Acta Universitatis Apulensis
ISSN: 1582-5329
http://www.uab.ro/auajournal/

No. 41/2015
pp. 89-105
doi: 10.17114/j.aua.2015.41.07

THE RELATIONSHIPS BETWEEN CONTEXT CONSTRUCTIONS AND MATROIDS

H. Mao

Abstract. For the frequently used constructions of two contexts, we discuss the relationships between them and geometric lattices. We obtain that both concept lattices of two contexts are geometric if and only if the concept lattice of their disjoint union, or their direct sum, or their direct product is geometric. With the assistance of hard examples, for the constructions of two contexts such as apposition, subposition and semiproduct, we find out their relationships with geometric lattices respectively. Up to isomorphism, applying the one-to-one connection between geometric lattices and simple matroids, under matroid frameworks, we receive the relationships between matroids and the contexts of disjoint union, and direct sum for two contexts respectively. Finally, it concludes this paper and points out our future works.

2010 *Mathematics Subject Classification*: 68P15; 05B35; 06C10; 68P05

Keywords: context constructions, geometric lattice, matroid, closure operator.

1. Introduction

The theory of concept lattices, i.e. Formal Concept Analysis (simply, FCA), was proposed by R.Wille [1] and has been developed by many scholars [2],[3], [4]. It is a powerful tool for data mining, data analysis, knowledge discovery and information retrieval, and also for intra- and extra- mathematical theories and applications [2], [3], [4], [5], [6], [7], [8], [29]. The central notion of FCA is formal context. Compared with formal contexts, many-valued contexts are common in the real life (see [5], [6], [7], [8], [30]).

C. Carpineto and G .Romano [4] point out that generally, in order to assign concepts to a many-valued context, we can transform the many-valued context into a single-valued one, and then interpret the concepts of the derived context as the concepts of the many-valued context. To complete this, more generally, for each many-valued attribution, one can provide a conceptual scaling by Ganter and Wille

[2]. There are many other methods for scaling [8],[18],[19],[20]. Many of interesting context families have proved to be useful as scales. Hence, it shows much more important to detect the properties for some constructions such as frequently used sum and product constructions for two contexts ([2]). Usually, people discuss these constructions by means of definitions and lattice theory ([2],[3],[4],[9],[20]).

Matroid which was proposed by Whitney [21] plays an important role in many fields such as combinatorial optimization ([10],[11],[28]), and is widely used in greedy algorithm ([10],[11]). It is worth noting that many algorithms for formal contexts have the same ideas with greedy algorithm. Recently, some scholars study on FCA with matroids ([12],[13],[14],[22],[23],[24],[25],[26],[27],[31],[32]). All these imply that matroids will highlight many-valued contexts someday. Based on the beyond, we may be convinced that the first what we do for applying matroids on many-valued contexts is to discover the relationships between context constructions and matroids.

How to fulfill the above first duty is the work of this paper.

We find that every concept lattice is complete ([2]) and matroids have correspondent relation with geometric lattices ([10],[11]). Thereby, up to isomorphism, we may realize our duty if we discover the relationships between context constructions and geometric lattices. After that, we generalize some of the above results relative to geometric lattices to matroid frameworks.

Actually, all the contexts in real life are finite. According to the mathematics induction, we only need to put our efforts on the constructions for two contexts. We will do as this line in this paper.

The rest of this paper is organized as follows. We review some facts of concept lattices and matroids in Section 2. In Section 3, we search the relationships between geometric lattices and the frequently used context constructions for two contexts. Afterwards, under matroid frameworks, Section 4 discusses the relationships between matroids and some constructions for two contexts. The final part concludes this paper and points out some works on contexts constructions with matroid theory left rooms for the future.

2. Preliminaries

This section introduces some notations and properties of concept lattices and matroids. For more detail, concept lattices are seen [2],[3],[4] and matroids are cf. [10],[11]. For lattice theory in this paper, we refer to [15],[16],[17].

Definition 1. *(1) [2, pp.17-18] A context $\mathbb{K} := (O, P, I)$ consists of two sets O and P and a relation I between O and P. The elements of O are called the objects and the elements of P are called the attributes of the context.*

A concept of $\mathbb{K}$ *is a pair* (A, B) *with* $A \subseteq O, B \subseteq P, A' = B$ *and* $B' = A$*, where* $A' := \{p \in P \mid oIp \text{ for all } o \in A\}$ *and* $B' := \{o \in O \mid oIp \text{ for all } p \in B\}$*. We call* A *the extent and* B *the intent of* (A, B)*.*

(2) [2, p.246] An isomorphism between contexts $\mathbb{K}_1 := (O, P, I)$ *and* $\mathbb{K}_2 := (H, N, J)$ *is a pair* (α, β) *of bijective maps* $\alpha : O \to H, \gamma : P \to N$ *with* $oIp \Leftrightarrow \alpha(o)J(\gamma(p))$*.*

We denote $\mathbb{K}_1 \cong \mathbb{K}_2$ *if they are isomorphic.*

In [2], the authors formulated the following frequently used constructions for two contexts each.

Definition 2. *Let* $\mathbb{K} := (O, P, I), \mathbb{K}_1 := (O_1, P_1, I_1)$ *and* $\mathbb{K}_2 := (O_2, P_2, I_2)$ *be contexts. Let* $\dot{O}_j := \{j\} \times O_j, \dot{P}_j := \{j\} \times P_j$ *and* $\dot{I}_j := \{((j, o), (j, p)) \mid (o, p) \in I_j\}$ *for* $j \in \{1, 2\}$*.*

(1) [2, pp.41-42] $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2 := (\dot{O}_1 \cup \dot{O}_2, \dot{P}_1 \cup \dot{P}_2, \dot{I}_1 \cup \dot{I}_2)$ *is the disjoint union of* $\mathbb{K}_1$ *and* $\mathbb{K}_2$*.*

If $O = O_1 = O_2$*, then* $\mathbb{K}_1 \mid \mathbb{K}_2 := (O, \dot{P}_1 \cup \dot{P}_2, \dot{I}_1 \cup \dot{I}_2)$ *is the apposition of* $\mathbb{K}_1$ *and* $\mathbb{K}_2$*, dually, if* $P = P_1 = P_2$*, then* $\frac{\mathbb{K}_1}{\mathbb{K}_2} := (\dot{O}_1 \cup \dot{O}_2, P, \dot{I}_1 \cup \dot{I}_2)$ *is the subposition of* $\mathbb{K}_1$ *and* $\mathbb{K}_2$*.*

(2) [2, p.46] The direct sum of two contexts is defined by
$\mathbb{K}_1 + \mathbb{K}_2 := (\dot{O}_1 \cup \dot{O}_2, \dot{P}_1 \cup \dot{P}_2, \dot{I}_1 \cup \dot{I}_2 \cup (\dot{O}_1 \times \dot{P}_2) \cup (\dot{O}_2 \times \dot{P}_1))$*.*

The semiproduct is defined by $\mathbb{K}_1 \bowtie \mathbb{K}_2 := (O_1 \times O_2, \dot{P}_1 \cup \dot{P}_2, \nabla)$ *with*
$$(o_1, o_2)\nabla(j, p) :\Leftrightarrow o_j I_j p \text{ for } j \in \{1, 2\}.$$

The direct product is given by $\mathbb{K}_1 \times \mathbb{K}_2 := (O_1 \times O_2, P_1 \times P_2, \nabla)$ *with*
$$(o_1, o_2)\nabla(p_1, p_2) :\Leftrightarrow o_1 I_1 p_1 \text{ or } o_2 I_2 p_2.$$

Lemma 1 *(1) [2, pp.18-20] Let* $\mathbb{K}$ *be a context. Then* $\mathcal{B}(\mathbb{K})$ *denotes the set of all concepts of* $\mathbb{K}$ *is a complete lattice with hierarchical order and is called concept lattice of* $\mathbb{K}$*.*

If A, A_1, A_2 *are sets of objects and* B, B_1, B_2 *are sets of attributes, then*

(i) $A \subseteq A''$; $B \subseteq B''$; *(ii)* $A_1 \subseteq A_2 \Rightarrow A_1' \supseteq A_2'$; $B_1 \subseteq B_2 \Rightarrow B_1' \supseteq B_2'$*.*

(2) [2, p.246] Isomorphic contexts have isomorphic concept lattices.

(3) [2, p.46] Let $\mathbb{K}_j$ *be contexts* $(j = 1, 2)$*. Then the extents of* $\mathbb{K}_1 \bowtie \mathbb{K}_2$ *are precisely the set of the form* $A_1 \times A_2$*, each set* A_j *being an extent of* $\mathbb{K}_j, (j = 1, 2)$*.*

The definition of a *matroid* M is referred to [11], p.7. In [11], p.7, related to M, we can find out the definitions of a *closed set, closure operator* and so on. Additionally, a *simple matroid* is referred to [11], pp.12-13.

Definition 3. *[11, p.9] Two matroids* M_1 *and* M_2 *on* S_1 *and* S_2 *respectively are isomorphic if there is a bijection* $\varphi : S_1 \to S_2$ *which preserves independence.*

Lemma 2 *(1) [11, p.8] (Closure axioms) A function $\sigma : 2^S \to 2^S$ is the closure operator of a matroid on S if and only if for X, Y subsets of S, and $x, y \in S$;*
(S1) $X \subseteq \sigma X$;
(S2) $Y \subseteq X \Rightarrow \sigma Y \subseteq \sigma X$;
(S3) $\sigma X = \sigma\sigma X$;
(S4) if $x, y \notin \sigma X$ *and* $y \in \sigma(X \cup x)$, *then* $x \in \sigma(X \cup y)$.

(2) [11, p.9] Let M be a matroid with σ as its closure operator. If $\sigma(A) = A$, then A is a closed set of M, and vice versa.

(3) [11, pp.48-54] If M is a matroid on S, we can associate with M a partially ordered set $\mathcal{L}(M)$ whose elements are the closed sets of M ordered by inclusion.

A finite lattice L is isomorphic to the lattice of closed sets of a matroid if and only if it is geometric.

The correspondence between a geometric lattice L and the matroid $M(L)$ on the set of atoms of L is a bijection between the set of finite geometric lattices and the set of simple matroids.

(4) [11, p.61] The closure operators σ, σ_T of $M, M|T$ (the restriction of M to $T \subseteq S$) respectively are linked by $\sigma_T(A) = (\sigma A) \cap T, (A \subseteq T)$.

(5) [11, pp.72-73] Let M_1, M_2 be matroids on disjoint sets S_1, S_2. The direct sum of M_1 and M_2, written $M_1 + M_2$ is the matroid on $S_1 \cup S_2$. The closure function σ is given for $A \subseteq S_1 \cup S_2$ by $\sigma A = \sigma_1(A \cap S_1) \cup \sigma_2(A \cap S_2)$, where σ_i is the closure function of M_i $(i = 1, 2)$.

Definition 4. *(1) [4, p.6 & 15-17] For two (disjoint) ordered sets $(S_1, \leq)$ and $(S_2, \leq)$, the direct product of them is $(S_1 \times S_2, \leq)$, where $(S_1 \times S_2)$ is the Cartesian product of S_1 and S_2 and the order relation on $S_1 \times S_2$ is such that $(x_1, x_2) \leq (y_1, y_2) \Longleftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$.*

(2) [11, p.47 & 15-17] A finite lattice L is semimodular if for all $x, y \in L$:

$$x \text{ and } y \text{ cover } x \wedge y \Rightarrow x \vee y \text{ covers } x \text{ and } y.$$

(3) [15, p.80 & 10, 11, 16, 17] A finite lattice is geometric if it is semimodular and every point is the join of atoms.

Lemma 3 *(1) [16, p.234 & 15,17] Every interval in a geometric lattice is geometric.*

(2) [15, p.8] The direct product of any two lattices is a lattice with

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2, y_1 \vee y_2),\ (x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2, y_1 \wedge y_2).$$

From the above reviewed and some well known results, we obtain the following ideas.

(2.1) Let $\mathcal{B}(O)$ be the family of extents in $\mathbb{K}$ with the same hierarchical order to $\mathcal{B}(\mathbb{K})$. Then $\mathcal{B}(O) \cong \mathcal{B}(\mathbb{K})$ (see [2]). Generally, $\mathcal{B}(O)$ is called *extent lattice* of $\mathbb{K}$.

(2.2) According to Lemma 1(1) and Lemma 2, we may state that if a concept lattice $\mathcal{B}(\mathbb{K})$ is geometric, then there is a unique simple matroid $M(\mathcal{B}(\mathbb{K}))$ corresponding to $\mathcal{B}(\mathbb{K})$ up to isomorphism.

(2.3) By (2.1) and (2.2), $\mathcal{B}(\mathbb{K})$ is geometric if and only if $\mathcal{B}(O)$ is geometric. If $\mathcal{B}(O)$ is geometric, then it exists a simple matroid $M(\mathcal{B}(O))$ defined on O corresponding to $\mathcal{B}(O)$.

Up to isomorphism, $M(\mathcal{B}(\mathbb{K}))$ is $M(\mathcal{B}(O))$. Thus, under isomorphism, it is no confusion to denote $M(\mathcal{B}(O))$ as $M(\mathbb{K})$. In addition, it is no harm to indicate that $\mathbb{K}$ corresponds to $M(\mathbb{K})$ if $M(\mathbb{K})$ exists, and vice versa.

(2.4) A matroid is uniquely determined by its closure operator by Lemma 2(1).

(2.5) By (2.3), (2.4) and Definition 1(1), if $\mathbb{K}$ corresponds to a matroid $M(\mathbb{K})$, then $''$ (i.e. $A \to A''$ for $A \subseteq O$) of $\mathbb{K}$ is the closure operator $\sigma_{M(\mathbb{K})}$ of $M(\mathbb{K})$.

For convenient, we provide some notations.

(1) If M is a matroid, then σ_M denotes its closure operator.

(2) Sometimes, for a context $\mathbb{K}$, the operations $'$ and $''$ of $\mathbb{K}$ are denoted as $'^{\mathbb{K}}$ and $''^{\mathbb{K}}$ respectively.

(3) If a lattice L_1 is isomorphic to a lattice L_2, then it is in notation $L_1 \cong L_2$.

(4) If T is a set and $\psi : 2^T \to 2^T$ satisfies (S1)-(S3). Then ψ is called a closure operator on T.

3. Geometric lattices

This section will deal with the relationships between the formulated frequently used constructions for two contexts and geometric lattices.

All the discussion in what follows are up to isomorphism.

Let $\mathbb{K} = (O, P, I)$ be a context. We know the following ideas.

(3.1) The operator $''$ (i.e. $A \to A''$ for $A \subseteq O$) is a closure operator of $\mathbb{K}$ (see [2], p.66).

(3.2) The extent lattice $\mathcal{B}(O)$ of $\mathbb{K}$ is determined by the operator $''$ appeared in (3.1).

(3.3) By (3.1), $''$ is a closure operator on O.

After analyzing Definition 1(1), Definition 2, and the operator $''$, we receive the following views for two contexts $\mathbb{K}_j = (O_j, P_j, I_j), (j = 1, 2)$. Let $\dot{\mathbb{K}}_j = (\dot{O}_j, \dot{P}_j, \dot{I}_j), (j = 1, 2)$.

(3.4) $\dot{\mathbb{K}}_j \cong \mathbb{K}_j$, $(j = 1, 2)$.

(3.5) When we consider the property of $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$ and $\mathbb{K}_1 + \mathbb{K}_2$, by (3.4), for simplicity, we can suppose $O_1 \cap O_2 = P_1 \cap P_2 = \emptyset$.

(3.6) In light of (3.4) and Definition 1(2), if it is no confusion, then sometimes, we denote $''^{\mathbb{K}_j}$ as $''^{\dot{\mathbb{K}}_j}$, and also sometimes $''^{\dot{\mathbb{K}}_j}$ as $''^{\mathbb{K}_j}$ $(j = 1, 2)$.

In addition, we find some properties relative to $\mathbb{K}_1 + \mathbb{K}_2$ and $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$ even though they are easily obtained from definitions.

(3.7) If $X \subseteq \dot{O}_1 \cup \dot{O}_2$ is an extent of $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$, then

$X'^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} = (X \cap \dot{O}_1)'^{\mathbb{K}_1} \cup (X \cap \dot{O}_2)'^{\mathbb{K}_2}$,

$X''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} = (X \cap \dot{O}_1)''^{\dot{\mathbb{K}}_1} \cup (X \cap \dot{O}_2)''^{\dot{\mathbb{K}}_2}$. Furthermore, by (3.6), $X''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} = (X \cap \dot{O}_1)''^{\mathbb{K}_1} \cup (X \cap \dot{O}_2)''^{\mathbb{K}_2}$.

(3.8) Let $X \subseteq \dot{O}_1 \cup \dot{O}_2$. Then

$(X \cap \dot{O}_j)'^{\mathbb{K}_1+\mathbb{K}_2} = (X \cap \dot{O}_j)'^{\mathbb{K}_j} \cup \dot{P}_i$, where $j = 1, 2; i \in \{1, 2\} \setminus j$.

$X'^{\mathbb{K}_1+\mathbb{K}_2} = ((X \cap \dot{O}_1)'^{\mathbb{K}_1} \cup \dot{P}_2) \cap ((X \cap \dot{O}_2)'^{\mathbb{K}_2} \cup \dot{P}_1)$;

$X''^{\mathbb{K}_1+\mathbb{K}_2} = (X \cap \dot{O}_1)''^{\mathbb{K}_1} \cup (X \cap \dot{O}_2)''^{\mathbb{K}_2}$.

We start to discover the geometric properties of two context constructions.

Theorem 1. *Let $\mathbb{K}_1 = (O_1, P_1, I_1)$ and $\mathbb{K}_2 = (O_2, P_2, I_2)$ be two contexts. Then*

(1) Both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric if and only if $\mathcal{B}(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)$ is geometric.

(2) Both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric if and only if $\mathcal{B}(\mathbb{K}_1 + \mathbb{K}_2)$ is geometric.

Proof. (1) ($\Rightarrow$) From Lemma 2(3), we can suppose $M(\mathbb{K}_j)$ to be the correspondent matroid to $\mathcal{B}(\mathbb{K}_j), (j = 1, 2)$. By (3.4) and (3.5), we suppose $O_1 \cap O_2 = P_1 \cap P_2 = \emptyset$. In virtue of definition of $\mathbb{K}_1\dot{\cup}\mathbb{K}_2$ in Definition 2, we receive $\dot{O}_1 \cap \dot{O}_2 = \dot{P}_1 \cap \dot{P}_2 = \dot{I}_1 \cap \dot{I}_2 = \emptyset$.

We find that by Lemma 2(5) and $\dot{O}_1 \cap \dot{O}_2 = \dot{P}_1 \cap \dot{P}_2 = \emptyset$, there exist the direct sum $M(\mathbb{K}_1) + M(\mathbb{K}_2)$ of $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ such that $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}(A) = \sigma_{M(\mathbb{K}_1)}(A \cap \dot{O}_1) \cup \sigma_{M(\mathbb{K}_2)}(A \cap \dot{O}_2)$ for any $A \subseteq \dot{O}_1 \times \dot{O}_2$, where $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}$ is the closure operator of $M(\mathbb{K}_1) + M(\mathbb{K}_2)$. Thus, in terms of (2.5) and (3.7), we obtain $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}(A) = (A \cap \dot{O}_1)''^{\mathbb{K}_1} \cup (A \cap \dot{O}_2)''^{\mathbb{K}_2} = A''^{\mathbb{K}_1+\mathbb{K}_2}$. Thereby, $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}$ is $''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2}$. So, $M(\mathbb{K}_1) + M(\mathbb{K}_2)$ corresponds to $\mathcal{B}(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)$. Therefore, $\mathcal{B}(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)$ is geometric since Lemma 2(3).

($\Leftarrow$) Let $\mathcal{B}(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)$ be geometric.

Using Lemma 2(4), $M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j$ is a matroid with $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}$ as its closure operator such that $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}(X) = \sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)}(X) \cap \dot{O}_j$ for any $X \subseteq \dot{O}_j$.

In addition, the closure operator $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)}$ of $M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)$ is $''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2}$ in light of (2.5). Combining with (3.7), we attain $A''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} = A_1''^{\mathbb{K}_1} \cup A_2''^{\mathbb{K}_2}$ where $A_j = A \cap \dot{O}_j, (j = 1, 2)$ for any $A \subseteq \dot{O}_1 \cup \dot{O}_2$. Moreover, there is $A''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} \cap \dot{O}_j = A_j''^{\mathbb{K}_j}, (j = 1, 2)$. With (2.5), this follows $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)}(A) \cap \dot{O}_j = A_j''^{\mathbb{K}_j}, (j = 1, 2)$. So, there is $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)}(X) \cap \dot{O}_j = X''^{\mathbb{K}_j}$ for any $X \subseteq \dot{O}_j$, $(j = 1, 2)$. Thus, $''^{\mathbb{K}_j}$ is $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}$, $(j = 1, 2)$. Thereby, the extent lattice $\mathcal{B}(\dot{O}_j)$ of $\mathbb{K}_j$ is $L(M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j)$ where $L(M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j) = (\{X \subseteq \dot{O}_j \mid \sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}(X) = X\}, \subseteq)$. Indeed, Lemma 2(3) assures $L(M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j)$ to be geometric. Therefore, both of $\mathcal{B}(\dot{O}_j)$ and $\mathcal{B}(\dot{\mathbb{K}}_j)$ are geometric.

According to $\mathbb{K}_j \cong \dot{\mathbb{K}}_j$, we can point that $\mathcal{B}(\mathbb{K}_j)$ is geometric , $(j = 1, 2)$.

(2) ($\Rightarrow$) Since $\mathcal{B}(\mathbb{K}_j)$ is geometric, we obtain the existence of the correspondent matroid $M(\mathbb{K}_j)$, $(j = 1, 2)$. Comparing (3.7) with (3.8), using the similar method in the proof of "$\Rightarrow$" part of (1), we receive that $\mathcal{B}(\mathbb{K}_1 + \mathbb{K}_2)$ is geometric.

($\Leftarrow$) Let $A \subseteq \dot{O}_1 \cup \dot{O}_2$ and $B \subseteq \dot{P}_1 \cup \dot{P}_2$.

$A_1^{\prime\prime\mathbb{K}_1} \cap \dot{O}_2 = A_2^{\prime\prime\mathbb{K}_1} \cap \dot{O}_1 = \emptyset$ follows $A_j^{\prime\prime\mathbb{K}_j} = A_j$ if A_j is an extent of $\mathbb{K}_j$, $(j = 1, 2)$.

In virtue of (3.8), A is an extent of a concept of $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$ if and only if both $A \cap \dot{O}_1$ and $A \cap \dot{O}_2$ are extents of $\mathbb{K}_1$ and $\mathbb{K}_2$ respectively.

Since $\mathcal{B}(\mathbb{K}_1 + \mathbb{K}_2)$ is geometric, by (2.5), we confirm that $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)}$ is ${}^{\prime\prime\mathbb{K}_1+\mathbb{K}_2}$. Moreover, there is $A = \sigma_{M(\mathbb{K}_1+\mathbb{K}_2)}(A) = A^{\prime\prime\mathbb{K}_1+\mathbb{K}_2} = ((A \cap \dot{O}_1) \cup (A \cap \dot{O}_2))^{\prime\prime\mathbb{K}_1+\mathbb{K}_2} = A_1^{\prime\prime\mathbb{K}_1} \cup A_2^{\prime\prime\mathbb{K}_2}$, where $A_j = A \cap \dot{O}_j, (j = 1, 2)$. Hence, we receive $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)}(A_j) \cap \dot{O}_j = A_j^{\prime\prime\mathbb{K}_j}$ for any $A_j \subseteq \dot{O}_j, (j = 1, 2)$.

Owing to Lemma 2(4), $M(\mathbb{K}_1 + \mathbb{K}_2)|\dot{O}_j$ is a matroid with $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_j}$ as its closure operator such that $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_j}(X) = \sigma_{M(\mathbb{K}_1+\mathbb{K}_2)}(X) \cap \dot{O}_j$ for $X \subseteq \dot{O}_j$. Therefore, we attain $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|O_j}(X) = X^{\prime\prime\mathbb{K}_j}$ for any $X \subseteq \dot{O}_j, (j = 1, 2)$. Thus, on $\dot{O}_j, \sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_j}$ is ${}^{\prime\prime\mathbb{K}_j}, (j = 1, 2)$. Since Lemma 2(1) and (3.1)-(3.3), we may indicate that the matroid $M(\mathbb{K}_1 + \mathbb{K}_2)|\dot{O}_j$ corresponds to $\mathcal{B}(\dot{O}_j)$. Hence, by Lemma 2(3), $\mathcal{B}(\mathbb{K}_j)$ is geometric $(j = 1, 2)$.

Before investigating geometric property of $\mathcal{B}(\mathbb{K}_1 \times \mathbb{K}_2)$, we easily provide some properties related to $\mathbb{K}_1 \times \mathbb{K}_2$.

(3.9) Let $A_1 \subseteq O_1$ and $A_2 \subseteq O_2$. Then $(A_1 \times A_2)^{\prime\mathbb{K}_1\times\mathbb{K}_2} = (A_1^{\prime\mathbb{K}_1} \times P_2) \cup (P_1 \times A_2^{\prime\mathbb{K}_2})$.

Let $B_1 \subseteq P_1$ and $B_2 \subseteq P_2$. Then $(B_1 \times B_2)^{\prime\mathbb{K}_1\times\mathbb{K}_2} = (B_1^{\prime\mathbb{K}_1} \times O_2) \cap (O_1 \times B_2^{\prime\mathbb{K}_2})$.

(3.10) According to (3.9) and the definition of $\mathbb{K}_1 \times \mathbb{K}_2$ in Definition 2(2), we may ensure that if we discuss with $\mathbb{K}_1 \times \mathbb{K}_2$, then we can suppose $O_1 \cap O_2 = P_1 \cap P_2 = \emptyset$.

We will investigate the geometric property of $\mathbb{K}_1 \times \mathbb{K}_2$ as follows.

Theorem 2. *Let $\mathbb{K}_j = (O_j, P_j, I_j)$ be a context $(j = 1, 2)$. Then, both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric if and only if $\mathcal{B}(\mathbb{K}_1 \times \mathbb{K}_2)$ is geometric.*

Proof. ($\Rightarrow$) Combining (3.9) and (3.10) with Definition 4(1), we may be assured that in the extent lattice $\mathcal{B}(O_1 \times O_2)$ of $\mathbb{K}_1 \times \mathbb{K}_2$, $(A_1, A_2) \leq (A_3, A_4)$ if and only if $A_1 \leq A_3$ and $A_2 \leq A_4$. Thus, we receive $(A_1, A_2) \vee (A_5, A_6) = (A_1 \vee A_5, A_2 \vee A_6)$ and $(A_1, A_2) \wedge (A_5, A_6) = (A_1 \wedge A_5, A_2 \wedge A_6)$ in $\mathcal{B}(O_1 \times O_2)$.

Let $\mathcal{A}(O_j)$ be the atoms in $\mathcal{B}(O_j)$, $(j = 1, 2)$. Then, utilizing the geometric property of $\mathcal{B}(\mathbb{K}_j)$ and $\mathcal{B}(\mathbb{K}_j) \cong \mathcal{B}(O_j)$, there is $A = \vee_{a_i \in \mathcal{A}(O_i), a_i \leq A_i} a_i$ for any $A \subseteq O_j, (j = 1, 2)$. In addition, according to (3.9), we believe that (a_1, a_2) is an atom of $\mathcal{B}(O_1 \times O_2)$ where $a_j \in \mathcal{A}(O_j), (j = 1, 2)$.

Therefore, we may express that every point (A_1, A_2) in $\mathcal{B}(O_1 \times O_2)$ is the join of atoms, that is, $(A_1, A_2) = \vee_{a_1 \in \mathcal{A}(O_1), a_2 \in \mathcal{A}(O_2), a_i \leq A_i, i=1,2}(a_1, a_2)$.

Let $(A_1, A_2), (A_3, A_4) \in \mathcal{B}(O_1 \times O_2)$ satisfy that both (A_1, A_2) and (A_3, A_4) cover $(A_1, A_2) \wedge (A_3, A_4)$. However, $(A_1, A_2) \wedge (A_3, A_4) = (A_1 \wedge A_3, A_2 \wedge A_4)$ holds. Considering the semimodular property owned by $\mathcal{B}(O_1)$ and $\mathcal{B}(O_2)$ and Definition 4, we obtain that $A_1 \vee A_3$ covers A_1 and A_3, and $A_2 \vee A_4$ covers A_2 and A_4. Moreover, $(A_1 \vee A_3, A_2 \vee A_4)$ covers (A_1, A_2) and (A_3, A_4). Thus, $\mathcal{B}(O_1 \times O_2)$ is semimodular.

Summing up, $\mathcal{B}(O_1 \times O_2)$ is geometric. Thereby, $\mathcal{B}(\mathbb{K}_1 \times \mathbb{K}_2)$ is geometric.

($\Leftarrow$) Let $(A_1, a_2) \subseteq O_1 \times O_2$, where a_2 is an extent and an atom in $\mathcal{B}(\mathbb{K}_2)$; A_1 is an extent in $\mathcal{B}(\mathbb{K}_1)$. Then, (A_1, a_2) is an extent of $\mathcal{B}(\mathbb{K}_1 \times \mathbb{K}_2)$ in light of (3.9).

Let $0_{\mathbb{K}_1 e}$ and $1_{\mathbb{K}_1 e}$ be the extent of the minimum and the maximum of $\mathcal{B}(\mathbb{K}_1)$ respectively. Then, it is easily seen that in the extent lattice $\mathcal{B}(O_1 \times O_2)$ of $\mathbb{K}_1 \times \mathbb{K}_2$, there is $(0_{\mathbb{K}_1 e}, a_2) \leq (A_1, a_2) \leq (1_{\mathbb{K}_1 e}, a_2)$.

Let $(A, B) \in \mathcal{B}(O_1 \times O_2)$ satisfy $(0_{\mathbb{K}_1 e}, a_2) \leq (A, B) \leq (1_{\mathbb{K}_1 e}, a_2)$. Then, $0_{\mathbb{K}_1 e} \leq A \leq 1_{\mathbb{K}_1 e}$ and $a_2 \leq B \leq a_2$. So, we obtain $a_2 = B$. That is to say, the interval $[(0_{\mathbb{K}_1 e}, a_2), (1_{\mathbb{K}_1 e}, a_2)]$ in $\mathcal{B}(O_1 \times O_2)$ is $\{(A, a_2) \mid A \text{ is an extent in } \mathcal{B}(\mathbb{K}_1)\}$. On the other hand, the interval $[(0_{\mathbb{K}_1 e}, a_2), (1_{\mathbb{K}_1 e}, a_2)]$ in $\mathcal{B}(O_1 \times O_2)$ is geometric since Lemma 3(1) and the geometric of $\mathcal{B}(\mathbb{K}_1 \times \mathbb{K}_2)$.

Define a map $\psi : [(0_{\mathbb{K}_1 e}, a_2), (1_{\mathbb{K}_1 e}, a_2)] \to \mathcal{B}(O_1)$ as $(A, a_2) \mapsto A$. It is easily seen that ψ is an isomorphism. Thus, we confirm $[(0_{\mathbb{K}_1 e}, a_2), (1_{\mathbb{K}_1 e}, a_2)] \cong \mathcal{B}(O_1)$. Therefore, $\mathcal{B}(O_1)$ is geometric. So, $\mathcal{B}(\mathbb{K}_1)$ is geometric.

Analogously, $\mathcal{B}(\mathbb{K}_2)$ is geometric.

The following examples will indicate that the following expression is not true: both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric if and only if $\mathcal{B}(\mathbb{K}_1 | \mathbb{K}_2)$ is geometric.

Example 1. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two contexts shown as Table I and Table II respectively. By definition, the context of $\mathbb{K}_1 | \mathbb{K}_2$ is Table III. We may easily receive $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ as shown in Figure 1 and Figure 2 respectively. We easily confirm that both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric.*

Actually, we can obtain the Hasse diagram of $\mathcal{B}(\mathbb{K}_1 | \mathbb{K}_2)$ as Figure 3. From Definition 4, we assure that $\mathcal{B}(\mathbb{K}_1 | \mathbb{K}_2)$ is not geometric.

Table I $\mathbb{K}_1$

	n_1	n_2
u_1	$\times$	
u_2	$\times$	
u_3		$\times$

Table II $\mathbb{K}_2$

	n_3
u_1	$\times$
u_2	
u_3	

Table III *Context* $\mathbb{K}_1 | \mathbb{K}_2$

	$(1, n_1)$	$(1, n_2)$	$(2, n_3)$
u_1	$\times$		$\times$
u_2	$\times$		
u_3		$\times$	

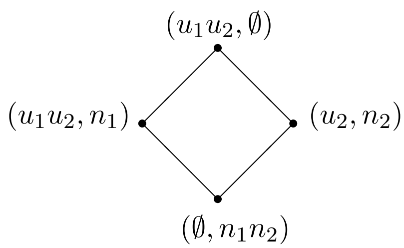

Fig 1
Hasse diagram of $\mathcal{B}(\mathbb{K}_1)$

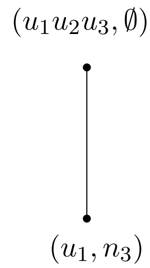

Fig 2
Hasse diagram of $\mathcal{B}(\mathbb{K}_2)$

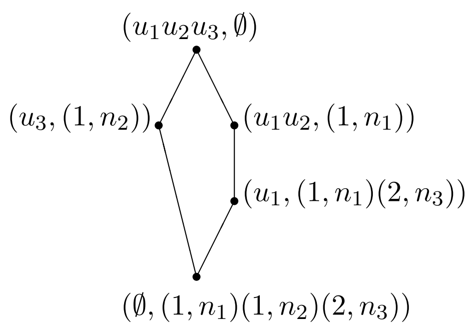

Fig 3
Hasse diagram of $\mathcal{B}(\mathbb{K}_1|\mathbb{K}_2)$

In fact, Example 1 points that "both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric" can not follow "$\mathcal{B}(\mathbb{K}_1|\mathbb{K}_2)$ is geometric".

Example 2. *Let $\mathbb{K}_3$ be shown as Table IV; $\mathbb{K}_4$ be shown as Table V; $\mathbb{K}_3|\mathbb{K}_4$ be shown as Table VI. It is not difficult to obtain the Hasse diagram of $\mathcal{B}(\mathbb{K}_3)$, $\mathcal{B}(\mathbb{K}_4)$ and $\mathcal{B}(\mathbb{K}_3|\mathbb{K}_4)$ respectively as Figure 4 , Figure 5 and Figure 6. From these figures, we indicate that though $\mathcal{B}(\mathbb{K}_3|\mathbb{K}_4)$ is geometric, $\mathcal{B}(\mathbb{K}_4)$ is not geometric.*

Table IV $\mathbb{K}_3$

	m_1	m_2
u_1	×	
u_2		×
u_3	×	×

Table V $\mathbb{K}_4$

	n_1	n_2	n_3
u_1	×		×
u_2	×		
u_3		×	

Table VI *Context* $\mathbb{K}_3|\mathbb{K}_4$

	$(1, m_1)$	$(1, m_2)$	$(2, n_1)$	$(2, n_2)$	$(2, n_3)$
u_1	$\times$		$\times$		$\times$
u_2		$\times$	$\times$		
u_3	$\times$	$\times$		$\times$	

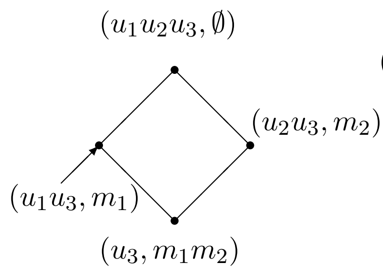

Fig 4
Hasse diagram of $\mathcal{B}(\mathbb{K}_3)$

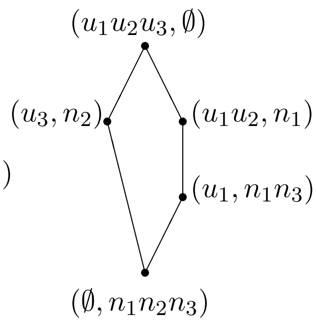

Fig 5
Hasse diagram of $\mathcal{B}(\mathbb{K}_4)$

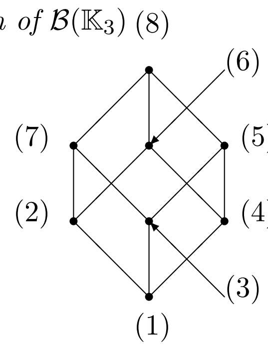

Fig 6
Hasse diagram of $\mathcal{B}(\mathbb{K}_3|\mathbb{K}_4)$

where (1)=$(\emptyset, (1, m_1)(1, m_2)(2, n_1)(2, n_2))$*; (2)=*$(u_1, (1, m_1)(2, n_1)(2, n_3))$*;*
(3)=$(u_3, (1, m_1)(1, m_2)(2, n_2))$*; (4)=*$(u_2, (1, m_2)(2, n_1))$*;*
(5)=$(u_2u_3, (1, m_2))$*; (6)=*$(u_1u_2, (2, n_1))$*;*
(7)=$(u_1u_3, (1, m_1)(2, n_2))$*; (8)=*$(u_1u_2u_3, \emptyset)$.

Actually, Example 2 points that "$\mathcal{B}(\mathbb{K}_3|\mathbb{K}_4)$ is geometric" can not follow "both $\mathcal{B}(\mathbb{K}_3)$ and $\mathcal{B}(\mathbb{K}_4)$ are geometric".

According to [15],[16],[17] or [10],[11], we may infer to the following views.

(3.11) In a finite geometric lattice, every point is a meet of some co-atoms. If L is geometric, then its dual L^d is geometric.

We will find the application of (3.11) in the discussions of relationships between context constructions and geometric lattices.

Let $\mathbb{K} = (O, P, I)$ be a context. Define $\mathbb{K}^d = (P, O, I^{-1})$ where $aIb \Longleftrightarrow bI^{-1}a$. From Definition 1(1) and (3.11), we may easily obtain that if $\mathcal{B}(\mathbb{K})$ is geometric, then its dual $\mathcal{B}(\mathbb{K}^d)$ is also geometric. In addition, we follow $\mathbb{K}^{dd} = \mathbb{K}^d$.

Therefore, when we consider geometric property with $\frac{\mathbb{K}_1}{\mathbb{K}_2} = (\dot{O}_1 \cup \dot{O}_2, P, \dot{I}_1 \cup \dot{I}_2)$, we can pay attention to geometric property of its dual $(\frac{\mathbb{K}_1}{\mathbb{K}_2})^d = (P, \dot{O}_1 \cup \dot{O}_2, (\dot{I}_1 \cup \dot{I}_2)^d) = (P, \dot{O}_1 \cup \dot{O}_2, \dot{I}_1^{-1} \cup \dot{I}_2^{-1}) = \mathbb{K}_1^d \mid \mathbb{K}_2^d$.

Taken Example 1, Example 2 and the above discussions, we may state that if both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric, that is, both $\mathcal{B}(\mathbb{K}_1^d)$ and $\mathcal{B}(\mathbb{K}_2^d)$ are geometric, then we will not follow the geometric property of $\mathcal{B}(\mathbb{K}_1^d \mid \mathbb{K}_2^d)$. This statement implies that $\mathcal{B}(\frac{\mathbb{K}_1}{\mathbb{K}_2})$ is perhaps not geometric. Conversely, if $\mathcal{B}(\frac{\mathbb{K}_1}{\mathbb{K}_2})$ is geometric, then $\mathcal{B}((\frac{\mathbb{K}_1}{\mathbb{K}_2})^d) = \mathcal{B}(\mathbb{K}_1^d \mid \mathbb{K}_2^d)$ is geometric. Considering with Example 2, we may accept that $\mathcal{B}(\mathbb{K}_1^d)$ and $\mathcal{B}(\mathbb{K}_2^d)$ are perhaps not geometric. Furthermore, we can not determine both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ to be geometric.

The following example shows that if both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric, then we can not confirm $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ to be geometric.

Example 3. *Let $\mathbb{K}_5 = (\{u_1, u_2\}, \{m_1, m_2\}, I_5)$ and $\mathbb{K}_6 = (\{v_1, v_2, v_3\}, \{n_1, n_2\}, I_6)$ be shown as Table VII and Table VIII respectively. Then the semiproduct of $\mathbb{K}_5$ and $\mathbb{K}_6$ is shown in Table IX. We may obtain the Hasse diagrams of $\mathcal{B}(\mathbb{K}_5)$ and $\mathcal{B}(\mathbb{K}_6)$ as Figure 7 and Figure 8 respectively. We also obtain the Hasse diagram of $\mathcal{B}(\mathbb{K}_5 \bowtie \mathbb{K}_6)$ as Figure 9. It is easily seen that both $\mathcal{B}(\mathbb{K}_5)$ and $\mathcal{B}(\mathbb{K}_6)$ are geometric, but $\mathcal{B}(\mathbb{K}_5 \bowtie \mathbb{K}_6)$ is not.*

Table VII $\mathbb{K}_5$

	m_1	m_2
u_1	×	
u_2	×	×

Table VIII $\mathbb{K}_6$

	n_1	n_2
v_1		×
v_2	×	×
v_3	×	

Table IX *Context* $\mathbb{K}_5 \bowtie \mathbb{K}_6$

	$(1, m_1)$	$(1, m_2)$	$(2, n_1)$	$(2, n_2)$
(u_1, v_1)	×			×
(u_1, v_2)	×		×	×
(u_1, v_3)	×		×	
(u_2, v_1)	×	×		×
(u_2, v_2)	×	×	×	×
(u_2, v_3)	×	×	×	

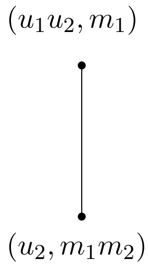

Fig 7
Hasse diagram of $\mathcal{B}(\mathbb{K}_5)$

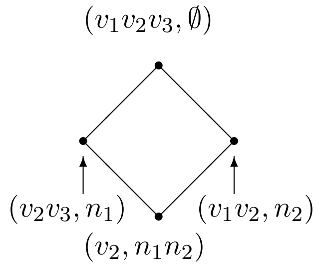

Fig 8
Hasse diagram of $\mathcal{B}(\mathbb{K}_6)$

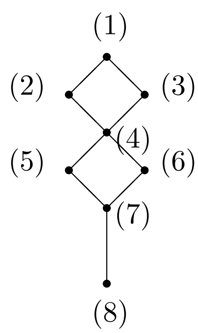

Fig 9
Hasse diagram of $\mathcal{B}(\mathbb{K}_5 \bowtie \mathbb{K}_6)$*)*

where: (1)=$(\{u_1v_1, u_1v_2, u_1v_3, u_2v_1, u_2v_2, u_2v_3\}, (1, m_1))$*;*
(2)=$(\{u_1v_1, u_1v_2, u_2v_1, u_2v_2\}, (2, n_2))$*;*
(3)=$(\{u_1v_2, u_1v_3, u_2v_2, u_2v_3\}, (2, n_1))$*;*
(4)=$(\{u_1v_2, u_2v_2\}, (1, n_1)(2, n_1)(2, n_2))$*;*
(5)=$(\{u_2v_1, u_2v_2, u_2v_3\}, (1, m_1)(1, m_2))$*;*
(6)=$(\{u_2v_1, u_2v_2\}, (1, m_1)(1, m_2)(1, n_2))$*;*
(7)=$(\{u_2v_2, u_2v_3\}, (1, m_1)(1, m_2)(1, n_1))$*;*
(8)=$(u_2v_2, (1, m_1)(1, m_2)(2, n_1)(2, n_2))$.

The following lemma describes that if $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ is geometric, then both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric.

Lemma 4 *Let* $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ *be geometric. Then, both* $\mathcal{B}(\mathbb{K}_1)$ *and* $\mathcal{B}(\mathbb{K}_2)$ *are geometric.*

Proof. Let a_2 be an atom in $\mathcal{B}(O_2)$. Then $A_1 \times a_2$ is an extent in $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ according to Lemma 1(3). Let $0_{\mathbb{K}_1e}, 1_{\mathbb{K}_1e}$ be the minimum and the maximum extent of $\mathbb{K}_1$. Let $X \times Y$ be an extent in $\mathbb{K}_1 \bowtie \mathbb{K}_2$ such that $0_{\mathbb{K}_1e} \times a_2 \leq X \times Y \leq 1_{\mathbb{K}_1e} \times a_2$. Then, we receive $0_{\mathbb{K}_1e} \subseteq X \subseteq 1_{\mathbb{K}_1e}$ and $a_2 \subseteq Y \subseteq a_2$. So, it follows $Y = a_2$. Thereby, the interval $[0_{\mathbb{K}_1e} \times a_2, 1_{\mathbb{K}_1e} \times a_2]$ in the extent lattice $\mathcal{B}(O_1 \times O_2)$ of $\mathbb{K}_1 \bowtie \mathbb{K}_2$ is $\{A \times a_2 \mid A \text{ is an extent of } \mathbb{K}_1\}$. In view of Lemma 3 and the geometric property of $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$, we believe that $[0_{\mathbb{K}_1e} \times a_2, 1_{\mathbb{K}_1e} \times a_2]$ is geometric.

Define $\mu : [0_{\mathbb{K}_1e} \times a_2, 1_{\mathbb{K}_1e} \times a_2] \to \mathcal{B}(O_1)$ as $A \times a_2 \mapsto A$. It is easily to check that μ is an isomorphism. Thus, $\mathcal{B}(O_1)$ is geometric. Furthermore, $\mathcal{B}(\mathbb{K}_1)$ is geometric.

Similarly, we attain that $\mathcal{B}(\mathbb{K}_2)$ is geometric.

Combining Example 3 and Lemma 4, we can state a result as follows.

Theorem 3. *(1) If $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ is geometric, then both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric.*

(2) If both $\mathcal{B}(\mathbb{K}_1)$ and $\mathcal{B}(\mathbb{K}_2)$ are geometric, then $\mathcal{B}(\mathbb{K}_1 \bowtie \mathbb{K}_2)$ can not be determined to be geometric.

4. Matroids

If we find the relationships between matroids and contexts constructions under matroid umbrella, then we infer that some problems for contexts constructions will be simpler sometimes. Therefor, this section will obtain some results under matroid umbrella. The results demonstrate that matroid is a different notation from geometric lattice though they have one-to-one relationships under isomorphism.

Theorem 4. *Let $\mathbb{K}_j = (O_j, P_j, I_j), (j = 1, 2)$ be two contexts. Then, both $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ exist if and only if $M(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2)$ exists.*

Proof. In virtue of (3.5), we can suppose $O_1 \cap O_2 = P_1 \cap P_2 = \emptyset$.

($\Rightarrow$) Owing to Lemma 2(4), we confirm that the direct sum $M(\mathbb{K}_1) + M(\mathbb{K}_2)$ of $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ exists and the closure operator $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}$ satisfies

$$\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}(X) = \sigma_{M(\mathbb{K}_1)}(X \cap \dot{O}_1) \cup \sigma_{M(\mathbb{K}_2)}(X \cap \dot{O}_2).$$

Using (2.5) and the existence of $M(\mathbb{K}_j)$, we obtain that the operation $''^{\mathbb{K}_j}$ of $\mathbb{K}_j$ is $\sigma_{M(\mathbb{K}_j)}$, $(j = 1, 2)$. Thus, we receive $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}(X) = (X \cap O_1)''^{\mathbb{K}_1} \cup (X \cap \dot{O}_2)''^{\mathbb{K}_2}$.

Considering with (3.7), we assure that the operation $''^{\mathbb{K}_1 \dot{\cup} \mathbb{K}_2}$ is $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}$.

Therefore, since (2.3) and Lemma 2(5), we may be assured that there is a matroid corresponding to $\mathbb{K}_1 \dot{\cup} \mathbb{K}_2$ with $\sigma_{M(\mathbb{K}_1)+M(\mathbb{K}_2)}$ as its closure operator. That is to say, $M(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2)$ exists.

($\Leftarrow$) The existence of $M(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2)$ expresses that in terms of Lemma 2(4), Lemma 2(5), and (2.4), the restriction matroid $M(\mathbb{K}_1 \dot{\cup} \mathbb{K}_2)|\dot{O}_j$ owns the closure operator

$\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}$ such that $\sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j}(X) = \sigma_{M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)}(X) \cap \dot{O}_j = X''^{\mathbb{K}_1\dot{\cup}\mathbb{K}_2} \cap \dot{O}_j = (X \cap \dot{O}_j)''^{\mathbb{K}_1}$, $(j = 1, 2)$.

Moreover, $M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2)|\dot{O}_j$ is a matroid corresponding to $\mathbb{K}_j$, $(j = 1, 2)$.

From the proof of Theorem 4, we demonstrate $M(\mathbb{K}_1\dot{\cup}\mathbb{K}_2) = M(\mathbb{K}_1) + M(\mathbb{K}_2)$.

Theorem 5. *Let $\mathbb{K}_1$ and $\mathbb{K}_2$ be two extents. Then both $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ exist if and only if $M(\mathbb{K}_1 + \mathbb{K}_2)$ exists.*

Proof. (⇒) The supposition that both $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ exist implies that the closure operator $\sigma_{M(\mathbb{K}_j)}$ is just $''^{\mathbb{K}_j}$ on $\dot{O}_j, (j = 1, 2)$ according to (2.5) .

Next, we check that $''^{\mathbb{K}_1+\mathbb{K}_2}$ satisfies (S1)-(S4).

By (3.1) and (3.3), we may easily decide that $''^{\mathbb{K}_1+\mathbb{K}_2}$ satisfies (S1)-(S3).

Let $y \notin (X)''^{\mathbb{K}_1+\mathbb{K}_2}$ and $y \in (X \cup x)''^{\mathbb{K}_1+\mathbb{K}_2}$. In view of (3.8), there is $y \in ((X \cup x) \cap \dot{O}_1)''^{\mathbb{K}_1}$ or $y \in ((X \cup x) \cap \dot{O}_2)''^{\mathbb{K}_2}$.

Since $\dot{O}_1 \cap \dot{O}_2 = \emptyset$, we can suppose $x \in \dot{O}_1$.

If $y \in ((X \cup x) \cap \dot{O}_2)''^{\mathbb{K}_2}$, then $y \in (X \cap \dot{O}_2)''^{\mathbb{K}_2}$, a contradiction to the given supposition.

If $y \in ((X \cup x) \cap \dot{O}_1)''^{\mathbb{K}_1}$, then $x \in ((X \cup y) \cap \dot{O}_1)''^{\mathbb{K}_1}$ since $''^{\mathbb{K}_1}$ is $\sigma_{M(\mathbb{K}_1)}$. So, we receive $x \in (X \cup y)''^{\mathbb{K}_1+\mathbb{K}_2}$ in virtue of $(X \cup y)'^{\mathbb{K}_1} \subseteq (X \cup y)''^{\mathbb{K}_1+\mathbb{K}_2}$.

Summarizing, $''^{\mathbb{K}_1+\mathbb{K}_2}$ satisfies (S4).

Therefore, using Lemma 2(1), $''^{\mathbb{K}_1+\mathbb{K}_2}$ is a closure operator of a matroid on $\dot{O}_1 \cup \dot{O}_2$. Furthermore, $M(\mathbb{K}_1 + \mathbb{K}_2)$ exists.

(⇐) In virtue of Lemma 2, we may decide that $M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_1$ is a matroid on $\dot{O}_1$ such that its closure operator $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_1}(A) = \sigma_{M(\mathbb{K}_1+\mathbb{K}_2)}(A) \cap \dot{O}_1$ for any $A \subseteq \dot{O}_1$. Moreover, we obtain $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_1}(A) = A''^{\mathbb{K}_1+\mathbb{K}_2} \cap \dot{O}_1 = (A \cap \dot{O}_1)''^{\mathbb{K}_1} \cup (A \cap \dot{O})_2)''^{\mathbb{K}_2} = (A \cap \dot{O}_1)''^{\mathbb{K}_1} = A''^{\mathbb{K}_1}$ for any $A \subseteq \dot{O}_1$. Therefore, $\sigma_{M(\mathbb{K}_1+\mathbb{K}_2)|\dot{O}_1}$ is $''^{\mathbb{K}_1}$. In other words, $M(\mathbb{K}_1 + \mathbb{K}_2)|\dot{O}_1$ corresponds to $\mathbb{K}_1$.

Analogously, $M(\mathbb{K}_1 + \mathbb{K}_2)|\dot{O}_2$ corresponds to $\mathbb{K}_2$.

Therefore, both $M(\mathbb{K}_1)$ and $M(\mathbb{K}_2)$ exist.

We extend some results in Section 3 to matroid umbrella. In fact, with the one-to-one relationships between geometric lattices and matroids, the other results in Section 3 can be generalized to matroid frameworks by similar approaches to the above.

5. Conclusion

In Section 3 and Section 4, we explore the relationships between context constructions for two contexts and geometric lattices, matroids respectively. We find that some frequently used constructions for two contexts have good relationships with geometric lattices, of course, with matroids. Additionally, we are well known that all these context constructions are good to be scale in solving with many-valued contexts.

With the assitance of this paper, we can directly apply matroids into the study on many-valued contexts as scales. Therefor, the future work is to use these relationships with algorithms and properties of matroids to discover some issues for many-valued contexts.

Though we prove that some context constructions do not own a direct relation with matroids such as apposition, subposition, and so on, we do not discover under what conditions, these constructions can have a relation connecting them and matroids. All these works are left rooms for the future.

Acknowledgements. This research is supported by NSF of China (61073121) and NSF of Hebei Province(A2013201119,A2014201033).

References

[1] R. Wille, *Restructuring lattice theory: an approach based on hierarchies of concept*, In: Rival I. (Eds.): Ordered Sets, Reidel, Dordrecht-Boston, 1982, 445-470.

[2] B. Ganter and R. Wille, *Formal Concept Analysis: Mathematical Foundations*, Springer-Verlag, Berlin, Heidelberg, 1999.

[3] B. Ganter, G. Stumme and R. Wille, *Formal Concept Analysis: Foundations and Applications*, Springer-Verlag, Berlin Heidelberg, 2005.

[4] C. Carpineto and G. Romano, *Concept Data Analysis: Theory and Applications*, John Wiley & Sons Led., England, 2004.

[5] T.B. Kaiser, *On the conncetion between many-valued contexts and general geometric structures*, Annals of Mathematics and Artificial Intelligence, 59, (2010), 169-185.

[6] R. Gugisch, *Many-valued context analysis using descriptions*, In: H.Delugach and G.Stumme (Eds.): ICCS 2002, LNAI 2120, 157-168, 2001, Springer-Verlag, Berlin Heidelberg 2001.

[7] J. Kötters, *Context graphs for many-valued contexts*, 17th International Conference on Conceptual Structures, State Univeristy Higer School of Economics, 26-31 July 2009, Moscow, Russia, http://ceur-ws.org/Vol-483/http://ceur-ws.org/Vol-483/ Accessed on May 5, 2013.

[8] J. Qi, L. Wei, H. Hong, *Concept lattice construction about many-valued contexts*, Proceedings of the 2011 International Conference on Machine Learning and Cybernetics, Gulin, July 10-13, 2011, 1124-1129.

[9] K. Deiters, M. Erné, *Sums, product and negations of contexts and complete lattices*, Algebra Universlis, 60, (2009), 469-496.

[10] J. Oxley, *Matroid Theory*, 2nd edition, Oxford University Press, New York, 2011.

[11] D.J.A. Welsh,*Matroid Theory*, Academic Press Inc., London, 1976.

[12] H. Mao, *The relation between matroid and concept lattice*, Advances in Mathematics (China), 35, 3(2006), 361-365. (in Chinese with English summary)

[13] W. Zhu, S. Wang,*Rough matroid*, 2011 IEEE International Conference on Granular Computing (GrC), Nov. 8-10, Kaohsiung, Taiwan, 2011, 817-824.

[14] S. Wang, W. Zhu, F. Min, *Transversal and function matroidal structures of covering-based rough sets*, In: J.T.Yao et al. (Eds.), Rough Sets and Knowledge Technology, 2011, LNCS 6954, (2011), 146-155.

[15] G. Birkhoff, *Lattice Theory*, 3rd. edition, American Mathematical Society, 1967.

[16] G. Grätzer, *General Lattice Theory*, 2nd. edition, Birkhäuser Verlag, Basel 1998.

[17] G. Grätzer, *Lattice Theory: Foundation*, Springer, Basel, 2011.

[18] P. Becker, J. Toscana, *Conceptual Scaling of Many-Valued Contexts*, http://citeseerx.ist.psu.edu/viewdoc Accessed on April 19, 2014.

[19] J. Liu, X.Q. Yao, *Formal concept analysis of incomplete information system*, Seventh International Conference on Fuzzy Systems and Knowledge Discovery, August 10-12, 2010.

[20] J. Kötters, *Context graphs for many-valued contexts*, http://ceur-ws.org/Vol-483/paper5.pdf Accessed on May 5, 2013.

[21] H. Whitney, *On the abstract properties of linear dependence*, American J. of Mathematics, 57, 3 (1935), 509-533.

[22] M. Wild, *Join epimorphisms which preserve certain lattice identities*, Algebra Universalis, 27, (1990), 398-410.

[23] J. Tang, K. She, W. Zhu, *Matroidal structure of rough sets from the viewpoint of graph theory*, J. of Applied Mathematics, vol.2012, Article ID 973920, 27 pages.

[24] H. Mao, *On closure axioms for a matroid using Galois connections*, Mathematical Communications, 14, 2(2009), 425-432.

[25] H. Mao,*Some relationships between matroids and concept lattices*, East-West J. of Mathematics, 14(1)(2012)43-53.

[26] H. Mao, *On lattice-theoretical construction of matroids*, Annals of the Alexandru Ioan Cuza University-Mathematics, 59, 1(2013), 201-208.

[27] S. Wang, Q. Zhu, W. Zhu, F. Min, *Matroidal structure of rough sets and its characterization to attribute reduction*, Knowledge-Based Systems, 36, (2012), 155-161.

[28] J. Cheng, M. Sitharam, *Maxwell-independence: a new rank estimate for the 3-dimensional generic rigidity matroid*, arXiv:1010.4052v8 [cs.CG] 8 Nov 2013.

[29] M.O. Aciego, J. Outrata (Eds.): *Proceedings of the Tenth International Conference on Concept Lattices and Their Applications*, La Rochelle, October 15-18, 2013. http://www.informatik.uni-trier.de/~ley/db/conf/cla/cla2013.html Accessed on April 16, 2013.

[30] N. Messai, M.D. Devigns, A. Napoli, M.S. Tabbone, *Many-valued concept lattices for concetpual clustering and information retrieval*, In: M.Ghallab et al.(Eds.), ECAI 2008, 127-131.

[31] A. Huang, H. Zhao, W. Zhu, *Nullity-based matroid of rough sets and its application to attribute reduction*, Information Science, 263(2014)153-165.

[32] W. Zhu, S. Wang, *Rough matroids based on relations*, Information Sceinces, 232, (2013), 241-252.

Hua Mao
Department of Mathematics,
Hebei University,
Baoding, China
email: *yushengmao@263.net*