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DEFORMATION OF INDEFINITE TRANS-SASAKIAN AND LSP SASAKIAN MANIFOLDS

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Abstract. The concern of the present paper is to show that indefinite trans-Sasakian manifold and indefinite LSP-Sasakian manifold remain invariant under some deformation, along with an example of each of the manifolds. Also we shall observe some of the properties of these manifolds which remain invariant under the given deformation.

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1. Introduction

A new class of almost contact manifold namely trans-Sasakian manifold was introduced by J.A.Oubina in 1985[8] and further investigation about the local structures of trans-Sasakian manifolds were carried by J.C.Marrero[9]. In 2010, lightlike hypersurfaces of indefinite trans-Sasakian manifolds were introduced by F.Massamba[7]. Also, K.Yano have studied deformation of a Sasakian manifold in [10].
In this paper we shall introduce a particular type of D-deformation on an indefinite trans-Sasakian manifold and indefinite LSP-Sasakian manifold, and studied various properties on them with examples.

2. Preliminaries

Let us recall some of the properties of indefinite trans-Sasakian manifold and indefinite LSP-Sasakian manifold. Let M_1 be an $(2n+1)$-dimensional indefinite almost contact metric manifold with indefinite almost contact metric structure (ϕ, ξ, η, g), where ϕ is a tensor of type $(1,1)$ having rank $2n$, ξ is a vector field, η is a 1-form and g is a associated Riemannian metric, satisfying following properties :

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta \otimes \phi = 0, \quad \phi\xi = 0, \ \eta(\xi) = 1, \tag{1}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2}$$

$$g(X, \xi) = \epsilon\eta(X), \tag{3}$$

for all vector fields X, Y on M_1 . It is easy to observe that $g(\xi, \xi) = \epsilon = \pm 1$. An indefinite almost contact metric structure (ϕ, ξ, η, g) is called an indefinite trans-Sasakian structure of type (α, β) if

$$(\nabla_X \phi)Y = \alpha[g(X, Y)\xi - \epsilon\eta(Y)X] + \beta[g(\phi X, Y)\xi - \epsilon\eta(Y)\phi X], \tag{4}$$

for functions α and β on M_1, where ∇ is the Levi-Civita connection on M_1. On indefinite trans-Sasakian manifold we obtain,

$$\nabla_X \xi = -\alpha\epsilon\phi X + \beta\epsilon[X - \eta(X)\xi], \tag{5}$$

for any $X \in TM_1$ where TM_1 is the Lie algebra of vector fields on M_1.

Let $(M_1, \phi, \xi, \eta, g)$ be an indefinite trans-Sasakian manifold and μ be an mapping such that $\mu = f\xi$ for any function $f \in C^\infty(M_1)$ such that $1 + f \neq 0$. D-deformation is defined in [10], as D be the distribution defined by $\eta = 0$ along with $\mu = f\xi$.

Now we give a brief description about indefinite Lorentzian para-Sasakian manifold.
An n-dimensional differentiable manifold is called indefinite Lorentzian para- Sasakian manifold if the following conditions hold:

$$\phi^2 X = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \quad \eta(\xi) = -1, \tag{6}$$

$$g(\phi X, \phi Y) = g(X, Y) + \epsilon\eta(X)\eta(Y), \tag{7}$$

$$g(X, \xi) = \epsilon\eta(X), \tag{8}$$

for all vector fields X, Y on M_2 [5] and where ϵ is 1 or -1 according as ξ is space-like or time-like vector field.
An indefinite almost metric structure (ϕ, ξ, η, g) is called an indefinite Lorentzian para-Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi + \epsilon\eta(Y)X + 2\epsilon\eta(X)\eta(Y)\xi, \tag{9}$$

∇ is the Levi-Civita $(L - C)$ connection for a semi-Riemannian metric g.
Also for $X \in TM_2$ we have

$$\nabla_X \xi = \epsilon\phi X. \tag{10}$$

An indefinite LP sasakian manifold M_2 is said to be Lorentzian special para-Sasakian manifold if it satisfies

$$F(X,Y) = g(X,Y) + \epsilon\eta(X)\eta(Y), \tag{11}$$

where $F(X,Y) = g(\phi X, Y)$ is a symmetric $(0,2)$ tensor. It can also be verified that

$$F(X,Y) = g(\phi X,Y) = g(X,\phi Y) = g(\phi X,\phi Y) = F(\phi X,\phi Y). \tag{12}$$

In an indefinite LSP sasakian manifold in addition to above properties, also satisfies following properties :

$$\eta(\phi X) = 0, \tag{13}$$

$$rank(\phi) = n-1, \tag{14}$$

$$R(X,Y)\xi = \epsilon[\eta(Y)X - Y\eta(X)]. \tag{15}$$

Let (M_2,ϕ,ξ,η,g) be an indefinite LSP-Sasakian manifold and μ be a mapping such that $\mu = f\xi$ for any function $f \in C^\infty(M_2)$ such that $1+f \neq 0$. As earlier said D-deformation is defined by $\eta = 0$ along with $\mu = f\xi$.

3. Some results on indefinite trans-Sasakian manifold under D-deformation

Theorem 1. *In an indefinite trans-Sasakian manifold the following relations hold :*

$$L_\mu g = g((Xf)\xi, Y) + g(X,(Yf)\xi) + 2f\beta\epsilon g(\phi X,\phi Y), \tag{16}$$

$$[\mu,\xi] = -(f\xi)\xi, \tag{17}$$

$$1+\eta(\mu) = 1+f \neq 0. \tag{18}$$

Proof. Let (M_1,ϕ,ξ,η,g) is an indefinite trans-Sasakian manifold and μ be a vector field over M_1, where $\mu = f\xi$ for some $f \in C^\infty(M_1)$.

Then,

$$(L_\mu g)(X,Y) = L_\mu g(X,Y) - g(L_\mu X,Y) - g(X,L_\mu Y).$$

$$(L_\mu g)(X,Y) = \mu g(X,Y) - g([\mu,X],Y) - g(X,[\mu,Y]).$$

Again using $T(\xi,X) = 0$ and $(\nabla_\xi g)(X,Y) = 0$, we obtain that

$$(L_\mu g)(X,Y) = g(\nabla_X\mu,Y) + g(X,\nabla_Y\mu),$$

$$= g(\nabla_X f\xi,Y) + g(X,\nabla_Y f\xi)$$

$$= g(f\nabla_X\xi, Y) + g((Xf)\xi, Y) + g(X, f\nabla_Y\xi) + g(X, (Yf)\xi).$$

Using (2.5) and $g(X, \phi Y) = -g(\phi X, Y)$ and $\eta = 0$ and having some brief calculations we obtain

$$(L_\mu g)(X, Y) = g((Xf)\xi, Y) + g(X, (Yf)\xi) + 2\beta\epsilon f g(X, Y).$$

Using (2.2) we can write

$$(L_\mu g)(X, Y) = g((Xf)\xi, Y) + g(X, (Yf)\xi) + 2\beta\epsilon f g(\phi X, \phi Y)$$

, since $\eta = 0$. Now,

$$[\mu, \xi] = [f\xi, \xi] = f[\xi, \xi] - (f\xi)\xi = -(f\xi)\xi,$$

since $[\xi, \xi] = 0$. Again,

$$1 + \eta(\mu) = 1 + \eta(f\xi) = 1 + \epsilon g(f\xi, \xi) = 1 + \epsilon f g(\xi, \xi) = 1 + \epsilon^2 f = 1 + f.$$

Corollary 2. *On using $f = a$ where a is a constant function we can obtain*

$$L_\mu g = 2\beta\epsilon a g(\phi X, \phi Y),$$

$$[\mu, \xi] = 0,$$

$$1 + \eta(\mu) = 1 + a \neq 0.$$

Proof. The proof follows from [4].

The indefinite trans-Sasakian structure after D-deformation denoted by $(\tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{\xi})$ is given by

$$\tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{\xi}), \tag{19}$$

$$\tilde{\eta} = \frac{1}{1 + \eta(\mu)}\eta, \tag{20}$$

$$\tilde{\xi} = \xi + \mu, \tag{21}$$

$$\tilde{g}(X, Y) = \frac{1}{1 + \eta(\mu)} g(X - \tilde{\eta}(X)\tilde{\xi}, Y - \tilde{\eta}(Y)\tilde{\xi}) + \epsilon\tilde{\eta}(X)\tilde{\eta}(Y). \tag{22}$$

4. Invariance of an indefinite trans-Sasakian manifold under D-deformation

Now we give the following theorem:

Theorem 3. *Let D be the deformation on an indefinite trans-Sasakian manifold $(M_1, \phi, \xi, \eta, g)$ then $(M_1, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{\xi})$ is also an indefinite trans-Sasakian manifold .*

Proof. Since D is a the distribution defined by $\eta = 0$ i.e. for any $X \in D$, $\eta(X) = 0$. Now our task is to verify the properties of an indefinite trans-Sasakian manifold w.r.t the deformation μ. Using (2.1), (3.5) and (3.6) we obtain

$$\tilde{\eta}(\tilde{\xi}) = \frac{1}{1+\eta(\mu)}\eta(\xi+\mu) = \frac{1}{1+\eta(\mu)}(1+\eta(\mu)) = 1. \tag{23}$$

Next using (3.4), (3.6) and (3.8) we infer

$$\tilde{\phi}(\tilde{\xi}) = \phi(\xi - \tilde{\eta}(\xi)\tilde{\xi}) = 0. \tag{24}$$

Let X be a vector field which belongs to D, then

$$\tilde{\eta}(X) = \frac{1}{1+\eta(\mu)}\eta(X) = 0, \tag{25}$$

since $\eta = 0$. For $X, Y \in D$ we obtain from above definitions (3.4), (3.5), (3.6) and (3.7)

$$\tilde{\phi}(X) = \phi(X), \tag{26}$$

$$\tilde{g}(X,Y) = \frac{1}{1+\eta(\mu)}g(X,Y), \tag{27}$$

since $\tilde{\eta} = 0$ for all $X, Y \in D$. Using (3.5) and (3.11), for $X \in D$ we have,

$$\tilde{\eta}(\tilde{\phi})(X) = \tilde{\eta}\phi(X) = \frac{1}{1+\eta(\mu)}\eta\phi(X) = 0. \tag{28}$$

Again using (3.11), for $X \in D$ we get

$$\tilde{\phi}^2(X) = \tilde{\phi}(\tilde{\phi}(X)) = \phi^2(X) = -X + \tilde{\eta}(X)\tilde{\xi}. \tag{29}$$

According to the definition of $\tilde{g}$ and from (3.15) we obtain for $X, Y \in D$,

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \frac{1}{1+\eta(\mu)}g(\tilde{\phi}X - \tilde{\eta}(\tilde{\phi}X)\tilde{\xi}, Y - \tilde{\eta}(\tilde{\phi}Y)\tilde{\xi}) + \epsilon\tilde{\eta}(\tilde{\phi}X)\tilde{\eta}(\tilde{\phi}Y). \tag{30}$$

$$= \frac{1}{1+\eta(\mu)} g(\phi X, \phi Y) = \frac{1}{1+\eta(\mu)} g(X, Y).$$

Using (3.10) and (3.12) we have

$$\tilde{g}(\tilde{\phi} X, \tilde{\phi} Y) = \tilde{g}(X, Y) - \epsilon \tilde{\eta}(X) \tilde{\eta}(Y). \tag{31}$$

Again we can obtain on using (3.11) and (3.12),

$$\tilde{g}(\tilde{\phi} X, Y) + \tilde{g}(X, \tilde{\phi} Y) = \tilde{g}(\phi X, Y) + \tilde{g}(X, \phi Y) = \frac{1}{1+\eta(\mu)} [g(\phi X, Y) + g(X, \phi Y)] = 0.$$

Hence

$$\tilde{g}(\tilde{\phi} X, Y) + \tilde{g}(X, \tilde{\phi} Y) = 0. \tag{32}$$

Now replacing Y by $\tilde{\xi}$ in (3.7) and using (3.8) we get,

$$\tilde{g}(X, \tilde{\xi}) = \frac{1}{1+\eta(\mu)} [g(X - \tilde{\eta}(X)\tilde{\xi}, \tilde{\xi} - \tilde{\eta}(\tilde{\xi})\tilde{\xi})] + \epsilon \tilde{\eta}(X) \tilde{\eta}(\tilde{\xi}) = \epsilon \tilde{\eta}(X).$$

$$\tilde{g}(X, \tilde{\xi}) = \epsilon \tilde{\eta}(X). \tag{33}$$

For $X, Y \in D$ we obtain from (3.11),

$$(\nabla_X \tilde{\phi}) Y = \nabla_X \tilde{\phi}(Y) - \tilde{\phi}(\nabla_X Y) = (\nabla_X \phi) Y. \tag{34}$$

Now,

$$\alpha[\tilde{g}(X, Y)\tilde{\xi} - \epsilon \tilde{\eta}(Y) X] + \beta[\tilde{g}(\tilde{\phi} X, Y)\tilde{\xi} - \epsilon \tilde{\eta}(Y) \tilde{\phi} X] \tag{35}$$

$$= \alpha \tilde{g}(X, Y)\tilde{\xi} + \beta g(\tilde{\phi} X, Y)\tilde{\xi},$$

[using (3.10)]

$$= \frac{1}{1+\eta(\mu)} (\xi + f\xi)[\alpha g(X, Y) + \beta g(\phi X, Y)],$$

[using (3.6) and (3.7)]

$$= \frac{1}{1+\eta(\mu)} (1+f)[\alpha g(X, Y) + \beta g(\phi X, Y)]\xi,$$

$$= \frac{1}{1+\eta(\mu)} (1+f)(\nabla_X \tilde{\phi}) Y.$$

[using (3.20) and $\eta = 0$] Therefore from (3.20) and (3.21) we have,

$$(\nabla_X\tilde{\phi})Y = \frac{1+\eta(\mu)}{1+f}\alpha[\tilde{g}(X,Y)\tilde{\xi} - \epsilon\tilde{\eta}(Y)X] + \beta[\tilde{g}(\tilde{\phi}X,Y)\tilde{\xi} - \epsilon\tilde{\eta}(Y)\tilde{\phi}X]$$

$$(\nabla_X\tilde{\phi})Y = \alpha[\tilde{g}(X,Y)\tilde{\xi} - \epsilon\tilde{\eta}(Y)X] + \beta[\tilde{g}(\tilde{\phi}X,Y)\tilde{\xi} - \epsilon\tilde{\eta}(Y)\tilde{\phi}X]. \quad (36)$$

Hence we have proved that $(M_1, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an indefinite trans-Sasakian manifold.

5. Example of a 3-dimensional indefinite trans-Sasakian manifold which remains invariant under D-deformation

Example of a 3-dimensional indefinite trans-Saskian manifold is given in [13], where $M_1 = \{(x,y,z) \in R^3 : z \neq 0\}$, (x,y,z) are the standard coordinate of R^3, and

$$e_1 = e^z\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, e_2 = e^z\frac{\partial}{\partial y}, e_3 = \frac{\partial}{\partial z},$$

which are linearly independent vector fields at each point of M_1. Define a semi-Riemannian metric g on M_1 as

$$g(e_1,e_3) = g(e_2,e_3) = g(e_1,e_2) = 0,$$

$$g(e_1,e_1) = g(e_2,e_2) = g(e_3,e_3) = \epsilon,$$

where $\epsilon = \pm 1$.

Let η be a 1-form defined by $\eta(z) = \epsilon g(z, e_3)$, for any $z \in (TM_1)$ and ϕ be the tensor field of type $(1,1)$ defined by

$$\phi e_1 = e_2, \quad \phi e_2 = -e_1, \phi e_3 = 0.$$

Then by applying linearity of ϕ and g, we have

$$\eta(e_3) = 1, \quad \phi^2 Z = -Z + \eta(Z)e_3, \quad g(\phi Z, \phi U) = g(Z,U) - \epsilon\eta(Z)\eta(U)$$

for any $Z, U \in (TM_1)$. Hence for $e_3 = \xi$, $(\phi, \xi, \eta, g, \epsilon)$ defines an (ϵ)-almost contact metric structure M_1. Authors have shown that (ϕ, ξ, η, g) is an (ϵ)- trans-Sasakian manifold in [13]. Now under the deformation D,

$$\tilde{\phi}(e_1) = \phi(e_1 - \eta(e_1)\tilde{\xi}) = \phi(e_1) = e_2, \eta(e_1) = \epsilon g(e_1, e_3) = 0.$$

Similarly,

$$\tilde{\phi}(e_2) = \phi(e_2) = -e_1, \tilde{\phi}(e_3) = \phi(0) = 0.$$

So under D-deformation (ϵ)-trans-Sasakian structure on M_1 remains invariant.

6. Some results on indefinite Lorentzian special para-Sasakian manifold under D-deformation

Theorem 4. *In an indefinite Lorentzian special para-Sasakian manifold the following relations hold:*

$$L_\mu g = g((Xf)\xi, Y) + g(X, (Yf)\xi) + 2f\epsilon F(\phi X, \phi Y), \tag{37}$$

$$[\mu, \xi] = -(f\xi)\xi, \tag{38}$$

$$1 - \eta(\mu) = 1 + f \neq 0. \tag{39}$$

$$R(\xi, \mu)\xi = 0. \tag{40}$$

$$F(\xi, \mu) = 0. \tag{41}$$

Proof. Let $(M_2, \phi, \xi, \eta, g)$ is an indefinite Lorentzian special para-Sasakian manifold and μ be a vector field over M_2, where $\mu = f\xi$ for some $f \in C^\infty(M_2)$.
Then,

$$(L_\mu g)(X,Y) = L_\mu g(X,Y) - g(L_\mu X, Y) - g(X, L_\mu Y).$$

$$(L_\mu g)(X,Y) = \mu g(X,Y) - g([\mu, X], Y) - g(X, [\mu, Y]).$$

Again using $T(\xi, X) = 0$ and $(\nabla_\xi g)(X,Y) = 0$, we can deduce that

$$(L_\mu g)(X,Y) = g(\nabla_X \mu, Y) + g(X, \nabla_Y \mu),$$

$$= g(\nabla_X f\xi, Y) + g(X, \nabla_Y f\xi)$$

$$= g(f\nabla_X \xi, Y) + g((Xf)\xi, Y) + g(X, f\nabla_Y \xi) + g(X, (Yf)\xi).$$

From (2.10) and $g(X, \phi Y) = g(\phi X, Y)$ and $\eta = 0$ and having some brief calculations we obtain

$$(L_\mu g)(X,Y) = g((Xf)\xi, Y) + g(X, (Yf)\xi) + 2f\epsilon F(X,Y),$$

Using (2.12) we can write

$$(L_\mu g)(X,Y) = g((Xf)\xi, Y) + g(X, (Yf)\xi) + 2f\epsilon F(\phi X, \phi Y),$$

since $\eta = 0$. Again,

$$[\mu, \xi] = [f\xi, \xi] = f[\xi, \xi] - (f\xi)\xi = -(f\xi)\xi,$$

$[\xi, \xi] = 0$.
We also have,

$$1 - \eta(\mu) = 1 - \eta(f\xi) = 1 - \epsilon g(f\xi, \xi) = 1 - \epsilon f g(\xi, \xi) = 1 - \epsilon^2 f\eta(\xi) = 1 + f.$$

Again we know,

$$R(\xi,\mu)\xi = \epsilon[\eta(\mu)\xi - \eta(\xi)\mu].$$

Again using $\mu = f\xi$ and (2.8) we get

$$R(\xi,\mu)\xi = 0.$$

From (2.11) we can write

$$F(\xi,\mu) = g(\xi,\mu) + \epsilon\eta(\xi)\eta(\mu) = 0,$$

as $\mu = f\xi$ and from (2.6) and (2.8).

We can give a corollary from the above theorem:

Corollary 5. *In the above theorem if f is a constant function a then*

$$L_\mu g = 2aF(\phi X, \phi Y),$$

$$[\mu,\xi] = 0,$$

$$1 - \eta(\mu) = 1 + a \neq 0.$$

$$F(\xi,\mu) = 0.$$

$$R(\xi,\mu)\xi = 0.$$

Proof. The proof follows very trivially.

Let us consider the indefinite Lorentzian special para-Sasakian structure after D-deformation denoted by $(\tilde{\phi},\tilde{\xi},\tilde{\eta},\tilde{\xi})$ is defined by

$$\tilde{\phi}(X) = \phi(X - \tilde{\eta}(X)\tilde{\xi}), \tag{42}$$

$$\tilde{\eta} = \frac{1}{1-\eta(\mu)}\eta, \tag{43}$$

$$\tilde{\xi} = \xi + \mu, \tag{44}$$

$$\tilde{g}(X,Y) = \frac{1}{1-\eta(\mu)}g(X + \tilde{\eta}(X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi}) + \epsilon\tilde{\eta}(X)\tilde{\eta}(Y), \tag{45}$$

$$\tilde{F}(X,Y) = \frac{1}{1-\eta(\mu)}F(X + \tilde{\eta}(X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi}) + \epsilon\tilde{\eta}(X)\tilde{\eta}(Y). \tag{46}$$

7. Invariance of an indefinite Lorentzian special para-Sasakian manifold under D-deformation

Now we prove the following theorem:

Theorem 6. *Let D be the deformation on an indefinite Lorentzian special para-Sasakian manifold $(M_2, \phi, \xi, \eta, g)$ then $(M_2, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is also an indefinite Lorentzian special para-Sasakian manifold.*

Proof. As D is a distribution defined by $\eta = 0$ i.e. for any $X \in D$, $\eta(X) = 0$. We shall now verify the properties of an indefinite Lorentzian special para-Sasakian manifold w.r.t the deformation μ.
Using (4.7) we obtain

$$\tilde{\eta}(\tilde{\xi}) = \frac{1}{1-\eta(\mu)}\eta(\xi+\mu),$$

$$= \frac{1}{1-\eta(\mu)}(-1+\eta(\mu)),$$

$$\tilde{\eta}(\tilde{\xi}) = -1. \tag{47}$$

as $\eta(\xi) = -1$. Again from (4.6), (4.7) and (4.11) we infer,

$$\tilde{\phi}(\tilde{\xi}) = \phi(\xi - \tilde{\eta}(\xi)\tilde{\xi}),$$

$$\tilde{\phi}(\tilde{\xi}) = 0. \tag{48}$$

Let X be a vector field which belongs to D, so

$$\tilde{\eta}(X) = \frac{1}{1-\eta(\mu)}\eta(X) = 0, \tag{49}$$

since $\eta = 0$.
For $X, Y \in D$ we have from above definitions we get

$$\tilde{\phi}(X) = \phi(X), \tag{50}$$

$$\tilde{g}(X,Y) = \frac{1}{1-\eta(\mu)}g(X,Y), \tag{51}$$

since $\tilde{\eta} = 0$ for all $X, Y \in D$.

Using (4.7) and (4.14), for $X \in D$ we can obtain,

$$\tilde{\eta}(\tilde{\phi})(X) = \tilde{\eta}\phi(X) = \frac{1}{1-\eta(\mu)}\eta\phi(X) = 0,$$

$$\tilde{\eta}(\tilde{\phi})(X) = 0. \tag{52}$$

Again from (4.14), for $X \in D$ we infer

$$\tilde{\phi}^2(X) = \tilde{\phi}(\tilde{\phi}(X)) = \phi^2(X) = X + \tilde{\eta}(X)\tilde{\xi}. \tag{53}$$

From the definition of $\tilde{g}$ and from (4.9) and (4.13) we compute for $X, Y \in D$,

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \frac{1}{1-\eta(\mu)} g(\tilde{\phi}X - \tilde{\eta}(\tilde{\phi}X)\tilde{\xi}, Y - \tilde{\eta}(\tilde{\phi}Y)\tilde{\xi}) + \epsilon\tilde{\eta}(\tilde{\phi}X)\tilde{\eta}(\tilde{\phi}Y).$$

$$= \frac{1}{1-\eta(\mu)} g(\phi X, \phi Y) = \frac{1}{1-\eta(\mu)} g(X, Y).$$

$$\tilde{g}(\tilde{\phi}X, \tilde{\phi}Y) = \tilde{g}(X, Y) + \epsilon\tilde{\eta}(X)\tilde{\eta}(Y). \tag{54}$$

On replacing Y by $\tilde{\xi}$ in (4.9) and using (4.11),

$$\tilde{g}(X, \tilde{\xi}) = \frac{1}{1-\eta(\mu)} g(X + \tilde{\eta}(X)\tilde{\xi}, \tilde{\xi} + \tilde{\eta}(\tilde{\xi})\tilde{\xi}) + \epsilon\tilde{\eta}(X)\tilde{\eta}(\tilde{\xi}).$$

$$\tilde{g}(X, \tilde{\xi}) = \epsilon\tilde{\eta}(X). \tag{55}$$

Now,

$$\tilde{g}(\tilde{\phi}X, Y) = \frac{1}{1-\eta(\mu)} g(\tilde{\phi}X + \tilde{\eta}(\tilde{\phi}X)\tilde{\xi}, Y + \tilde{\eta}(Y)\tilde{\xi}),$$

Using (4.7), (4.14) and (4.17), we obtain

$$\tilde{g}(\tilde{\phi}X, Y) = \frac{1}{1-\eta(\mu)} g(X, \phi Y), \tag{56}$$

Again from (2.12), (4.19) and (4.20) we can calculate

$$\tilde{g}(\tilde{\phi}X, Y) = \tilde{g}(X, \tilde{\phi}Y), \tag{57}$$

since $\eta = 0$.

From (2.12), (4.10) and (4.13) we get

$$\tilde{F}(X, Y) = \frac{1}{1-\eta(\mu)} g(X, Y), \tag{58}$$

Using (4.15) and (4.23) we obtain

$$\tilde{F}(X, Y) = \tilde{g}(X, Y) + \epsilon\tilde{\eta}(X)\tilde{\eta}(Y). \tag{59}$$

For $X, Y \in D$ we obtain from (3.11),

$$(\nabla_X \tilde{\phi})Y = \nabla_X \tilde{\phi}(Y) - \tilde{\phi}(\nabla_X Y) = (\nabla_X \phi)Y. \tag{60}$$

We have,

$$\tilde{g}(X,Y)\tilde{\xi} + \epsilon\tilde{\eta}(Y)X + 2\epsilon\tilde{\eta}(Y)\tilde{\eta}(X)\tilde{\xi}, \tag{61}$$

$$= \tilde{g}(X,Y)\tilde{\xi},$$

[using (4.13)]

$$= \frac{1}{1-\eta(\mu)}(\xi + f\xi)g(X,Y),$$

[using (3.6) and (3.7)]

$$= \frac{1}{1-\eta(\mu)}(1+f)g(X,Y)\xi,$$

$$= \frac{1}{1-\eta(\mu)}(1+f)(\nabla_X \tilde{\phi})Y,$$

Hence using (4.24) and (4.25),

$$(\nabla_X \tilde{\phi})Y = \frac{1}{1-\eta(\mu)}(1+f)\tilde{g}(X,Y)\tilde{\xi} + \epsilon\tilde{\eta}(Y)X + 2\epsilon\tilde{\eta}(Y)\tilde{\eta}(X)\tilde{\xi}.$$

$$(\tilde{\nabla}_X \tilde{\phi})Y = \tilde{g}(X,Y)\tilde{\xi} + \epsilon\tilde{\eta}(Y)X + 2\epsilon\tilde{\eta}(X)\tilde{\eta}(Y)\tilde{\xi}. \tag{62}$$

Hence $(M_2, \tilde{\phi}, \tilde{\xi}, \tilde{\eta}, \tilde{g})$ is an indefinite Lorentzian special para-Sasakian manifold.

8. Example of a 3-dimensional indefinite Lorentzian special para-Sasakian manifold which remains invariant under D-deformation

Example of a 3-dimensional indefinite Lorentzian special para-Sasakian is given in [6], where $M_2 = \{(x,y,z) \in R^3 : z \neq 0\}$, (x,y,z) are the standard coordinate of R^3, and

$$e_1 = e^z \frac{\partial}{\partial y}, e_2 = e^z(\frac{\partial}{\partial x} + \frac{\partial}{\partial z}), e_3 = \frac{\partial}{\partial z},$$

which are linearly independent vector fields at each point of M_2. Define a semi-Riemannian metric g on M_2 is given as

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0,$$

$$g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = -\epsilon,$$

where $\epsilon = \pm 1$. Let η be a 1-form defined by $\eta(z) = \epsilon g(z, e_3)$, for any $z \in (TM_2)$ and ϕ be the tensor field of type $(1, 1)$ defined by

$$\phi e_1 = -e_1, \phi e_2 = -e_2, \phi e_3 = 0.$$

Then by applying linearity of ϕ and g, we have

$$\eta(e_3) = -1, \phi^2 Z = Z + \eta(Z)e_3, g(\phi Z, \phi U) = g(Z, U) + \epsilon\eta(Z)\eta(U),$$

for any $Z, U \in (TM_2)$. Hence for $e_3 = \xi$, $(\phi, \xi, \eta, g, \epsilon)$ defines an (ϵ)-almost contact metric structure M_2. Authors have shown that (ϕ, ξ, η, g) is an indefinite Lorentzian para-Sasakian manifold in [14].We can also verify that

$$F(X, Y) = g(X, Y) + \epsilon\eta(X)\eta(Y).$$

Therefore (ϕ, ξ, η, g) is an indefinite Lorentzian special para-Sasakian manifold. Now under the deformation D,

$$\tilde{\phi}(e_1) = \phi(e_1 + \eta(e_1)\tilde{\xi}) = \phi(e_1 + \eta(e_1)\xi) = \phi(e_1)$$

$$= -e_1, \because \eta(e_1) = \epsilon g(e_1, e_3).$$

Similarly,

$$\tilde{\phi}(e_2) = \phi(e_2) = -e_2, \tilde{\phi}(e_3) = \phi(0) = 0.$$

So under D-deformation indefinite Lorentzian special para-Sasakian structure on M_2 remains invariant.

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