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ON NEIGHBORHOOD AND PARTIAL SUMS FOR CERTAIN SUBCLASS OF MEROMORPHIC FUNCTIONS DEFINED BY CONVOLUTION

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Abstract. For certain meromorphic p-valent functions ϕ and ψ, we study a class $G(b,\phi,\psi)$ of functions $f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}a_nz^n, (a_n\geq 0)$, defined in the punctured unit disc U^*, satisfying $\Re\left|\frac{1}{b}\left(\frac{(f*\phi)(z)^{(m)}}{(f*\psi)(z)^{(m)}}-1\right)\right|<1 \ \ (0\leq\alpha\leq 1, z\in U^*)$, coefficient estimate, distortion theorem, radii of starlikeness, convexity , Neighborhood and partial sums for function class $G(b,\phi,\psi)$.

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1. Introduction

Let $\sum_p$ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z)=\frac{1}{z^p}+\sum_{n=p}^{\infty}a_nz^n \quad (\ a_n\geq 0;\ p\in N=\{1,2,3,......\}), \tag{1}$$

which are analytic and p-valent punctured unit disk $U^*=\{z:0<|z|<1\}$. A function $f\in\sum_p$ is said to be meromorphically p-valent starlike of order ρ $(0\leq\rho<p)$ in U^* if and only if

$$\Re\left\{-\frac{zf'(z)}{f(z)}\right\}>\ \rho \qquad (z\in U^*;0\leq\rho<p)\ . \tag{2}$$

On the other hand, a function $f\in\sum_p$ is said to be meromorphically p-valent convex of order ρ $(0\leq\rho<p)$ in U^* if

$$\Re\left\{-\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\rho \qquad (z\in U^*;0\leq\rho<p). \tag{3}$$

The Hadamard product (or convolution) of the function f defined by (1) with the functions g given, by

$$g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n \qquad (b_n \geq 0; p \in \mathbb{N}),$$

can be expressed as follows:

$$(f * g)(z) = z^{-p} + \sum_{n=p}^{\infty} a_n b_n z^n \quad = (g * f)(z).$$

Suppose the functions $\phi(z)$ and $\psi(z)$ are given by

$$\begin{aligned} \phi(z) &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \lambda_n z^n \\ \psi(z) &= \frac{1}{z^p} + \sum_{n=p}^{\infty} \mu_n z^n \quad . \end{aligned}$$

We now introduce a new subclass $G(b, \phi, \psi)$ of meromorphiclly p-valent functions $\sum_p$, which inclosed functions $f(z)$ satisfying the following inequality:

$$\left| \frac{1}{b} \left(\frac{\left(\left(f * \phi\right)(z)\right)^{(m)}}{\left(\left(f * \psi\right)(z)\right)^{(m)}} - 1 \right) \right| < 1 \qquad (z \in U^*, m \in \mathbb{N}_0, b \in \mathbb{C} \setminus \{0\}), \qquad (4)$$

where

$$\left(\left(f * \psi\right)(z)\right)^{(m)} \neq 0,$$

and

$$f^{(m)}(z) = (-1)^{(m)} (p)_m \, z^{-(p+m)} + \sum_{n=p}^{\infty} \beta(n, m) \, a_n z^{n-m},$$

$$\beta(n, m) = \frac{n!}{(n-m)!} = \begin{cases} 1 & (\ m = 0) \\ n(n-1)...(n-m+1) & (\ m \neq 0) \end{cases}$$

and $(\alpha)_j$ is the Pochammer symbol (or shifted factorial) defined, in terms of the Gamma function, by

$$(X)_j = \frac{\Gamma(x+j)}{\Gamma(x)} = \begin{cases} 1 & if \quad (j = 0) \\ x(x+1) \ldots (x+j-1) & (j \in N) \end{cases} .$$

Suitably specializing the parameters we note that

(i) $G(b, zg', g) = G(b, g)$ where $G(b, g)$ is defined as follows:

$$G(b,g) = \left\{ f \in \sum_p : \left| \frac{1}{b} \frac{((f * zg)(z))^{(m+1)}}{((f * g)(z))^{(m)}} - 1 \right| < 1 \right\}$$

(ii) $G(b, z^{-p} + \sum_{k=0}^{\infty} \left(\frac{\ell+\lambda(k+p)}{\ell}\right)^m a_k z^k) = G(b, \ell, \lambda)$where $G(b, \ell, \lambda)$ is defined by

$$G(b,\ell,\lambda) = \left\{ f \in \sum_p : \left| \frac{1}{b} \frac{\left(z\left(I_p^m(\lambda,\ell) f(z)\right)'\right)^{(m)}}{\left(I_p^m(\lambda,\ell) f(z)\right)^{(m)}} - 1 \right| < 1 \right\}$$

$$\left(z \in U^*, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, b \in \mathbb{C} \setminus \{0\}, \lambda \geq 0, \ell > 0\right),$$

where $I_p^m(\lambda, \ell)$ introduced by El-Ashwah [4].

(iii) $G(b, z^{-p} + \sum_{k=p+1}^{\infty} (1 + p + n)^k a_k z^k) = G(b, n)$where $G(b, n)$ is defined as

$$G(b,n) = \left\{ f \in \sum_p : \left| \frac{1}{b} \frac{\left(z\left(D_\lambda^n f(z)\right)'\right)^{(m)}}{\left(D_\lambda^n f(z)\right)^{(m)}} - 1 \right| < 1 \right\}$$

$$z \in U^*, m \in \mathbb{N}_0, b \in \mathbb{C} \setminus \{0\}, \lambda > 0,$$

where the operator D_λ^n was introduced and studied by Aouf and Hossen [1], Liu and Owa [6] and Srivastava and Patel [12].

The object of the present paper is to investigate coefficient estimate, distortion theorem, radii of convexity and starlikeness, we also apply the familiar concept of neighborhood of analytic functions to meromorphically p-valent functions in the class $\sum_p$.

2. Coefficient Inequalities

In this section we begin by proving a characterization property which provides a necessary condition for a function $f \in \sum_p$ belong to the class $G(b, \phi, \psi)$ of meromorphically p-valent functions with positive coefficients

Theorem 1. *Let $f \in \sum_p$ be given by* (1). *If*

$$\sum_{n=P}^{\infty} \beta(n,m) \{ |\lambda_n - \mu_n| + |b| \mu_n \} a_n \leq |b| (p)_m . \tag{5}$$

then $f \in G(b, \phi, \psi)$.

Proof. Suppose that (5) is true. Then we have

$$\left|\frac{((f*\phi)(z))^{(m)}}{((f*\psi)(z))^{(m)}}-1\right|$$

$$=\left|\frac{\sum_{n=p}^{\infty}\beta(n,m)(\lambda_n-\mu_n)a_nz^{n+p}}{(-1)^m(p)_m+\sum_{n=p}^{\infty}\beta(n,m)\mu_na_nz^{n+p}}\right|$$

$$\leq\frac{\sum_{n=p}^{\infty}\beta(n,m)|\lambda_n-\mu_n|a_n}{\left[(p)_m-\sum_{n=p}^{\infty}\beta(n,m)\mu_na_n\right]}.$$

The last expression is bounded above by $|b|$ provided

$$\sum_{n=p}^{\infty}\beta(n,m)\{|\lambda_n-\mu_n|+|b|\mu_n\}\leq|b|(p)_m$$

which is true by the hypothesis. Hence $f\in G(b,\phi,\psi)$.

3. Distortion Theorem

We denote $\breve{G}(b,\phi,\psi)$ to be the class of functions $f\in G(b,\phi,\psi)$ whose coefficients satisfy the condition (5).

Theorem 2. *If the function $f(z)$ defined by (1) is in the class $\breve{G}(b,\phi,\psi)$, then for $0<|z|=r<1$, $\mu_n\geq 0$ and the sequences $<\mu_n>$, $<\frac{\lambda_n}{\mu_n}>$ are nondecreasing, we have*

$$\frac{1}{r^p}-\frac{|b|\ (p)_m}{\beta(p,m)\ \{\ |\lambda_p-\mu_p|+|b|\ \mu_p\}}r^p\leq|f(z)|\leq\frac{1}{r^p}+\frac{|b|\ (p)_m}{\beta(p,m)\ \{\ |\lambda_p-\mu_p|+|b|\ \mu_p\}}r^p. \tag{6}$$

The bounds in (6) is attained for the functions $f(z)$ given by

$$f(z)=\frac{1}{z^p}+\frac{|b|\ (p)_m}{\beta(p,m)\ \{\ |\lambda_n-\mu_n|+|b|\ \mu_n\}}z^n. \tag{7}$$

Proof. Since $f\in\breve{G}(b,\phi,\psi)$ from the inequality (6). We observe that $|\lambda_n-\mu_n|+|b|$ μ_n nondecreasing we have

$$\sum_{n=p}^{\infty}|a_n|\leq\frac{|b|\ (p)_m}{\beta(p,m)\ \{\ |\lambda_p-\mu_p|+|b|\ \mu_p\}}. \tag{8}$$

Thus for $0 < |z| = r < 1$, and making use of (8) we have

$$\begin{aligned} |f(z)| &\leq \left|\frac{1}{z^p}\right| + \sum_{n=p}^{\infty} |a_n| \, |z|^n \qquad (9) \\ &\leq \frac{1}{r^p} + r^p \sum_{n=p}^{\infty} |a_n| \\ &\leq \frac{1}{r^p} + \frac{|b| \; (\;p)_m}{\beta(p,m) \;\; \{\; |\lambda_p - \mu_p| + |b| \;\; \mu_p\}} r^p \end{aligned}$$

and

$$\begin{aligned} |f(z)| &\geq \left|\frac{1}{z^p}\right| - \sum_{n=p}^{\infty} |a_n| \, |z|^n \qquad (10) \\ &\geq \frac{1}{r^p} - r^p \sum_{n=p}^{\infty} |a_n| \\ &\geq \frac{1}{r^p} - \frac{|b| \; (\;p)_m}{\beta(p,m) \;\; \{\; |\lambda_p - \mu_p| + |b| \;\; \mu_p\}} r^p \end{aligned}$$

which readily yields the following distortion inequalities. This completes the proof of Theorem 2.

4. Radii of Starlikeness and Convexity

We next determine the radii of meromorphiclly p-valent starlikeness and meromorphiclly p-valent convexity of the class $\breve{G}(b, \phi, \psi)$,which are given by

Theorem 3. *Let a function $f \in \sum_p$ of the form* (1) *belong to the class $\breve{G}\,(b, \phi, \psi)$. Then*

(i) f is meromorphiclly p-valent starlike of order $\gamma\,(0 \leq \gamma < p)$ in the disk $|z| < r_1$, where

$$r_1 = \inf_{n \geq p} \left\{ \frac{(p-\gamma)\,|b| \;\; (\;p)_m}{(n+\gamma)\,[\beta\,(n,m) \;\;\; \{\;\; |\lambda_n - \mu_n| + |b| \;\; \mu_n\}]} \right\}^{\frac{1}{n+p}} \qquad (11)$$

(ii) f is meromorphiclly p-valent convex of order $\gamma\,(0 \leq \gamma < p)$ in the disk $|z| < r_2$, where

$$r_2 = \inf_{n \geq p} \left\{ \frac{(p-\gamma)\,|b| \;\; (\;p)_m}{n\,(n+\gamma)\,[\beta\,(n,m) \;\;\; \{\;\; |\lambda_n - \mu_n| + |b| \;\; \mu_n\}]} \right\}^{\frac{1}{n+p}}. \qquad (12)$$

Proof. To prove (i), we observe from the inequality that the function f of the form (1) is meromorphiclly p-valent starlike of order $\gamma\,(0 \leq \gamma < p)$ if for

$$\left|\frac{\frac{z\,f\,'(z)}{f\,(z)}+p}{\frac{z\,f\,'(z)}{f\,(z)}-p+2\gamma}\right| \leq 1 \qquad (0 \leq \gamma < p; p \in \mathbb{N}).$$

For $|z| = r$, the inequality is true if

$$\sum_{n=p}^{\infty}\left(\frac{n+\gamma}{p-\gamma}\right)a_n r^{n+p} \leq 1. \tag{13}$$

Comparing with coefficient inequality , we conclude that the function f is meromorphiclly p-valent starlike of order$\gamma\,(0 \leq \gamma < p)$ in the disk $|z| < r_1$ with r_1 given precisely by .The proof of (ii) is similar to that of (i) detailed above ;it is , therefore, being omitted here.

5. Neighborhood Theorem

Next, following the earlier investigations by Goodman [5], Ruscheweyh [8] and Atlantis et al.[2] (see also [3],[7] and [11]), we define the (n, δ)- neighborhood of a function $f(z) \in \sum_p$by (see, for details, [2])

$$N_\delta\,(f) := \left\{ g \in \sum_p : g(z) = \frac{1}{z^p} + \sum_{n=p}^{\infty} b_n z^n \;\text{ and }\; \sum_{n=p}^{\infty} \frac{\beta\,(n,m)\,[|\lambda_n - \mu_n| + |b|\,\mu_n]}{|b|\;\;(p)_m}\,|a_n - b_n| \leq \gamma \right\}.$$

Theorem 4. *Let $\delta > 0$ and $f\,(z) \in \sum_p$ given by (1)satisfies the inclusion property*

$$\frac{f(z) + \epsilon z^{-p}}{1+\epsilon} \in G(b, \phi, \psi) \tag{14}$$

for any complex number ϵ such that $|\epsilon| < \delta$, $N_\delta\,(f) \subset G(b, \phi, \psi)$.

Proof. It is easily to seen from (4) that $f\,(z) \in G(b, \phi, \psi)$ if and only if for any complex number σ with $|\sigma| = 1$,we have

$$\left|\frac{1}{b}\left(\frac{((f*\phi)\,(z))^{(m)}}{((f*\psi)\,(z))^{(m)}} - 1\right)\right| \neq \sigma \qquad (z \in U^*)\,, \tag{15}$$

which is equivalent to

$$\frac{(f*Q)\,(z)}{z^{-P}} \neq 0 \qquad (z \in U^*)\,, \tag{16}$$

where

$$Q(z) = z^{-p} + \sum_{n=p}^{\infty} c_n z^n$$

$$\left(c_n = \frac{\delta(n,m) \ \{ \ -(\lambda_n - \mu_n) + b \ \sigma \ \mu_n \}}{\sigma \ b \ (\ p)_m} \right) \tag{17}$$

It is follows from (16) that

$$|c_n| = \left| \frac{\beta(n,m) \ \{ \ |\lambda_n - \mu_n| + \sigma \ |b| \ \mu_n \}}{\sigma \ |b| \ (\ p)_m} \right| \leq \frac{\beta(n,m) \ \{ \ |\lambda_n - \mu_n| + \ |b| \ \mu_n \}}{|b| \ (\ p)_m}$$

if $f\ (z) \in \sum_p$ given by (1) satisfies the inclusion property (14) , then (16) yields

$$\frac{(f * Q)(z)}{z^{-P}} \geq \delta \qquad (z \in U^*) \quad . \tag{18}$$

Now , if we suppose that

$$g(z) = z^{-p} + \sum_{n=p}^{\infty} b_n z^n \in N_\delta(f), \tag{19}$$

we easily see that

$$\begin{aligned} \left| \frac{(g-f)(z) * Q(z)}{z^{-p}} \right| &= \left| \sum_{n=p}^{\infty} (b_n - a_n) c_n z^{n+p} \right| \\ &\leq \sum_{n=p}^{\infty} |b_n - a_n| \, |c_n| \, z^{n+p} \\ &\leq |z| \sum_{n=p}^{\infty} \frac{\beta(n,m) \ \{ \ |\lambda_n - \mu_n| + \ |b| \ \mu_n \}}{|b| \ (\ p)_m} \\ &\quad \times |b_n - a_n| \leq \delta \end{aligned}$$

then

$$\left| \frac{(g)(z) * Q(z)}{z^{-p}} \right| = \left| \frac{f + g - f \ (z) * Q(z)}{z^{-p}} \right|$$

$$\geq \left| \frac{f \ (z) * Q(z)}{z^{-p}} \right| - \left| \frac{(g-f) \ (z) * Q(z)}{z^{-p}} \right| > 0$$

thus, for any complex number σ such that $|\sigma| = 1$, we have

$$\frac{(g)(z) * Q(z)}{z^{-p}} \neq 0 \qquad (z \in U^*)$$

which implies that $g(z) \in G(b, \phi, \psi)$.

6. Partial sums of function class $G(b,\phi,\psi)$

In this section, applying methods used by Silverman [9] and Silvia [10], we investigate the ratio of a function of the form (1) to its sequence of partial sums $f_{m+p-1}(z) = \frac{1}{z^p} + \sum_{k=p}^{m+p-1} a_k z^k$.

Theorem 5. *If the function f of the form* (1) *be in the class $\breve{G}(b,\phi,\psi)$, then*

$$\Re\left(\frac{f(z)}{f_{m+p-1}(z)}\right) > 1 - \frac{1}{c_{m+p}} \qquad (z \in U^*\ , m \in N)\,, \tag{20}$$

and

$$c_k \geq \begin{cases} 1 & k = 2,3,4,...,m \\ c_{m+p} & k = m+p, m+2p, ... \end{cases}$$

where

$$c_k = \frac{\beta(n,m)\ \ \{\ |\lambda_n - \mu_n| + \ \ |b|\ \ \mu_n\}}{|b|\ \ (\ p)_m} \tag{21}$$

The result in (20) *is sharp for every m, with the external function*

$$f(z) = z^{-p} + \frac{z^{m+p}}{c_{m+p}} \tag{22}$$

Proof. Define the function $w(z)$, we may write

$$\begin{aligned} \frac{1+w(z)}{1-w(z)} &= c_{m+p}\left\{\frac{f(z)}{f_{m+p-1}(z)} - \left(1 - \frac{1}{c_{m+p}}\right)\right\} \\ &= \left\{\frac{1 + \sum_{k=p}^{m+p-1} a_k z^{p+k} + c_{m+p}\sum_{k=m+p}^{\infty} a_k z^{p+k}}{1 + \sum_{k=p}^{m+p-1} a_k z^{p+k}}\right\}. \end{aligned} \tag{23}$$

Then, from (23) we can obtain

$$w(z) = \frac{c_{m+p}\sum_{k=m+p}^{\infty} a_k z^{p+k}}{2 + 2\sum_{k=p}^{m+p-1} a_k z^{p+k} + c_{m+p}\sum_{k=m+p}^{\infty} a_k z^{p+k}}$$

and

$$|w(z)| \leq \frac{c_{m+p}\sum_{m+p}^{\infty} |a_k|}{2 - 2\sum_{k=p}^{m+p-1} |a_k| - c_{m+p}\sum_{k=m+p}^{\infty} |a_k|}.$$

Now $|w(z)| \leq 1$ if

$$2c_{m+p}\sum_{k=m+p}^{\infty} |a_k| \leq 2 - 2\sum_{k=p}^{m+p-1} |a_k|\,,$$

which is equivalent to

$$\sum_{k=p}^{m+p-1} |a_k| + c_{m+p} \sum_{k=m+p}^{\infty} |a_k| \leq 1. \tag{24}$$

It is suffices to show that the left and hand side of (24) is bounded above by $\sum_{k=p}^{\infty} c_k |a_k|$,which is equivalent to

$$\sum_{k=p}^{m+p-1} (c_k - 1) |a_k| + \sum_{k=m+p}^{\infty} (c_k - c_{m+p}) |a_k| \geq 0.$$

To see that the function given by (22) gives the sharp result, we observe that for $z = re^{i\frac{\pi}{m+2p}}$,

$$\frac{f\ (z)}{f_{m+p-1}(z)} = 1 + \frac{z^{m+2p}}{c_{m+p}}, \tag{25}$$

then we have

$$\frac{f\ (z)}{f_{m+p-1}(z)} = 1 - \frac{1}{c_{m+p}}$$

This completes the proof of Theorem 5.

We next determine bounds for $f_{m+p-1}(z)/f(z)$.

Theorem 6. *If the function f of the form* (1) *be in the class $\breve{G}(b, \phi, \psi)$, then*

$$\Re\left(\frac{f_{m+p-1}(z)}{f(z)}\right) > \frac{c_{m+p}}{1 + c_{m+p}} \qquad (z \in U^*\ , m \in N)\ . \tag{26}$$

The result is sharp with the function given by (22) .

Proof. We may write

$$\begin{aligned} \frac{1 + w(z)}{1 - w(z)} &= (1 + c_{m+p}) \left(\frac{f_{m+p-1}(z)}{f\ (z)} - \frac{c_{m+p}}{1 + c_{m+p}}\right) \\ &= \frac{1 + \sum_{k=p}^{m+p-1} a_k z^{p+k} - c_{m+p} \sum_{k=m+p}^{\infty} a_k z^{p+k}}{1 + \sum_{k=p}^{m+p-1} a_k z^{p+k}} \end{aligned} \tag{27}$$

and

$$w(z) = \frac{(1 + c_{m+p}) \sum_{k=m+p}^{\infty} a_k z^{p+k}}{2 + 2\sum_{k=p}^{m+p-1} a_k z^{p+k} + (1 + c_{m+p}) \sum_{k=m+p}^{\infty} a_k z^{p+k}}$$

where
$$|w(z)| \leq \frac{(1+c_{m+p})\sum_{m+p}^{\infty}|a_k|}{2-2\sum_{k=p}^{m+p-1}|a_k|+(1-c_{m+p})\sum_{m+p}^{\infty}|a_k|}. \tag{28}$$

The last inequality is equivalent to

$$\sum_{k=p}^{m+p-1}|a_k| + c_{m+p}\sum_{k=m+p}^{\infty}|a_k| \leq 1. \tag{29}$$

It is suffices to show that the left and hand side of (29) is bounded above by $\sum_{k=p}^{\infty} c_k |a_k|$,which is equivalent to

$$\sum_{k=p}^{m+p-1}(c_k-1)|a_k| + \sum_{k=m+p}^{\infty}(c_k-c_{m+p})|a_k| \geq 0.$$

This completes the proof of Theorem 6.

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