = z - \sum_{k=j+1}^{\infty} \left(a_{k,1}^2 + a_{k,2}^2\right) z^k \tag{7.22}$$

belongs to the class $\mathbb{Q}_j(m, n, \lambda, \eta(m, n, \lambda, \alpha, \beta), \beta)$, where

$$\eta(m, n, \lambda, \alpha, \beta) = 1 - \{2j(1+\beta)(1-\alpha)^2\}.$$

$$.\{(j+1)^n[j(1+\beta)+(1-\alpha)]^2[1+((j+1)^m-1)\lambda]-2(1-\alpha)^2\}^{-1}. \tag{7.23}$$

The result is sharp for the functions $f_\nu(z)$ $(\nu = 1, 2)$ defined by (7.14).

Proof. By virtue of Theorem 1, we obtain

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n[k(1+\beta)-(\alpha+\beta)][1+(k^m-1)\lambda]}{1-\alpha}\right]^2 a_{k,1}^2$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha} a_{k,1} \right]^2 \leq 1 \tag{7.24}$$

and

$$\sum_{k=j+1}^{\infty} \left[\frac{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha} \right]^2 a_{k,2}^2$$

$$\leq \left[\sum_{k=j+1}^{\infty} \frac{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha} a_{k,2} \right]^2 \leq 1. \tag{7.25}$$

It follows from (7.24) and (7.25) that

$$\sum_{k=j+1}^{\infty} \frac{1}{2} \left[\frac{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha} \right]^2 \left(a_{k,1}^2+a_{k,2}^2\right) \leq 1. \tag{7.26}$$

Therefore, we need to fined the largest $\eta = \eta(m, n, \lambda, \alpha, \beta)$ such that

$$\frac{k^n \left[k\left(1+\beta\right)-\left(\eta+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\eta} \leq$$

$$\frac{1}{2} \left[\frac{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]\left[1+\left(k^m-1\right)\lambda\right]}{1-\alpha} \right]^2 \quad \left(k \geq j+1\right),$$

that is,

$$\eta \leq 1 - \{2\left(k-1\right)\left(1+\beta\right)\left(1-\alpha\right)^2\}.$$

$$.\left\{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]^2\left[1+\left(k^m-1\right)\lambda\right]\right\} - 2\left(1-\alpha\right)^2\}^{-1}$$

$$\left(k \geq j+1\right). \tag{7.27}$$

Since

$$D\left(k\right) = 1 - \{2\left(k-1\right)\left(1+\beta\right)\left(1-\alpha\right)^2\}.$$

$$.\left\{k^n \left[k\left(1+\beta\right)-\left(\alpha+\beta\right)\right]^2\left[1+\left(k^m-1\right)\lambda\right]\right\} - 2\left(1-\alpha\right)^2\}^{-1} \tag{7.28}$$

is an increasing function of k $\left(k \geq j+1\right)$, we readily have

$$\eta \leq D(j+1) = 1 - \{2j\left(1+\beta\right)\left(1-\alpha\right)^2\}.$$

$$.\{(j+1)^n \left[j\left(1+\beta\right)+\left(1-\alpha\right)\right]^2\left[1+\left(\left(j+1\right)^m-1\right)\lambda\right] - 2\left(1-\alpha\right)^2\}^{-1}$$

and Theorem 11 follows at once. ∎

Remark 3. Putting $m = 1$ in Theorem 11, we obtain similar result for the class $U_j(n, \lambda, \alpha, \beta)$.

8. Applications of fractional calculus

We begin with the statements of the following definitions of fractional calculus (that is , fractional derivative and fractional integral) which were defined by Owa [7](and, subsequently, by Srivastava and Owa [14]).

Definition 1. The fractional integral of order μ is defined, for a function $f(z)$ by

$$D_z^{-\mu} f(z) = \frac{1}{\Gamma(\mu)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{1-\mu}} d\zeta \ \ (\mu > 0), \tag{8.1}$$

where $f(z)$ is an analytic function in a simply-connected region of the $z-$plane containing the origin, and the multiplicity of $(z-\zeta)^{\mu-1}$ is removed by requiring log $(z-\zeta)$ to be real when $z-\zeta > 0$.

Definition 2. The fractional derivative of order μ is defined, for a function $f(z)$, by

$$D_z^{\mu} f(z) = \frac{1}{\Gamma(1-\mu)} \frac{d}{d(z)} \int_0^z \frac{f(\zeta)}{(z-\zeta)^{\mu}} d\zeta \quad (0 \le \mu < 1), \tag{8.2}$$

where $f(z)$ is constrained, and the multiplicity of $(z-\zeta)^{-\mu}$ is removed, as in Definition 1.

Definition 3. Under the hypotheses of Definition 2, the fractional derivative of order $\eta + \mu$ is defined

$$D_z^{n+\mu} f(z) = \frac{d^n}{dz^n} D_z^{\mu} f(z) \qquad (0 \le \mu < 1; n \in \mathbb{N}_0). \tag{8.3}$$

Theorem 12. *Suppose that the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$. Then*

$$\left| D_z^{-\mu}(D^i f(z)) \right| \ge \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{ 1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i} \ \ [j(1+\beta)+(1-\alpha)]\left[1+\left[(1+j)^m - 1\right]\lambda\right]\Gamma(j+2+\mu)} |z|^j \right\} \tag{8.4}$$

and

$$\left| D_z^{-\mu}(D^i f(z)) \right| \le \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{ 1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i} \ \ [j(1+\beta)+(1-\alpha)]\left[1+\left[(1+j)^m - 1\right]\lambda\right]\Gamma(j+2+\mu)} |z|^j \right\} \tag{8.5}$$

$$(\mu > 0; 0 \leq i \leq n; z \in \mathbb{U}).$$

The result is sharp.

Proof. Let

$$F(z) = \Gamma(2+\mu) z^{-\mu} D_z^{-\mu}(D^i f(z)) \tag{8.6}$$

$$= z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} k^i a_k z^k = z - \sum_{k=j+1}^{\infty} \Psi(k) k^i a_k z^k,$$

where

$$\Psi(k) = \frac{\Gamma(k+1)\Gamma(2+\mu)}{\Gamma(k+1+\mu)} \qquad (k \geq j+1). \tag{8.7}$$

Since

$$0 < \Psi(k) \leq \Psi(j+1) = \frac{\Gamma(j+2)\Gamma(2+\mu)}{\Gamma(j+2+\mu)}, \tag{8.8}$$

therefore, by using (3.5) and (8.8), we see that

$$|F(z)| \geq |z| - \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k$$

$$\geq |z| - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i} \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2+\mu)} |z|^{j+1} \tag{8.9}$$

and

$$|F(z)| \leq |z| + \Psi(j+1)|z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k$$

$$\leq |z| + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i} \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2+\mu)} |z|^{j+1}, \tag{8.10}$$

which proves the inequalities (8.4) and (8.5) of Theorem 12. The equalities in (8.4) and (8.5) are attained for the function $f(z)$ given by

$$D_Z^{-\mu}(D^i f(z)) = \frac{z^{1+\mu}}{\Gamma(2+\mu)}.$$

$$\cdot \left\{ 1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^{n-i} \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2+\mu)} z^j \right\} \tag{8.11}$$

or, equivalently, by

$$D^i f(z) = z - \frac{(1-\alpha)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)]\left[1+[(1+j)^m-1]\lambda\right]} z^j. \tag{8.12}$$

Thus we complete the proof of Theorem 12. ∎

Taking $i = 0$ in Theorem 12, we have

Corollary 7. Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m, n, \lambda, \alpha, \beta)$. Then

$$\left|D_z^{-\mu} f(z)\right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)]\left[1+[(1+j)^m-1]\lambda\right]\Gamma(j+2+\mu)}|z|^j\right\} \tag{8.13}$$

and

$$\left|D_Z^{-\mu} f(z)\right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)]\left[1+[(1+j)^m-1]\lambda\right]\Gamma(j+2+\mu)}|z|^j\right\} \tag{8.14}$$

$$(\mu > 0; z \in U).$$

The equalities in (8.13) and (8.14) are attained for the function $f(z)$ given by (3.3).

Remark 4. Putting $m = 1$ in Corollary 7, we obtain the following corollary which corrects the result obtained by Shanmugam et al. [12, Theorem 6.6].

Corollary 8. Let the function $f(z)$ defined by (1.4) be in the class $U_j(n, \lambda, \alpha, \beta)$.Then

$$\left|D_z^{-\mu} f(z)\right| \geq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)](1+j\lambda)\Gamma(j+2+\mu)}|z|^j\right\} \tag{8.15}$$

and

$$\left|D_Z^{-\mu} f(z)\right| \leq \frac{|z|^{1+\mu}}{\Gamma(2+\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2+\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)](1+j\lambda)\Gamma(j+2+\mu)}|z|^j\right\}. \tag{8.16}$$

The result is sharp.

Theorem 13. *Let the function* $f(z)$ *defined by* (1.4) *be in the class* $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$*. Then*

$$\left|D_z^{\mu}(D^i f(z))\right| \geq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}.$$
$$.\left\{1-\frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}|z|^j\right\} \tag{8.17}$$

and

$$\left|D_z^{\mu}(D^i f(z))\right| \leq \frac{|z|^{1-\mu}}{\Gamma(2-\mu)}.$$
$$.\left\{1+\frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}|z|^j\right\} \tag{8.18}$$
$$(0\leq\mu<1; 0\leq i\leq n-1; z\in\mathbb{U}).$$

The result is sharp.

Proof. Let

$$\begin{aligned} G(z) &= \Gamma(2-\mu)\, z^{\mu} D_z^{\mu}(D^i f(z)) \\ &= z-\sum_{k=j+1}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)}k^i a_k z^k \\ &= z-\sum_{k=j+1}^{\infty}\Theta(k)\, k^{i+1}a_k z^k, \end{aligned}$$

where

$$\Theta(k)=\frac{\Gamma(k+1)\Gamma(2-\mu)}{\Gamma(k+1-\mu)} \qquad (k\geq j+1). \tag{8.20}$$

It is easily seen from (8.20) that

$$0<\Theta(k)\leq\Theta(j+1)=\frac{\Gamma(j+2)\Gamma(2-\mu)}{\Gamma(j+2-\mu)}. \tag{8.21}$$

Consequently, with the aid of (3.5) and (8.21), we have

$$|G(z)|\geq|z|-\Theta(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty}k^{i+1}a_k$$

$$\geq |z| - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}|z|^{j+1} \tag{8.22}$$

and

$$|G(z)| \geq |z| + \Theta(j+1)|z|^{j+1}\sum_{k=j+1}^{\infty} k^{i+1}a_k$$

$$\leq |z| + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}|z|^{j+1} \tag{8.23}$$

Now (8.17) and (8.18) follow from (8.22) and (8.23), respectively. Since the equalities in (8.17) and (8.18) are attained for the function $f(z)$ given by

$$D_z^{\mu}(D^i f(z)) = \frac{z^{1-\mu}}{\Gamma(2-\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n-i}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}z^j\right\} \tag{8.24}$$

or for the function $D^i f(z)$ given by (8.12), the proof of Theorem 13 is thus completed. ∎

Taking $i = 0$ in Theorem 13, we have

Corollary 9. Let the function $f(z)$ defined by (1.4) be in the class $\mathbb{Q}_j(m,n,\lambda,\alpha,\beta)$. Then

$$|D_z^{\mu}\ f(z)| \geq \frac{z^{1-\mu}}{\Gamma(2-\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}z^j\right\} \tag{8.25}$$

and

$$|D_z^{\mu}\ f(z)| \leq \frac{z^{1-\mu}}{\Gamma(2-\mu)}.$$

$$.\left\{1 + \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^{n}\ \ [j(1+\beta)+(1-\alpha)][1+[(1+j)^m-1]\lambda]\Gamma(j+2-\mu)}z^j\right\} \tag{8.26}$$

The equalities in (8.25) and (8.26) are attained for the function $f(z)$ given by (3.3).

Remark 5. Putting $m = 1$ in Corollary 9, we obtain the following corollary which corrects the result obtained by Shanmugam et al.[12, Theorem 6.7].

Corollary 10. Let the function $f(z)$ defined by (1.4) be in the class $U_j(n,\lambda,\alpha,\beta)$.Then

$$|D_z^{\mu}\ f(z)| \geq \frac{z^{1-\mu}}{\Gamma(2-\mu)}$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)](1+j\lambda)\Gamma(j+2-\mu)} z^j\right\} \quad (8.27)$$

and

$$|D_z^{\mu}\ f(z)| \geq \frac{z^{1-\mu}}{\Gamma(2-\mu)}.$$

$$.\left\{1 - \frac{(1-\alpha)\Gamma(j+2)\Gamma(2-\mu)}{(j+1)^n\ \ [j(1+\beta)+(1-\alpha)](1+j\lambda)\Gamma(j+2-\mu)} z^j\right\}. \quad (8.28)$$

The result is sharp.

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