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DIRECT METHOD OF SOLVING OPTIMAL CONTROL PROBLEMS

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Abstract. In this paper, we elaborate a new method for solving an optimal terminal control problem for a linear dynamic system. The control is bounded constant piecewise function. This method consists of three procedures : method of discretization, interior-points method and final procedure. Simulations demonstrate the effectiveness of the proposed method and the results obtained were illustrated by numerical example.

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1. Introduction

Methods for solving optimal control problems existing in literatures are of two types : direct and indirect methods. The direct method is based on the discretization [1]-[5], [9], [11]. Only this method gives an approximate solution. The indirect method is based on the Pontryagin's maximum principle [8], [10], [12]. But this principle gives only necessary condition.

The aim of this paper is to show that the solution of continuous problem is different the solution of its discrete problem. For solving our problem, we have elaborated a new approach based in three procedures :

1. Discretization.
2. Interior-points method [5].
3. Final procedure.

The method of resolution is based in first part on a discretization of the control at regular intervals. The second part, we applied a method of interior points adapted from the simplex method (adaptive method), this method was elaborated by the authors R.Gabasov and F.Kirillov during the years 1980 [5]. Here we prove that the solution of the discrete problem is not optimal for the initial problem. Therefore, using the solution of the discrete problem, we have introduced another step called final procedure based on Newton's method, who we have given an optimal solution for the initial problem. An implementation under the Matlab environment has been developed for this method. In the implementation, at first we have checked the controllability of our problem [7]. Thereafter, we have elaborated the three procedures: discretization, adaptive method and final procedure. Finally, the efficiency of this method are given by numerical example.

2. Statement of the problem, Basic Concepts and Definitions

Consider the following terminal optimal control problem. In the class of piecewise continuous functions $u(.) = (u(t), t \in [0, t_f])$:

$$J(u(t)) = c'x(t_f) \longrightarrow \max, \tag{1}$$

$$\dot{x} = Ax + bu, x(0) = x_\circ = 0, \tag{2}$$

$$Hx(t_f) = g, \tag{3}$$

$$d_1 \leq u(t) \leq d_2, t \in [0, t_f] = T, \tag{4}$$

where, $J(u(t))$ is quality criterion, $x(t)$ is the state n -vector of the dynamic system at time t, $x_\circ$ is a given initial state vector of the system, $u(t)$ is the controller action (input signal) at time t is constant by piecewise and bounded by giving numbers d_1 and d_2, d_1 and d_2 are real numbers.t_f is finally time. $Hx(t_f)$ is the output signal of the system at time t_f equal to m -vectors g. $A \in \Re^{n\times n}$ et $H \in \Re^{m\times n}$, $b \in \Re^m$, $c' \in \Re^n$ are given constant matrices and vectors respectively. I = 1, 2, ..., m and J = 1, 2, ..., n is index sets. The symbol $'$ denotes transposition.

Definition 1. *A control $u(t), t \in T$, and a corresponding trajectory $x(t), t \in T$, satisfying the constraints (2)-(4) are called admissible of the problem (1)-(4).*

Definition 2. *An admissible control $u^\circ(.) = (u^\circ(t), t \in T)$, and a corresponding trajectory $x^\circ(t), t \in T$, are called optimal if the criterion* (1) *attains the maximum value :*

$$J(u^\circ(.)) = \max_u J(u(.)),$$

where the maximum is found over all the admissible controls.

An admissible control $u^{\varepsilon}(.) = (u^{\varepsilon}(t), t \in T)$*, and a corresponding trajectory* $x^{\varepsilon}(t), t \in T$ *, is called* ε*-optimal if*

$$J(u^{\circ}(.)) - J(u^{\varepsilon}(.)) = c' x^{\circ}(t_f) - c' x^{\varepsilon}(t_f) \leq \varepsilon,$$

where $u^{\circ}(.)$ *is an optimal control of the problem* $(1) - (4)$ *and* ε *is a nonnegative number, fixed in advance.*

By using the Cauchy formula, the solution of system (2) can be written in the form

$$x(t) = F(t)(x_{\circ} + \int_0^t F(\tau)^{-1} b u(\tau) d\tau), t \in T, \tag{5}$$

where $F(t) = e^{At}$, $t \in T$ is the square $n \times n$ matrix defined by the relations $\dot{F}(t) = AF(t)$, $F(0) = I_n$, I_n : identity matrix.

Using formula (5), the problem $(1) - (4)$ takes the form of a problem of one variable u :

$$\begin{cases} J(u(t)) = \int_0^{t_f} C(t) u(t) dt \longrightarrow \max\limits_{u(t)}, \\ \int_0^{t_f} p(t) u(t) dt = g, \\ d_1 \leq u(t) \leq d_2, t \in T, \end{cases} \tag{6}$$

where $C(t) = c' F(t_f) F(t)^{-1} b$, $p(t) = HF(t_f) F(t)^{-1} b, t \in T$.

Theorem 1. *[7] The Kalman rank condition states that a linear autonomous system of the form*

$$\dot{x}(t) = Ax(t) + bu(t)$$

is controllable if and if the controllability matrix

$$C = [b, Ab, A^2 b, ..., A^{n-1} b]$$

is of rank n*.*

3. Support-control

We assume that the problem $(1) - (4)$ is controllable.

Definition 3. *In the set T, we choose an arbitrary subset $\tau_B = \{\tau_j, j = \overline{1,m}\}$ formed of isolated times. The set τ_B is called a support of the problem* (6) *if $det\phi_B \neq 0$, where $\phi_B = \{p(\tau_j), j = \overline{1,m}\}$.*

Using the support moments, we construct the subset $T_B = \bigcup\limits_{j=1}^{m} [\underline{\tau}_j, \overline{\tau}_j]$, $\underline{\tau}_j \bigcap \overline{\tau}_j = \emptyset$, $\underline{\tau}_j = \tau_j$ or $\overline{\tau}_j = \tau_j$.

Definition 4. *The set of the moments T_B is called generalized support if the support matrix*

$$\phi(T_B) = \left(\int\limits_{\underline{\tau}_j}^{\overline{\tau}_j} p(t)dt, j = \overline{1,m} \right)$$

is non-degenerate.

Remark 1. *The support notion is very linked to the controllability notion of the problem* $(1) - (4)$.

Definition 5. *A pair $\{u, \tau_B\}$ formed from an admissible control $u = (u(t), t \in T)$ and a support τ_B called support-control of the problem* $(1) - (4)$.

4. The maximum principle

4.1. Formula of increment of quality criterion

Let $\{u, \tau_B\}$ be a support-control. Consider another admissible control $\bar{u}(t) = u(t) + \triangle u(t)$, $t \in T$ and the corresponding trajectory $\bar{x}(t) = x(t) + \triangle x(t)$, $t \in T$.

Using the support τ_B, we construct the following vectors : $y' = (C(\tau_j), j = \overline{1,m})\phi_B^{-1}$, $\Delta(t) = -\psi'(t)b,\ t \in T$ are called multipliers and estimates (co-control) vectors , where $\psi(t)$ is the solution of the system :

$$\dot{\psi} = -A'\psi, \psi(t_f) = C - H'y \tag{7}$$

called a conjugate system.

Then, the increment formula has the following form :

$$\triangle J(u) = J(\bar{u}) - J(u) = -\int\limits_{0}^{t_f} \Delta(t)\triangle u(t)dt. \tag{8}$$

Of the admissibility $u(t)$ and $\bar{u}(t)$, we have :

$$d_1 - u(t) \leq \triangle u(t) \leq d_2 - u(t). \tag{9}$$

According the equations (8) and (9), the maximum of the increment of quality criterion is reached for :

$$\begin{cases} \triangle u(t) = d_1 - u(t), & si \quad \Delta(t) > 0, \\ \triangle u(t) = d_2 - u(t), & si \quad \Delta(t) < 0, \\ d_1 - u(t) \leq \triangle u(t) \leq d_2 - u(t), & si \quad \Delta(t) = 0, t \in T. \end{cases}$$

and equal to

$$\beta = \beta(u, \tau_B) = \int\limits_{T^+} \Delta(t)(u(t) - d_1)dt + \int\limits_{T^-} \Delta(t)(u(t) - d_2)dt,$$

where $T^+ = \{t \in T/\Delta(t) > 0\}$, $T^- = \{t \in T/\Delta(t) < 0\}$.
The quantity $\beta(u, \tau_B)$ is called a suboptimality estimate of the support-control $\{u, \tau_B\}$.

Hence, the following inequality is always checked :

$$J(\bar{u}) - J(u) \leq \beta(u, \tau_B), \forall \bar{u},$$

and for $\bar{u} = u^\circ$, we have:

$$J(u^\circ) - J(u) \leq \beta(u, \tau_B).$$

Of this last inequality, the following result is deduced.

Theorem 2. *[5](Optimality criterion) Following relations :*

$$\begin{cases} \triangle u(t) = d_1 - u(t), & si \quad \Delta(t) > 0, \\ \triangle u(t) = d_2 - u(t), & si \quad \Delta(t) < 0, \\ d_1 - u(t) \leq \triangle u(t) \leq d_2 - u(t), & si \quad \Delta(t) = 0, t \in T. \end{cases} \tag{10}$$

are sufficient, and in the case of not degeneracy, they are necessary for the optimality of support-control $\{u, \tau_B\}$.

Remark 2. *We can write the optimality criterion in the extremal form*

$$\psi'(t)bu(t) = \max_{d_1 \leq u(t) \leq d_2} \psi'(t)bu(t), t \in T. \tag{11}$$

Let us introduce the function (Hamiltonian)

$$H(x,\psi,u,t) = \psi'(Ax(t) + bu(t)).$$

In terms of the Hamiltonian, relation (11) can be written as follows :

$$H(x,\psi,u,t) = \max_{d_1 \le u(t) \le d_2} H(x,\psi,u,t), t \in T \tag{12}$$

The optimality criterion of support-control can be formulated as follows.

Maximum principle *For optimality of the admissible control $u(t)$, $t \in T$, it is sufficient to have a support τ_B such that along the support-control $\{u, \tau_B\}$ and the corresponding trajectories $x(t)$, $\psi(t)$, $t \in T$, of system* (8) *Hamiltonian attain maximum value* (12)*.*
If $\{u, \tau_B\}$ is non-degenerate support-control then for optimality of the admissible control $u(t)$, $t \in T$, it is necessary that along $\{u, \tau_B\}$, $x(t)$, $\psi(t)$, $t \in T$, relation (12) *be satisfied.*

Now the criterion of suboptimality can be formulated as follows.

Theorem 3. *[5](Suboptimality criterion or ε-optimality criterion) At any $\varepsilon > 0$. An admissible control $u(t)$, $t \in T$ is ε-optimal if and only if there exists a support τ_B of the problem such that following ε- maximum conditions are satisfied along the corresponding solution $\psi(t)$, $t \in T$:*

$$H(x(t),\psi(t),u(t)) = \max_{d_1 \le u(t) \le d_2} H(x(t),\psi(t),u(t)) - \varepsilon(t)$$

with

$$\int_0^{t_f} \varepsilon(t)dt \le \varepsilon, t \in T.$$

4.2. Direct method for constructing the optimal controls

4.2.1. Introduction

The method suggested is iterative. We construct an algorithm based on three procedures :

1. Discretization.

2. Adaptive method.

3. Final procedure.

The discretization is present by a passage of the continuous problem to the discrete problem.

The adaptive method is called adaptive due to its property to use all the initial and current information for effective construction of suboptimal admissible points. This method be long to the same class as primal simplex method [5]. However the simplex method uses not arbitrary points but special basic points all the non-support (non-basic) components of which are critical. The only non-support component of the admissible point is changed at iterations of the simplex-method. The support (basis of the simplex method) is changed together with admissible point and its degree of non-optimality can increase at iterations. To stop solving of the simplex method uses (in the case of the existence of a solution) only the optimality criterion since it has no suboptimality criterion at all. Moreover, the optimality criterion for initial problem is tested by the equation (3). The final procedure is based on Newton's method, this method is applicable in the case where the difference of the last equation is sufficiently small. In this stage, we have introduced another step for algorithm called dual method, she permits of decrease this difference.

4.2.2. The discretization

Subdivide interval T to N subintervals $[\tau_j, \tau^j]$, with $\tau^j - \tau_j = h$, $h = (tf - 0)/N$, be the quantization step, N be a positive integer, and such that $T = \bigcup\limits_{j=1}^{N} [\tau_j, \tau^j]$.

As $u(t)$ is constant by piecewise, then denote

$$u(t) \equiv u_j, t \in [\tau_j, \tau^j], j = \overline{1, N}.$$

Using this last formulas and equation (5), the initial problem becomes the problem of the following linear programming :

$$\begin{cases} J(u) = \sum\limits_{j=1}^{N} C_j u_j \longrightarrow \max, \\ \sum\limits_{j=1}^{N} q_j u_j = g, \\ d_1 \leq u_j \leq d_2, j = \overline{1, N} \end{cases} \tag{13}$$

where $C_j = \int\limits_{\tau_j}^{\tau^j} C(t)dt$, $q_j = \int\limits_{\tau_j}^{\tau^j} p(t)dt$. This problem will be solved by the adaptive method [5].

Finite iteration of the adaptive method : A method for solving the extremal problem (13) is called finite if for any initial information it reveals unsolvability of the problem or constructs an ε-optimal admissible point after a finite number of operations. Every iteration of the adaptive method consists of a finite number of elementary arithmetical and logical operations. Therefore to prove in the theorem below that this method finite termination it is sufficient to show the finiteness of its iteration number.

Theorem 4. *[5] For $d_1 < \infty$, $d_2 < \infty$, $n < \infty$, $m < \infty$ and $\varepsilon \geq 0$ the adaptive method starting with an arbitrary initial support-point constructs an ε-optimal admissible point of very dually non-degenerate problem* (13) *in finite number of iterations.*

Let $\{u^\circ, J_B^\circ = \{j = \overline{1,m}\}\}$ an ε-optimal solution of the problem (13) given by the adaptive method.

Using this solution, we construct the solution of the initial problem. For this, let us put $\tau_B = \{\tau_j, j \in J_B^\circ\}$ and the control $u(t) = \{u_j^\circ, j \in J\}$, $t \in T$.

By using a support τ_B, we construct the co-control $\Delta(t) = -\psi'(t)b$, $t \in T$, where $\psi(t)$ is given by the formula (8). The support τ_B is used to find the quasi-control associated to $\omega(t)$, $t \in T$.

A quasi-control is defined as $\omega(t)$, $t \in T$

$$\omega(t) = \begin{cases} d_1, & si \quad \Delta(t) \geq 0, \\ d_2, & si \quad \Delta(t) < 0, t \in T \end{cases} \tag{14}$$

and the corresponding quasi-trajectory $\chi(t)$ of the system (5).

If $H\chi(t_f) = g$, then the control $\omega(t)$, $t \in T$ is optimal for the problem (1) – (4). If $H\chi(t_f) \neq g$, then, we construct the vector:

$$\lambda(\tau_B) = \phi_(\tau_B)^{-1}.(g - H\chi(t_f)). \tag{15}$$

Let $\mu > 0$ parameter of the method. If $\|\lambda(\tau_B)\| > \mu$, then change the support τ_B to the support $\bar{\tau}_B$ by the dual method. If $\|\lambda(\tau_B)\| < \mu$, then we pass to the final procedure.

4.3. Dual method

Let $t_\circ \in \tau_B$, such that $|\lambda(t_\circ)| = \max\limits_{t\in\tau_B} |\lambda(t)| > \mu$. The change of the support τ_B to $\bar{\tau}_B$ consists to change the co-control $\Delta(t)$ to $\bar{\Delta}(t) = \Delta(t) + \sigma\delta(t)$, $t \in T$, where $\delta(t)$ is the direction and σ is the dual step along this direction.

Compute the function :

$$\begin{cases} -\Delta(t)\backslash\delta(t), & \Delta(t)\delta(t) < 0 \\ 0, & otherwise, \end{cases}$$

and we construct the set :

$$T(\sigma) = \{t \in T, \sigma(t) < \sigma\}.$$

Hence, the speed of decrease of the dual quality criterion is equal to :

$$\alpha(\sigma) = -|\lambda(t_\circ)| + (d_2 - d_1) \int\limits_{T(\sigma)} |\delta(t)|dt.$$

By constructing $\alpha(0) < 0$ and $\alpha(\sigma) \leq \alpha(\bar{\sigma})$ if $\sigma < \bar{\sigma}$.

If $\alpha(\sigma) < 0$ for $\sigma > 0$ then, the problem $(1) - (4)$ does not admit admissible controls. Otherwise, we construct a step $\sigma_\circ \geq 0$ such that $\alpha(\sigma_\circ - \gamma) < 0, \alpha(\sigma_\circ + 0) \geq 0$ for all such that $0 < \gamma \leq \sigma_\circ$. By using $\sigma_\circ$, we obtain $t_* \in T \setminus \tau_B$, a moment such that:

$$\Delta(t_*) + \sigma_\circ \delta(t_*) = 0, \delta(t_*) \neq 0.$$

Then the new support is $\bar{\tau}_B = (\tau_B \backslash t_\circ) \bigcup t_*$. If $\lambda(\bar{\tau}_B) = 0$, then the quasi-control (14) constructed by the new support $\bar{\tau}_B$ is optimal of the problem $(1) - (4)$.

If $\|\lambda(\bar{\tau}_B)\| > \mu$, we perform the next iteration starting from the support-control $\{\omega, \bar{\tau}_B\}$.

If $\|\lambda(\bar{\tau}_B)\| < \mu$, we go to the final procedure.

4.4. The final procedure

We assume that the quasi-control ω and the corresponding quasi-trajectory χ, constructed by the support τ_B, and we have the condition $\|\lambda(\tau_B)\| < \mu$.

Let us $T^\circ = \{t \in T : \Delta(t) = 0\}$ the set of isolated times t_j , $j = \overline{1, m}$ and assume that $\dot{\Delta}(t_j) \neq 0, j = \overline{1, m}$.

The final procedure consists in constructing the solution $\tau_B^\circ = \{\tau_j^\circ, j = \overline{1, m}\}$ of the system of m nonlinear equations :

$$(d_2 - d_1) \sum_{j=1}^{m} sign\dot{\Delta}(t_j) \int\limits_{\tau_j}^{t_j} p(t)dt = g - H\chi(t_f), \tag{16}$$

Obtain from the constraint
$g - H\chi(t_f) = g - HF(t_f)x_\circ - \int\limits_T p(t)\omega(t)dt$, $g - H\chi(t_f) = g - HF(t_f)x_\circ - \int\limits_{T_H} p(t)\omega(t)dt -$

$\int\limits_{T_B} p(t)\omega(t)dt$. Identifying the part non-support T_H to zero and the part support T_B to μ.

We solve the system (16) by the Newton method using an initial approximation $\tau_B^\circ = \{\tau_j^{(\circ)}, j = \overline{1,m}\}$, $\tau_B^\circ = \tau_B = \{\tau_j, j = \overline{1,m}\}$.

The $(k+1)^{th}$ approximation $\tau_B^{(k+1)}$, at a step $k+1 \geq 1$, is computed as :

$$\tau_B^{(k+1)} = \tau_B^{(k)} + \frac{1}{d_2 - d_1}\{sign\dot{\Delta}(t_j)\lambda_j(\tau_B^{(k)}), j = \overline{1,m}\},$$

where $\lambda(\tau_B^{(k)})$ is a vector computed by the relation (15).

Then the function $\omega^\circ(t) = \omega(t)$, $t \in T$, computed by the support τ_B° is an optimal control of the problem (1) – (4).

5. Numerical example

The method analyzed in last section has been experimentally implemented. We shall present in this section preliminary computational results that demonstrate its efficiency on optimal control problem.

To illustrate of the results obtained here, we consider the terminal problem of the following optimal control :

$$\begin{cases} J(u) = x_2(3) \longrightarrow \max, \\ \dot{x}_1(t) = x_2(t), \quad x_1(0) = 0, \\ \dot{x}_2(t) = x_3(t), \quad x_2(0) = 0, \\ \dot{x}_3(t) = u(t), \quad x_3(0) = 0, \\ x_1(3) = 1, \\ |u(t)| \leq 1, t \in T = [0,3], \end{cases}$$

where $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, $c = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}$, $g = 1$.

Starting with three empty stats, we demand to determine the optimal control $u(t)$ allows to maximize $x_2(3)$ under the constraint $x_1(3) = 1$. By using the algorithm implemented under Matlab. Begin with discretizing this problem by using N periods. Therefore, we solved by the adaptive method. Finally, we have applied a final procedure. Here, we have realized a numerical comparison between the optimal solution obtaining by the adaptive method and the optimal solution of the initial problem. The following table contains the numerical results : where $J_d(u)$: is the quality criterion of the problem (1) – (4). $J_c(u)$: is the quality criterion of the problem (13).

Table 1: Numerical results for the two problem (1) - (4) and (13)

N	J°_{B}	$J_d(u)$	τ_B	$J_c(u)$	$H\chi(t_f)$	Time(s)
10	2	1.9588	0.6000	1.2600	0.1080	2.4836
100	16	1.9810	0.4800	1.8504	0.8343	3.2062
1000	152	1.9812	0.4560	1.9719	0.9882	19.1502
10000	1514	1.9812	0.4542	1.9811	0.9999	944.8169
20000	3028	1.9812	0.4542	1.9811	0.9999	3370.5000

From these results, we find that the optimal solution of the problem discretized does not optimal of the continuous problem , for different values of the number N, the quantity $H\chi(t_f)$ is not equal to g and the quality criterion value does not wait the optimal value, but where the number N is very high.

6. Conclusion

A new method of solving of a terminal problem of a linear dynamic system with controls constants by piecewise, is proposed. She is consisted of a fusion between the approach of the linear programming (Adaptive method) and nonlinear programming (Newton's method).

References

[1] Aidene mohamed, Louadj Kahina, *Adaptive Method for Solving Optimal Problem wiht State and Control Variables*, Mathematical Problems in Engineering, Volume 2012.

[2] Aidene mohamed, Louadj Kahina, *A problem of optimal control with free initial state*, ESAIM, Proe. 35, 184-190, 2012. Journal, ISSN 1270-900x.

[3] N. V. Balashevich, R. Gabasov, and F. M. Kirillova, *Numerical Methods for Open Loop and Closed Loop Optimization of Piecewise Linear Systems*, Computational Mathematics and Mathematical Physics, Vol. 41, No. 11, 2001, pp.1578-1593.

[4] Gabasov R, *Adaptive method of solving linear programming problem*, Preprint series of University of Karlsruhe, Institut of statistics and mathematics, 1994.

[5] Gabasov, R. and Kirillova, F. M., *Linear Programming Methods 1, 2 and 3*, Edition of University, Minsk (en russe), (1977, 1978 and 1980).

[6] M. Z. Ghanem, *Direct Method for Solving Optimal Control Problems with Kinks*, Journal of Optimisation Theory and Applications: Vol. 14, No. 4, 1974

[7] E.B.Lee and L.Markus, *Foundations of optimal control theory. The SIAM series in Applied Mathematics*, John Wiley and Sons, New-York-London-Sydney, 1967.

[8] L.Pontryagin et al, *Mathematical theory of optimal processes*, Eds Mir, Moscou 1974.

[9] S. Sager, H.G. Bock, G. Reinelt, *Direct Methods With Maximal Lower Bound for Mixed-Integer Optimal Control Problems*, Mathematical Programing Ser. A, 118: 109-149, 2009.

[10] S.P. Sethi and G.L.Thompson, *Optimal Control Theory: Applications to Management Science and Economics*, Second Edition, Springer, 2000, 504 pages.

[11] V. V. Ternovskii, M. M. Khapaev, *Direct Numerical Method for Solving Optimal Control Problems*, Doklady Mathematics, 2008, Vol. 77, No. 3, pp. 428431.

[12] E. Trélat, *Contrôle optimal : Théorie et applications*, Vuibert, Collection Mathématique Concr`tes, 2005, 246 pages.

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