Acta Universitatis Apulensis No. 40/2014
ISSN: 1582-5329 pp. 199-217
http://www.uab.ro/auajournal/ doi: 10.17114/j.aua.2014.40.17

UNIVALENT HARMONIC FUNCTIONS WITH DOMAINS CONVEX IN HORIZONTAL (VERTICAL) DIRECTIONS

P. Sharma, Om P. Ahuja, V.K. Gupta

Abstract. In this paper, using the shear construction method, we define two subclasses of harmonic univalent functions in the unit disk that are the harmonic shear of analytic functions and convex in the horizontal or vertical direction. For these classes, certain equivalent conditions and convolution conditions are obtained. Finally, inequalities that are both necessary and sufficient for the harmonic shears of analytic functions involving Wright's generalized hypergeometric functions are derived.

2000 *Mathematics Subject Classification*: 30C45, 30C55.

Keywords: Univalent harmonic functions, Convex functions, Wright's generalized hypergeometric function, Subordination.

1. Introduction and preliminaries

Let $\mathcal{S}_H$ denotes a class of functions f which are harmonic, univalent and orientation preserving in the open unit disc $\Delta = \{z : |z| < 1\}$ and are normalized by $f(0) = h(0) = f_z(0) - 1 = 0$. Since Δ is simply connected, a function $f \in \mathcal{S}_H$ has the canonical representation given by $h + \overline{g}$, where h and g are the members of linear space $A(\Delta)$ of all analytic functions in Δ and where h and g can be written as a power series representation

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, g(z) = \sum_{n=1}^{\infty} b_n z^n, |b_1| < 1. \tag{1}$$

We call h the analytic part and g the co-analytic part of f. A necessary and sufficient condition for a harmonic function of the form $f = h + \overline{g}$ to be locally univalent and sense preserving in Δ is that $|g'(z)| < |h'(z)|$ for all z in Δ. The analytic dilatation of a harmonic mapping $f = h + \overline{g}$ is defined by $\omega(z) = (g'(z)/h'(z))$. Thus if f is locally univalent and sense preserving, then $|\omega(z)| < 1$ in Δ.

A subclass $T\mathcal{S}_H$ of $\mathcal{S}_H$ is well known in the literature. A function $f = h + \overline{g}$ is said to be in the class $T\mathcal{S}_H$ if h and g are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| \, z^n \text{ and } g(z) = \sum_{n=1}^{\infty} |b_n| \, z^n, |b_1| < 1. \tag{2}$$

In case $g(z) = 0$, $\forall z \in \Delta$, the class $\mathcal{S}_H$ reduces to a well known class $\mathcal{S}$ of univalent functions and the class $\mathcal{TS}_H$ reduces to $\mathcal{T}$ introduced and studied by Silverman [18, 19]. We further denote a subclass $\mathcal{TS}_H^0$ of $\mathcal{T}S_H$ for which $f_{\overline{z}}(0) = 0$.

Recall that a domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex in the direction α $(0 \leq \alpha < 2\pi)$, if for all $a \in \mathbb{C}$, the set $\mathbb{D} \cap \{a + te^{i\alpha} : t \in \mathbb{R}\}$ is either connected or empty. In particular, a domain $\mathbb{D} \subset \mathbb{C}$ is said to be convex in the horizontal direction (or a CHD domain) if its intersection with each horizontal line is connected (or empty). The domains which are convex in every direction are called convex domains.

We say a univalent harmonic function f is convex in the direction α $(0 \leq \alpha < 2\pi)$ if the domain $f(\mathbb{D})$ is convex in the direction α. In particular, a univalent harmonic function f is called a CHD map if its range is a CHD domain.

Construction of a univalent harmonic mapping f with prescribed dilatation ω can be done effectively by a method known as the "*shear construction*" method which was devised by Clunie and Sheil-Small [8] (see also [9, 10, 11, 15]). The basic shear construction theorem of a harmonic univalent function discovered by Clunie and Sheil-Small [8] is as follows.

Theorem A: *For analytic functions h and g, assume the harmonic function $f = h + \overline{g}$ is locally univalent in a simply connected domain $\mathbb{D}$. Then a univalent function f maps $\mathbb{D}$ onto a CHD domain if and only if the analytic function $h - g$ is univalent and maps $\mathbb{D}$ onto a CHD domain.*

We note that, if φ is a CHD map, for a given dilatation ω, the harmonic shear $f = h + \overline{g} = h - g + 2\Re\{g\}$ of φ can be obtained by solving the differential equations:

$$h' - g' = \varphi', \ \omega h' - g' = 0 \tag{3}$$

with normalizations $h(0) = \varphi(0)$ and $g(0) = 0$. Also, if κ is a map convex in vertical direction, for a given dilatation ω, the harmonic shear $f = h + \overline{g} = h - g + 2\Re\{g\}$ of κ is obtained by solving the differential equations:

$$h' + g' = \kappa', \ \omega h' - g' = 0 \tag{4}$$

with normalizations $h(0) = \kappa(0)$ and $g(0) = 0$.

We also have following result of Clunie and Sheil-Small [8].

Theorem B: *A functions $f = h + \overline{g}$ is harmonic convex if and only if the analytic functions $h - e^{i\alpha}g$, $0 \leq \alpha < 2\pi$, are convex in the direction $\frac{\alpha}{2}$ and f is suitably normalized.*

We mention here two examples given in [8] of constructing harmonic functions by shearing the analytic functions in the horizontal direction and in the direction of $\frac{\pi}{2}$ that is in the vertical direction, respectively, as follows:

Example 1. *The harmonic Koebe function $K = h + \overline{g}$ where*

$$h(z) = \frac{z - \frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, g(z) = \frac{\frac{1}{2}z^2 + \frac{1}{6}z^3}{(1-z)^3}, z \in \Delta$$

can be constructed by the horizontal shear of the Koebe function $k(z) = \frac{z}{(1-z)^2}$ with the dilation $\omega(z) = z$.

Example 2. *The harmonic function $L = h + \overline{g}$ where*

$$h(z) = \frac{z - \frac{1}{2}z^2}{(1-z)^2}, g(z) = \frac{-\frac{1}{2}z^2}{(1-z)^2}, z \in \Delta$$

can be constructed by the vertical shear of the function $l(z) = \frac{z}{1-z}$ with the dilation $\omega(z) = -z$.

Example 3. *The harmonic function $f = z - \frac{1}{2}\overline{z}^2$ is the harmonic shear of a polynomial function $\phi(z) = z + \frac{z^2}{2}$ with the dilation $\omega(z) = -z$.*

Example 4. *The harmonic function $f = z + \frac{1}{3}\overline{z}^2$ is the harmonic shear of a polynomial function $\phi(z) = z + \frac{z^3}{3}$ with the dilation $\omega(z) = z^2$.*

Note that in Examples 1 and 2, $h(z)$ and $g(z)$ are the solutions of the system of equations given by (3) and (4), respectively, for the cases $\varphi(z) = k(z), \omega(z) = z$ and $\kappa(z) = l(z), \omega(z) = -z$.

We consider the following two subclasses $\mathcal{T}[A, B]$ and $\mathcal{C}[A, B]$ of $\mathcal{T}$

Definition 1. *[16] A function $h \in \mathcal{T}$ of the form given in (2) is said to be in $\mathcal{T}[A, B]$ if, for some constant A and B such that $-1 \leq B < A \leq 1$, it satisfies*

$$\sum_{n=2}^{\infty} \left\{ (n-1)\frac{1-B}{A-B} + 1 \right\} |a_n| \leq 1;$$

and is said to be in the class $\mathcal{C}[A, B]$, if $zh' \in \mathcal{T}[A, B]$.

We observe that the functions of the classes $\mathcal{T}[A,B]$ and $\mathcal{C}[A,B]$ are univalent. Note that the class $\mathcal{T}[1,-1]=\mathcal{T}^*$ was studied in [18, 19].

In view of Theorems A and B and by Examples 1 and 2, we now define classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$.

Definition 2. *Let a function ϕ_α defined by*

$$\phi_\alpha(z)=H_\alpha(z)-e^{2i\alpha}G_\alpha(z) \tag{5}$$

be convex in the direction $\alpha\in\{0,\pi/2\}$, where

$$H_\alpha(z)=z-\sum_{n=2}^{\infty}\frac{|a_n|}{1-e^{2i\alpha}|b_1|}z^n,\ G_\alpha(z)=\sum_{n=2}^{\infty}\frac{|b_n|}{1-e^{2i\alpha}|b_1|}z^n \tag{6}$$

are analytic in Δ, $|b_1|<1$ and $\alpha\in\{0,\pi/2\}$). Then the harmonic shear $F_\alpha=H_\alpha+\overline{G_\alpha}$ of ϕ_α, is said to be in the class $\mathcal{T}_H[A,B]$ if $\phi_\alpha\in\mathcal{T}[A,B]$. Further, we say that $F_\alpha=H_\alpha+\overline{G_\alpha}$ is in the class $\mathcal{C}_H[A,B]$ if $z\phi_\alpha'(z)\in\mathcal{T}[A,B]$.

We note that the analytic function ϕ_α considered in (5) may also be expressed as

$$\phi_\alpha(z)=\frac{h(z)-e^{2i\alpha}g(z)}{1-e^{2i\alpha}|b_1|}$$

where h and g are of the form (2).

Here it is worth mentioning that for a CHD map ϕ_0 defined by

$$\phi_0(z)=H_0(z)-G_0(z),$$

where

$$H_0(z)=z-\sum_{n=2}^{\infty}\frac{|a_n|}{1-|b_1|}z^n, G_0(z)=\sum_{n=2}^{\infty}\frac{|b_n|}{1-|b_1|}z^n \tag{7}$$

are analytic in Δ, $|b_1|<1$, there exists a dilatation ω_0, such that the harmonic shear $F_0=H_0+\overline{G_0}$ of ϕ_0 may be obtained by solving the differential equations:

$$H_0'-G_0'=\phi_0',\ \omega_0H_0'-G_0'=0.$$

Also, for a map $\phi_{\pi/2}$ convex in vertical direction, defined by

$$\phi_{\pi/2}(z)=H_{\pi/2}(z)+G_{\pi/2}(z),$$

where

$$H_{\pi/2}(z)=z-\sum_{n=2}^{\infty}\frac{|a_n|}{1+|b_1|}z^n, G_{\pi/2}(z)=\sum_{n=2}^{\infty}\frac{|b_n|}{1+|b_1|}z^n \tag{8}$$

are analytic in Δ, there exists a dilatation $\omega_{\pi/2}$, such that the harmonic shear $F_{\pi/2} = H_{\pi/2} + \overline{G_{\pi/2}}$ of $\phi_{\pi/2}$ may be obtained by solving the differential equations:

$$H'_{\pi/2} + G'_{\pi/2} = \phi'_{\pi/2},\ \omega_{\pi/2} H'_{\pi/2} - G'_{\pi/2} = 0.$$

In this paper, with the use of *shear construction* method we study classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$ of harmonic univalent functions $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$ which are the harmonic shear of analytic functions, convex in the direction $\alpha \in \{0, \pi/2\}$ (that is convex in the horizontal direction or vertical direction) and are in the classes $\mathcal{T}[A,B]$ and $\mathcal{C}[A,B]$, respectively. Coefficient conditions that are both necessary and sufficient for functions in the classes $T_H[A,B]$ and $\mathcal{C}_H[A,B]$ are obtained. Convolution conditions for these classes with the use of equivalent conditions are also derived. Finally, inequalities which are both necessary and sufficient for the harmonic shears of analytic functions involving Wright's generalized hypergeometric functions are obtained.

2. Main Lemma

Based on Definition 1 and motivated by the equivalent conditions of the class $\mathcal{T}[A,B]$ found in [16], we can easily prove the equivalent conditions of the class $\mathcal{T}_H[A,B]$ as given in the next lemma. However, we first recall well-known definition of subordinate function.

A function f_1 is subordinate to f_2 in Δ if there exists an analytic function w with $w(0) = 0$ and $|w(z)| < 1$ such that $f_1(z) = f_2(w(z))$ for $|z| < 1$; this is written as $f_1 \prec f_2$. Furthermore, if the function f_2 is univalent in Δ, then we have following equivalence:

$$f_1(z) \prec f_2(z) \Leftrightarrow f_1(0) = f_2(0) \text{ and } f_1(\Delta) \subset f_2(\Delta).$$

Lemma 1. *Let $\phi_\alpha(z) = H_\alpha(z) - e^{2i\alpha} G_\alpha(z)$ be convex in the direction $\alpha \in \{0, \pi/2\}$, where H_α and G_α are given by (6). Then its harmonic shear $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$, convex in the same direction α, is in $\mathcal{T}_H[A,B]$ if and only if it satisfies any one of the following conditions:*

$$\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} \prec \frac{1+Az}{1+Bz}, -1 \le B < A \le 1, z \in \Delta. \tag{9}$$

$$\left| \frac{\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} - 1}{A - B\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)}} \right| < 1, -1 \le B < A \le 1, z \in \Delta. \tag{10}$$

$$\Re\left(\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)}\right) > \frac{1-A}{1-B}, -1 \le B < A \le 1, z \in \Delta. \tag{11}$$

$$\left|\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} - 1\right| < \frac{A-B}{1-B}, -1 \le B < A \le 1, z \in \Delta. \tag{12}$$

Proof. Let $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$ where H_α and G_α are of the form (6). By Definition 2, $F_\alpha \in \mathcal{T}_H[A,B]$ is convex in the direction $\alpha \in \{0, \pi/2\}$ if and only if ϕ_α belongs to $\mathcal{T}[A,B]$ where ϕ_α is of the form

$$\phi_\alpha(z) = z - \sum_{n=2}^{\infty} d_n z^n \tag{13}$$

and

$$d_n = \frac{|a_n| + e^{2i\alpha}|b_n|}{1 - e^{2i\alpha}|b_1|}, n \ge 2. \tag{14}$$

But by Definition 1, $\phi_\alpha \in \mathcal{T}[A,B]$ if and only if the condition

$$\sum_{n=2}^{\infty}\left\{(n-1)\frac{1-B}{A-B} + 1\right\} d_n \le 1 \tag{15}$$

holds, where d_n is given by (14). It now suffices to prove that (15) is equivalent to (9) to (12). Note that for a Schwarz function w analytic in Δ with $w(0) = 0$, $|w(z)| < 1$ in Δ, (9) can be given by

$$\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}, -1 \le B < A \le 1, z \in \Delta$$

which equivalently be expressed by (10). For $\phi_\alpha(z)$ of the form (13), we observe that (10) is equivalent to

$$\left|\frac{-\sum_{n=2}^{\infty}(n-1)d_n z^n}{(A-B)z - \sum_{n=2}^{\infty}(A-Bn)d_n z^n}\right| < 1, z \in \Delta.$$

By using the fact $\Re(z) \le |z|$, and since, for real value of z, the quantity within the mod sign in the above inequality is real, we have as $z \to 1^-$,

$$\frac{\sum_{n=2}^{\infty}(n-1)d_n}{A - B - \sum_{n=2}^{\infty}(A-Bn)d_n} \le 1$$

which establishes (15). Again, (11) is equivalent to

$$\left|\frac{\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} - 1}{1 - 2\beta + \frac{z\phi'_\alpha(z)}{\phi_\alpha(z)}}\right| < 1, z \in \Delta$$

where $\beta = \frac{1-A}{1-B}$ and for $\phi_\alpha(z)$ of the form (13). The last inequality is equivalent to

$$\left|\frac{-\sum_{n=2}^{\infty}(n-1)d_n z^n}{2(1-\beta)z-\sum_{n=2}^{\infty}(1-2\beta+n)d_n z^n}\right|<1, z\in\Delta.$$

Using the above used argument, as $z \to 1^-$along real line, we obtain (15). Further, for $\phi_\alpha(z)$ of the form (13), the condition (12) is equivalent to

$$\left|\frac{-\sum_{n=2}^{\infty}(n-1)d_n z^n}{z-\sum_{n=2}^{\infty}d_n z^n}\right|<\frac{A-B}{1-B}, z\in\Delta.$$

which again using the above used argument, yields (15). This proves Lemma 1.

Note that the condition (9) is considered in [7]. Classes $\mathcal{T}_H[1,-1] =: \mathcal{TS}_H^{0*}$ and $\mathcal{C}_H[1,-1] =: \mathcal{K}_H^0$ are studied in [20] and [21]. A class of close-to-convex harmonic functions, by using transforming (shearing) a convex analytic functions, is studied by Jahangiri and Silverman [12]. Recently, results on growth, distortion and coefficient bounds are obtained in [5] for harmonic functions (constructed by shearing method) convex in both the horizontal and vertical directions.

3. Certain Equivalent Conditions

We first derive the coefficient inequalities which are both necessary and sufficient for the functions $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$ (convex in the direction $\alpha \in \{0, \pi/2\}$ where H_α and G_α are of the form (6), to be in the classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$. We assume throughout this section that the coefficients of H_α and G_α in (6) satisfy the condition: $|b_n| < |a_n|, n \geq 2$.

Theorem 2. *Let $\phi_\alpha(z) = H_\alpha(z) - e^{2i\alpha}G_\alpha(z) \in \mathcal{T}[A,B]$ be convex in the direction $\alpha \in \{0,\pi/2\}$, where H_α and G_α are given by (6). Also, suppose $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$ is the harmonic shear of ϕ_α and convex in the same direction α. Then $F_\alpha \in \mathcal{T}_H[A,B]$ if and only if the coefficient inequality*

$$\sum_{n=2}^{\infty}\left\{(n-1)\frac{1-B}{A-B}+1\right\}|a_n| + e^{2i\alpha}\sum_{n=1}^{\infty}\left\{(n-1)\frac{1-B}{A-B}+1\right\}|b_n| \leq 1 \qquad (16)$$

holds. The result is sharp.

Proof. Following initial lines of the proof of Lemma 1, we have $F_\alpha \in \mathcal{T}_H[A,B]$ if and only if (15) holds. In view of (14), the inequality (15) can also be given by (16). Sharpness of (16) can easily be verified for the function given by

$$f(z) = z - \sum_{n=2}^{\infty}\frac{(A-B)|x_n|}{(n-1)(1-B)+A-B}z^n + \sum_{n=2}^{\infty}\frac{(A-B)|y_n|}{(n-1)(1-B)+A-B}\overline{z^n},$$

where for $\alpha \in \{0, \pi/2\}$, $\sum_{n=2}^{\infty} |x_n| + e^{2i\alpha} \sum_{n=1}^{\infty} |y_n| = 1$.

In particular, taking $\alpha = 0$ and $\alpha = \pi/2$, respectively, in Theorem 2, we get following results for CHD map and for the map convex in vertical direction.

Corollary 3. *Let $\phi_0(z) = H_0(z) - G_0(z) \in \mathcal{T}[A, B]$ be a CHD map, where H_0 and G_0 are given by (7). Also, suppose $F_0 = H_0 + \overline{G_0} \in \mathcal{TS}_H^0$ is the harmonic shear of ϕ_0 and convex in the horizontal direction. Then $F_0 \in \mathcal{T}_H[A, B]$ if and only if the coefficient inequality*

$$\sum_{n=2}^{\infty} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} |a_n| + \sum_{n=1}^{\infty} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} |b_n| \leq 1$$

holds. The result is sharp.

Corollary 4. *Let $\phi_{\pi/2}(z) = H_{\pi/2}(z) + G_{\pi/2}(z) \in \mathcal{T}[A, B]$ be convex in vertical direction, where $H_{\pi/2}$ and $G_{\pi/2}$ are given by (8). Also, suppose $F_{\pi/2} = H_{\pi/2} + \overline{G_{\pi/2}} \in \mathcal{TS}_H^0$ is the harmonic shear of $\phi_{\pi/2}$ and convex in the vertical direction. Then $F_{\pi/2} \in \mathcal{T}_H[A, B]$ if and only if the coefficient inequality*

$$\sum_{n=2}^{\infty} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} |a_n| - \sum_{n=1}^{\infty} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} |b_n| \leq 1$$

holds. The result is sharp.

Theorem 5. *Under the hypothesis of Theorem 2, the function $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in $\mathcal{C}_H[A, B]$ if and only if the coefficient inequality*

$$\sum_{n=2}^{\infty} n\left\{(n-1)\frac{1-B}{A-B} + 1\right\} |a_n| + e^{2i\alpha} \sum_{n=1}^{\infty} n\left\{(n-1)\frac{1-B}{A-B} + 1\right\} |b_n| \leq 1 \tag{17}$$

holds. The result is sharp.

Proof. By Definition 2, $F_\alpha \in \mathcal{C}_H[A, B]$ if and only if $z\phi'_\alpha(z)$ convex in the direction α, belongs to $\mathcal{T}[A, B]$ where $z\phi'_\alpha(z)$ is of the form

$$z\phi'_\alpha(z) = z - \sum_{n=2}^{\infty} n d_n z^n$$

and d_n is given by (14). Hence, by Definition 1, it follows that $F \in \mathcal{C}_H[A, B]$ if and only if

$$\sum_{n=2}^{\infty} n\left\{(n-1)\frac{1-B}{A-B} + 1\right\} \frac{|a_n| + e^{2i\alpha}|b_n|}{1 - e^{2i\alpha}|b_1|} \leq 1. \tag{18}$$

This inequality is equivalent to (17). Sharpness can be verified for the functions given by

$$F(z) = z - \sum_{n=2}^{\infty} \frac{(A-B)\,|x_n|}{n\,\{(n-1)(1-B)+A-B\}} z^n + \sum_{n=2}^{\infty} \frac{(A-B)\,|y_n|}{n\,\{(n-1)(1-B)+A-B\}} \overline{z^n}$$

where for $\alpha \in \{0, \pi/2\}$, $\sum_{n=2}^{\infty} |x_n| + e^{2i\alpha} \sum_{n=1}^{\infty} |y_n| = 1$.This completes the proof of Theorem 5.

In particular, taking $\alpha = 0$ and $\alpha = \pi/2$, respectively, in Theorem 5, we get following results for CHD map and for the map convex in vertical direction.

Corollary 6. *Under the same hypothesis of Corollary 3, the function $F_0 = H_0 + \overline{G_0} \in \mathcal{C}_H[A,B]$ if and only if the coefficient condition*

$$\sum_{n=2}^{\infty} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}|a_n| + \sum_{n=1}^{\infty} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}|b_n| \le 1$$

holds. The result is sharp.

Corollary 7. *Under the same hypothesis of Corollary 4, the function $F_{\pi/2} = H_{\pi/2} + \overline{G_{\pi/2}} \in \mathcal{C}_H[A,B]$ if and only if the coefficient condition*

$$\sum_{n=2}^{\infty} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}|a_n| - \sum_{n=1}^{\infty} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}|b_n| \le 1$$

holds. The result is sharp.

Remark 1. *(1) Taking $A = 1, B = -1$ in Theorems 2 and 5, our results coincide with the results obtained in [20] and [21] for the classes $\mathcal{TS}_H^{0*}$ and $\mathcal{K}_H^0$.*

(2) Taking $A = 1, B = -1$ and the coefficients $b_n = 0$ $(n \in \mathbb{N})$ in Theorem 2, our result coincides with the one obtained by Silverman [18].

Further, on using the condition (11), we obtain other equivalent conditions for functions belonging to the classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$, respectively, as follows:

Theorem 8. *Under the hypothesis of Theorem 2, $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in $\mathcal{T}_H[A,B]$ if and only if*

$$\Re\left[\left(\frac{1-B}{A-B}\right)\left(H'_\alpha(z) - e^{2i\alpha}G'_\alpha(z)\right) - \left(\frac{1-A}{A-B}\right)\left(\frac{H_\alpha(z) - e^{2i\alpha}G_\alpha(z)}{z}\right)\right] > 0$$

holds in Δ.

Proof. By (11), $F \in \mathcal{T}_H[A, B]$ if and only if

$$\Re\left(\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)}\right) > \frac{1-A}{1-B}.$$

Using $\phi_\alpha(z)$ from (5) the result follows.

Taking $A = 1, B = -1$ in Theorem 8, we get following result for the class $\mathcal{TS}_H^{0*}$.

Corollary 9. *Let $\phi_\alpha(z) = H_\alpha(z) - e^{2i\alpha}G_\alpha(z) \in \mathcal{T}^*$ be convex in the direction $\alpha \in \{0, \pi/2\}$, where H_α and G_α are given by (6). Suppose $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{TS}_H^0$ is the harmonic shear of ϕ_α convex in the same direction α. Then $F_\alpha \in \mathcal{TS}_H^{0*}$ if and only if*

$$\Re\left[\left(H'_\alpha(z) - e^{2i\alpha}G'_\alpha(z)\right)\right] > 0$$

in Δ.

Similar to Theorem 8, we get the following result for the class $\mathcal{C}_H[A, B]$.

Theorem 10. *Under the hypothesis of Theorem 2, the function $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in $\mathcal{C}_H[A, B]$ if and only if*

$$\Re\left[\left(\frac{1-B}{A-B}\right)\left(\left(zH'_\alpha(z)\right)' - e^{2i\alpha}\left(zG'_\alpha(z)\right)'\right) - \left(\frac{1-A}{A-B}\right)\left(H'_\alpha(z) - e^{2i\alpha}G'_\alpha(z)\right)\right] > 0$$

in Δ.

Taking $A = 1, B = -1$ in Theorem 10, we get following result for the class $\mathcal{K}_H^0$.

Corollary 11. *Under the hypothesis of corollary 9, the function $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in $\mathcal{K}_H^0$ if and only if*

$$\Re\left[\left(\left(zH'_\alpha(z)\right)' - e^{2i\alpha}\left(zG'_\alpha(z)\right)'\right)\right] > 0$$

holds in Δ.

Remark 2. *Similar to the Corollaries 3 to 7, we can get results from Theorems 8 and 10 and from Corollaries 9 and 11 for the functions convex in horizontal as well as in vertical direction by taking $\alpha = 0$ and $\alpha = \pi/2$, respectively.*

4. Convolution Conditions

Using (9), we study convolution conditions for functions in the classes $\mathcal{T}_H[A,B]$ and $\mathcal{C}_H[A,B]$.

Theorem 12. *Under the hypothesis of Theorem 2, the function $F_\alpha = H_\alpha + \overline{G_\alpha}$ is in $\mathcal{T}_H[A,B]$ if and only if for some ϵ $(|\epsilon| = 1)$,*

$$\left[\frac{1}{z}\left[H_\alpha(z) * \left(\frac{z + \frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right)\right] - \epsilon\overline{\frac{e^{2i\alpha}}{z}\left[G_\alpha(z) * \left(\frac{z + \frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right)\right]}\right] \neq 0 \tag{19}$$

in Δ.

Proof. Since

$$F_\alpha \in \mathcal{T}_H[A,B] \Leftrightarrow \phi_\alpha \in \mathcal{T}[A,B]$$

where $\phi_\alpha(z)$ is given by (5). Hence, by (9), $\phi_\alpha \in \mathcal{T}[A,B]$ if and only if

$$\frac{z\phi'_\alpha(z)}{\phi_\alpha(z)} \neq \frac{1-Ae^{i\theta}}{1-Be^{i\theta}}, -\pi \leq \theta < \pi, z \in \Delta.$$

On writing $z\phi'_\alpha(z) = \phi_\alpha(z) * \frac{z}{(1-z)^2}$, and $\phi_\alpha(z) = \phi_\alpha(z) * \frac{z}{1-z}$, for $z \in \Delta$, we get $\phi_\alpha \in \mathcal{T}[A,B]$ if and only if

$$\frac{1}{z}\left[\left\{1 + \frac{\left(1-Ae^{i\theta}\right)z}{(A-B)e^{i\theta}}\right\}\frac{z}{(1-z)^2} * \phi_\alpha(z)\right] \neq 0, \tag{20}$$

Using $\phi_\alpha(z)$ from (5), we obtain

$$\frac{1}{z}\left[\left\{1 + \frac{\left(1-Ae^{i\theta}\right)z}{(A-B)e^{i\theta}}\right\}\frac{z}{(1-z)^2} * H_\alpha(z) - e^{2i\alpha}\left\{1 + \frac{\left(1-Ae^{i\theta}\right)z}{(A-B)e^{i\theta}}\right\}\frac{z}{(1-z)^2} * G_\alpha(z)\right] \neq 0.$$

Now using the fact that if $z_1 - z_2 \neq 0$ and $|z_1| \neq |z_2|$, then $z_1 - \epsilon\overline{z_2} \neq 0$, $|\epsilon| = 1$, we get the convolution condition (19). This proves Theorem 12.

Taking $A = 1, B = -1$ in Theorem 12, we get following result for the class $\mathcal{TS}_H^{0*}$.

Corollary 13. *Under the hypothesis of corollary 9, the function $F_\alpha = H_\alpha + \overline{G_\alpha}$ $\in \mathcal{TS}_H^{0*}$ if and only if for some ϵ $(|\epsilon| = 1)$, the condition*

$$\left[\frac{1}{z}\left[H_\alpha(z) * \left(\frac{z + \frac{1-e^{i\theta}}{2e^{i\theta}}z^2}{(1-z)^2}\right)\right] - \epsilon\overline{\frac{e^{2i\alpha}}{z}\left[G_\alpha(z) * \left(\frac{z + \frac{1-e^{i\theta}}{2e^{i\theta}}z^2}{(1-z)^2}\right)\right]}\right] \neq 0$$

holds in Δ.

Theorem 14. *Under the hypothesis of Theorem 2, the function* $F_\alpha = H_\alpha + \overline{G_\alpha}$ *is in* $\mathcal{C}_H[A,B]$ *if and only if for some* ϵ $(|\epsilon| = 1)$,

$$\frac{1}{z}\left[H_\alpha(z) * \left(\frac{z+\left(1+\frac{2\left(1-Ae^{i\theta}\right)}{(A-B)e^{i\theta}}\right)z^2}{(1-z)^3}\right)\right] - \epsilon\,\overline{\frac{e^{2i\alpha}}{z}\left[G_\alpha(z) * \left(\frac{z+\left(1+\frac{2\left(1-Ae^{i\theta}\right)}{(A-B)e^{i\theta}}\right)z^2}{(1-z)^3}\right)\right]} \neq 0 \tag{21}$$

in Δ.

Proof. Since

$$F_\alpha \in \mathcal{C}_H[A,B] \Leftrightarrow z\phi'_\alpha \in \mathcal{T}[A,B]$$

where $\phi_\alpha(z)$ is given by (5). Hence, similar to the proof of Theorem 12, $F_\alpha \in \mathcal{C}_H[A,B]$ if and only if

$$\frac{1}{z}\left[\left\{1+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z\right\}\frac{z}{(1-z)^2} * z\phi'_\alpha(z)\right] \neq 0,$$

which by (5) gives

$$\begin{aligned}
&\frac{1}{z}\left[\left\{1+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z\right\}\frac{z}{(1-z)^2} * \left\{zH'_\alpha(z) - e^{2i\alpha}zG'_\alpha(z)\right\}\right] \\
=\;& \frac{1}{z}\left[zH'_\alpha(z) * \left(\frac{z+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right) - e^{2i\alpha}zG'_\alpha(z) * \left(\frac{z+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right)\right] \\
=\;& \frac{1}{z}\left[H_\alpha(z) * z\left(\frac{z+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right)' - e^{2i\alpha}G_\alpha(z) * z\left(\frac{z+\frac{1-Ae^{i\theta}}{(A-B)e^{i\theta}}z^2}{(1-z)^2}\right)'\right] \\
=\;& \frac{1}{z}\left[H_\alpha(z) * \left(\frac{z+\left(1+\frac{2\left(1-Ae^{i\theta}\right)}{(A-B)e^{i\theta}}\right)z^2}{(1-z)^3}\right) - e^{2i\alpha}G_\alpha(z) * \left(\frac{z+\left(1+\frac{2\left(1-Ae^{i\theta}\right)}{(A-B)e^{i\theta}}\right)z^2}{(1-z)^3}\right)\right] \neq 0.
\end{aligned}$$

Again, by using the same fact that if $z_1 - z_2 \neq 0$ and $|z_1| \neq |z_2|$, then $z_1 - \epsilon\overline{z_2} \neq 0$, $|\epsilon| = 1$, we obtain the convolution condition (21).

Taking $A = 1, B = -1$ in Theorem 14, we get following result for the class $\mathcal{K}_H^0$.

Corollary 15. *Under the hypothesis of corollary 9, the function* $F_\alpha = H_\alpha + \overline{G_\alpha} \in \mathcal{K}_H^0$ *if and only if for some* ϵ $(|\epsilon| = 1)$,

$$\frac{1}{z}\left[H_\alpha(z) * \left(\frac{z + \left(1 + \frac{1-e^{i\theta}}{e^{i\theta}}\right) z^2}{(1-z)^3}\right)\right] - \epsilon\, \overline{\frac{e^{2i\alpha}}{z}\left[G_\alpha(z) * \left(\frac{z + \left(1 + \frac{1-e^{i\theta}}{e^{i\theta}}\right) z^2}{(1-z)^3}\right)\right]} \neq 0$$

in Δ.

Remark 3. *Similar to the Corollaries 3 to 7, we can get results from Theorems 12 and 14 and from Corollaries 13 and 15 for the functions convex in horizontal as well as in vertical direction by taking* $\alpha = 0$ *and* $\alpha = \pi/2$*, respectively.*

5. Applications to Wright's Functions

In this section we obtain results, similar to Theorems 2 and 5, for the harmonic functions defined by shearing of certain analytic functions which involve Wright's generalized hypergeometric (Wgh) functions.

The Wgh functions have an increasingly significant role in various types of applications (see [22, 23]). Generalized hypergeometric functions, generalized Mittag-Leffler functions and Bessel-Maitland (Wright generalized Bessel) functions are some special cases of Wgh functions; one may refer to [24, 25]. Several results on harmonic functions by involving hypergeometric functions have recently been studied in [1] to [4]. Involvement of the Wright generalized hypergeometric function (Wgh) in the harmonic functions has recently been investigated amongst others in [6, 13, 14, 17].

Let $A_i > 0$ $(i = 1, ..., p)$ and $B_i > 0$ $(i = 1, ..., q)$ such that $1 + \sum_{i=1}^{q} B_i - \sum_{i=1}^{p} A_i \geq 0$. Following the definition and terminology in [22], [24] and [26], a Wright's generalized hypergeometric (Wgh) function for non-negative integers p and q, $\alpha_i \in \mathbb{C}$ $\left(\frac{\alpha_i}{A_i} \neq 0, -1, -2, ...; i = 1, ..., p\right)$ and $\beta_i \in \mathbb{C}$ $\left(\frac{\beta_i}{B_i} \neq 0, -1, -2, ...; i = 1, ..., q\right)$ is defined by

$$ {}_p\psi_q\left([(\alpha_i, A_i)]\,; z\right) \equiv {}_p\psi_q\left[\begin{pmatrix}(\alpha_i, A_i)_{1,p} \\ (\beta_i, B_i)_{1,q}\end{pmatrix}; z\right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma\left(\alpha_i + nA_i\right)}{\prod_{i=1}^{q} \Gamma(\beta_i + nB_i)} \frac{z^n}{n!}, z \in \Delta. \quad (22)$$

By involving Wgh functions as defined by (22), consider an analytic function $\Phi_\alpha(z)$ defined by

$$\Phi_\alpha(z) = \frac{W_1(z) - e^{2i\alpha} W_2(z)}{1 - e^{2i\alpha} d_1}, z \in \Delta \quad (23)$$

where

$$W_1(z) = z\frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)}\ {}_p\psi_q\left[\begin{pmatrix}(\alpha_i, A_i)_{1,p}\\(\beta_i, B_i)_{1,q}\end{pmatrix}; z\right], \tag{24}$$

$$W_2(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\ {}_r\psi_s\left[\begin{pmatrix}(\gamma_i, C_i)_{1,r}\\(\delta_i, D_i)_{1,s}\end{pmatrix}; z\right] - 1 \tag{25}$$

and

$$d_1 = \frac{\prod_{i=1}^{r}(\gamma_i)_{C_i}}{\prod_{i=1}^{s}(\delta_i)_{D_i}} \tag{26}$$

for positive integers $A_i, B_i, C_i,$ and D_i and for $\alpha_i > -A_i \ \ (i = 1, ..., p)$, satisfying $\prod_{i=1}^{p}(\alpha_i)_{A_i} < 0$, and $\beta_i. > 0 \ \ (i = 1, ..., q)$, $\gamma_i > 0 \ \ (i = 1, ..., r)$, $\delta_i > 0 \ \ (i = 1, ..., s)$ with

$$\frac{\prod_{i=1}^{r}(\gamma_i)_{nC_i}}{\prod_{i=1}^{s}(\delta_i)_{nD_i}} < \frac{n\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right|\prod_{i=1}^{p}(\alpha_i + A_i)_{(n-2)A_i}}{\prod_{i=1}^{q}(\beta_i)_{(n-1)B_i}}, n \geq 2; \frac{\prod_{i=1}^{r}(\gamma_i)_{C_i}}{\prod_{i=1}^{s}(\delta_i)_{D_i}} < 1.$$

In view of the parametric constraints cosidered above and $\prod_{i=1}^{p}(\alpha_i)_{A_i} < 0$, we have

$$\prod_{i=1}^{p}\Gamma(\alpha_i) = \frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}{\prod_{i=1}^{p}(\alpha_i)_{A_i}} = -\frac{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right|}$$

and hence, the function $\Phi_\alpha(z)$ defined by (23) may also be written in the form

$$\Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \tag{27}$$

where

$$\mathcal{H}_\alpha(z) = z - \frac{\left|\prod_{i=1}^{p}(\alpha_i)_{A_i}\right|\ \prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i + A_i)}\sum_{n=2}^{\infty}\frac{\theta_n}{1 - e^{2i\alpha}d_1}\ z^n, \mathcal{G}_\alpha(z) = \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)}\sum_{n=2}^{\infty}\frac{\phi_n}{1 - e^{2i\alpha}d_1}z^n \tag{28}$$

and

$$\theta_n = \frac{\prod_{i=1}^{p} \Gamma\left(\alpha_i + (n-1)A_i\right)}{\prod_{i=1}^{q} \Gamma(\beta_i + (n-1)B_i)} \frac{1}{(n-1)!}, \quad \phi_n = \frac{\prod_{i=1}^{r} \Gamma\left(\gamma_i + nC_i\right)}{\prod_{i=1}^{s} \Gamma(\delta_i + nD_i)} \frac{1}{n!}, \tag{29}$$

d_1 is given by (26). Using $\Phi_\alpha(z)$ defined by (27), we get a harmonic shear $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha}$ and we obtain following results.

Theorem 16. *Under the parametric conditions stated as above, let $\mathcal{H}_\alpha$ and $\mathcal{G}_\alpha$ be functions of the form (28) with θ_n, ϕ_n given by (29). Let $\Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A, B]$ be convex in the direction $\alpha \in \{0, \pi/2\}$ and $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{TS}_H^0$ be its harmonic shear, convex in the same direction α.Then $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{T}_H[A, B]$ if and only if the inequality*

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\left|\prod_{i=1}^{p} \left(\alpha_i\right)_{A_i}\right| \prod_{i=1}^{q} \Gamma\left(\beta_i\right)}{\prod_{i=1}^{p} \Gamma\left(\alpha_i + A_i\right)} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \theta_n \\ &+e^{2i\alpha} \sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s} \Gamma(\delta_i)}{\prod_{i=1}^{r} \Gamma\left(\gamma_i\right)} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \phi_n \\ \leq\ & 1 \end{aligned} \tag{30}$$

is satisfied.

Proof. Similar to the proof of Theorem 2, we have $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{T}_H[A, B] \Leftrightarrow \Phi_\alpha(z) = \mathcal{H}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A, B]$. On using (27) and (28), by Definition 1, $\Phi_\alpha \in \mathcal{T}[A, B]$ if and only if

$$\frac{1}{1 - e^{2i\alpha}d_1} \sum_{n=2}^{\infty} \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \left[\frac{\left|\prod_{i=1}^{p} \left(\alpha_i\right)_{A_i}\right| \prod_{i=1}^{q} \Gamma\left(\beta_i\right)}{\prod_{i=1}^{p} \Gamma\left(\alpha_i + A_i\right)} \theta_n + e^{2i\alpha} \frac{\prod_{i=1}^{s} \Gamma(\delta_i)}{\prod_{i=1}^{r} \Gamma\left(\gamma_i\right)} \phi_n\right] \leq 1.$$

Substituting d_1 from (26) and simplifying the result follows.

The proof of next result is similar to Theorem 16.

Theorem 17. *Under the hypothesis of Theorem 16, the function* $\mathcal{F}_\alpha = \mathcal{H}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{C}_H[A,B]$ *if and only if the inequality*

$$\begin{aligned}
&\sum_{n=2}^{\infty} \frac{\left|\prod\limits_{i=1}^{p} (\alpha_i)_{A_i}\right| \prod\limits_{i=1}^{q} \Gamma(\beta_i)}{\prod\limits_{i=1}^{p} \Gamma(\alpha_i + A_i)} n \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \theta_n \\
&+ e^{2i\alpha} \sum_{n=1}^{\infty} \frac{\prod\limits_{i=1}^{s} \Gamma(\delta_i)}{\prod\limits_{i=1}^{r} \Gamma(\gamma_i)} n \left\{(n-1)\frac{1-B}{A-B} + 1\right\} \phi_n \\
\leq \; & 1
\end{aligned} \tag{31}$$

is satisfied.

We next consider an analytic function $\Psi_\alpha(z)$ defined by

$$\Psi_\alpha(z) = \frac{z\left(2 - \frac{W_1(z)}{z}\right) - e^{2i\alpha} W_2(z)}{1 - e^{2i\alpha} d_1}, z \in \Delta \tag{32}$$

where $W_1(z)$ and $W_2(z)$ are of the form (24) and (25), d_1 is given by (26) for positive integers A_i, B_i, C_i, D_i and for $\alpha_i > 0 \;\; (i = 1, ..., p)$, $\beta_i. > 0 \;\; (i = 1, ..., q)$, $\gamma_i > 0 \;\; (i = 1, ..., r)$, $\delta_i > 0 \;\; (i = 1, ..., s)$ with

$$\frac{\prod\limits_{i=1}^{r} (\gamma_i)_{nC_i}}{\prod\limits_{i=1}^{s} (\delta_i)_{nD_i}} < \frac{n \prod\limits_{i=1}^{p} (\alpha_i)_{(n-1)A_i}}{\prod\limits_{i=1}^{q} (\beta_i)_{(n-1)B_i}}, n \geq 1.$$

The function $\Psi_\alpha(z)$ may also be written in the form

$$\Psi_\alpha(z) = \mathcal{L}_\alpha(z) - e^{2i\alpha} \mathcal{G}_\alpha(z) \tag{33}$$

where

$$\mathcal{L}_\alpha(z) = z - \frac{\prod\limits_{i=1}^{q} \Gamma(\beta_i)}{\prod\limits_{i=1}^{p} \Gamma(\alpha_i)} \sum_{n=2}^{\infty} \frac{\theta_n}{1 - e^{2i\alpha} d_1} \; z^n, \tag{34}$$

$\mathcal{G}_\alpha(z)$, d_1 and θ_n are given as above. Using the method of proof in Theorems 16 and 17, we obtain next two theorems.

Theorem 18. *Under the parametric conditions stated as above, let* $\mathcal{L}_\alpha$ *and* $\mathcal{G}_\alpha$ *be given by (34) and (28), respectively, with* θ_n, ϕ_n *given by (29). Let* $\Psi_\alpha(z) = \mathcal{L}_\alpha(z) - e^{2i\alpha}\mathcal{G}_\alpha(z) \in \mathcal{T}[A,B]$ *be convex in the direction* $\alpha \in \{0, \pi/2\}$ *and* $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{TS}_H^0$ *be its harmonic shear, convex in the same direction* α*. Then, the function* $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{T}_H[A,B]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} \left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} \left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \le 1$$

is satisfied.

Theorem 19. *Under the hypothesis of Theorem 18, the function* $\mathcal{E}_\alpha = \mathcal{L}_\alpha + \overline{\mathcal{G}_\alpha} \in \mathcal{C}_H[A,B]$ *if and only if*

$$\sum_{n=2}^{\infty} \frac{\prod_{i=1}^{q}\Gamma(\beta_i)}{\prod_{i=1}^{p}\Gamma(\alpha_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\theta_n + e^{2i\alpha}\sum_{n=1}^{\infty} \frac{\prod_{i=1}^{s}\Gamma(\delta_i)}{\prod_{i=1}^{r}\Gamma(\gamma_i)} n\left\{(n-1)\frac{1-B}{A-B}+1\right\}\phi_n \le 1,$$

is satisfied.

Remark 4. *Taking* $A_i = 1\ (i=1,...,p), B_i = 1\ (i=1,...,q), C_i = 1\ (i=1,...,r)$, $D_i = 1\ (i=1,...,s)$, *in Theorems 16, 17, 18 and 19, we can easily get results for functions involving generalized hypergeometric functions and various special form of hypergeometric functions discussed in [6, 14, 16] etc.*

References

[1] O.P. Ahuja, *Planer harmonic convolution operators generated by hypergeometric functions*, Integ. Trans. Spec. Funct., 18 (3), (2007), 165-177.

[2] O.P. Ahuja, *Connections between various subclasses of planar harmonic mappings involving hypergeometric functions*, Appl. Math. Comput., 198 (2008), 305-316.

[3] O.P. Ahuja, *Harmonic starlikeness and convexity of integral operators generated by hypergeometric series*, Integ. Trans. Spec. Funct., 20 (8), (2009), 629-641.

[4] O.P. Ahuja, *Inclusion theorems involving uniformly harmonic starlike mappings and hypergeometric functions*, Analele Universitatii Oradea, Fasc. Matematica, 18 (2011), 5-18.

[5] O.P. Ahuja, *Use of theory of conformal mappings in harmonic univalent mappings with directional convexity*, Bull. Malays. Math. Sci. Soc., (2) 35 (3), (2012), 775-784.

[6] O.P. Ahuja and P. Sharma, *Inclusion theorems involving Wright's generalized hypergeometric functions and harmonic univalent functions*, Acta Universitatis Apulensis, 32 (2012), 111-128.

[7] H.A. AL-Kharsani, *An application of subordination on harmonic function*, J.I.P.A.M, 8 (3), (2007), Article 109, 8 pp.

[8] J. Clunie and T. Sheil-small, *Harmonic univalent functions*, Ann. Acad. Aci. Fenn. Ser. A Math., 9 (1984), 3-25.

[9] M. Dorff, J. Szynal, *Harmonic shears of elliptic integrals*, Rocky Mountain J. Math., 35 (2005), 485-499.

[10] K. Driver, P. Duren, *Harmonic shears of regular polygons by hypergeometric functions*, J. Math. Anal. Appl., 239 (1999), 72-84.

[11] W. Hengartner, G. Schober, *On schlicht mappings to domains convex in one direction*, Comment. Math. Helv., 45 (1970), 303-314.

[12] J.M. Jahangiri and H. Silverman, *Hamonic close-to-convex mappings*, J. Appl. Math. Stoch. Anal., 15 (1), (2002), 23-28.

[13] G. Murugusundaramoorthy, R.K. Raina, *On a subclass of harmonic functions associated with the Wright's generalized hypergeometric functions*, Hacettepe Journal of Mathematics and Statistics, 38 (2) (2009), 129-136.

[14] R.K. Raina and P. Sharma, *Harmonic univalent functions associated with Wright's generalized hypergeometric functions*, Integ. Trans. Spec. Funct., 22 (8), (2011), 561-572.

[15] L. Schaubroeck, *Growth, distortion and coefficient bounds for plane harmonic mappings convex in one direction*, Rocky Mountain J. Math., 31 (2001), 625-639.

[16] P. Sharma, *Univalent Wright's generalized hypergeometric functins*, Journal of Inequalities and Special Functions, 3 (1), (2012), 28-39.

[17] P. Sharma, *Some Wgh inequalities for univalent harmonic analytic functions*, Applied Mathematics, 1 (2010), 464-469.

[18] H. Silverman, *A survey with open problems on univalent functions whose coeffcients are negative*, Rocky Mountain J. Math., 21 (3), (1991), 1099-1124.

[19] H. Silverman, *Univalent functions with negative coefficients*, Proc. Amer. Math. Soc., 51 (1975), 109-116.

[20] H. Silverman, *Harmonic univalent functions with negative coefficients*, J. Math. Anal. Appl., 220 (1998), 283-289.

[21] H. Silverman and E.M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math., 28 (1999), 275-284.

[22] H.M. Srivastava, P.W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Horwood Ser. Math. Appl., Horwood, Chichester, 1985.

[23] H.M. Srivastava, R.K. Saxena, C. Ram, *A unified presentation of the Gamma-type functions occurring in diffraction theory and associated probability distributions*, Appl. Math. Comput., 162 (2005), 931-947.

[24] H.M. Srivastava, H.L. Manocha, *A Treatise on Generating Functions*, Halsted Press (Ellis Horwood limited, Chichester), 1984.

[25] H.M. Srivastava, *Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators*, Applicable Analysis and Discrete Mathematics, 1 (2007), 56-71.

[26] E.M. Wright, *The asymptotic expansion of the generalized hypergeometric function*, Proc. London Math. Soc., 46 (1946), 389-408.

Poonam Sharma
Department of Mathematics and Astronomy,
University of Lucknow,
Lucknow 226007, UP India
email: *sharma_poonam@lkouniv.ac.in*

Om P. Ahuja
Department of Mathematical Sciences
Kent State University
Burton, OH 44021-9500, USA
email: *oahuja@kent.edu*

Vimlesh K. Gupta
Department of Mathematics and Astronomy,
University of Lucknow,
Lucknow, 226007, UP INDIA
email: *vim987@gmail.com*