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FIXED POINT THEOREMS ON TWO COMPLETE FUZZY METRIC SPACES

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Abstract. In this paper, we prove two related fixed points theorems in two fuzzy metric spaces which are the generalization of Theorem 3 of [3] and theorem 4 of [4].

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1. Introduction and preliminaries

In 1965, The concept of fuzzy sets was introduced initially by Zadeh [13]. George and Veeramani [5] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [6] with the help of continuous t-norms. Recently, many authers have proved fixed point theorems involving fuzzy sets. Fisher [3], Telci [12], Popa [8] and Aliouche and Fisher [1] proved some related fixed point Theorems in compact and complete metric spaces. The aim of this paper is to prove a unique fixed point theorem for two and four mappings, which generalize the Theorem 3 of [3] and theorem 4 of [4]. We give also a fuzzy version of Theorem 3 of [3] and theorem 4 of [4].

Definition 1. *[11] A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a continuous t-norm if it satisfies the following conditions:*

1) $*$ *is associative and commutative,*

2) $*$ *is continuous,*

3) $a * 1 = a$ *for all* $a \in [0,1]$*,*

4) $a * b \le c * d$ *whenever* $a \le c$ *and* $b \le d$*, for each* $a, b, c, d \in [0,1]$*.*

Definition 2. *[5] A 3-tuple* $(X, M, *)$ *is called a fuzzy metric space if* X *is an arbitrary (non-empty) set,* $*$ *is a continuous t-norm and* M *is a fuzzy set on* $X^2 \times (0,\infty)$ *satisfying the following conditions for each* $x, y, z \in X$ *and* $t, s > 0$*,*

1) $M(x, y, t) > 0$,
2) $M(x, y, t) = 1$ *if and only if* $x = y$,
3) $M(x, y, t) = M(y, x, t)$,
4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
5) $M(x, y, .) : (0, \infty) \rightarrow [0, 1]$ *is continuous.*

Example 1. *[5] Let* (X, d) *be a metric space. Define* $a * b = ab$ *for all* $a, b \in [0, 1]$ *and let* M_d *fuzzy sets on* $X^2 \times (o, \infty)$ *defined as follows,* $M_d(x, y, t) = \frac{t}{t+d(x,y)}$, *then* $(X, M_d, *)$ *is a fuzzy metric space. We call this fuzzy metric induced by the metric* d*, the standard fuzzy metric. On the other hand note that there exists no metric on* X *satisfying the above* $M_d(x, y, t)$.

Definition 3. *[3] Let* $(X, M, *)$ *be a fuzzy metric space.*
1) For $t > 0$*, the open ball* $B(x, r, t)$ *with center* $x \in X$ *and radius* $0 < r < 1$ *is defined by:*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

2) Let $(X, M, *)$ *be a fuzzy metric space and* τ *be the set of all* $A \subset X$ *with* $x \in A$ *if and only if there exist* $t > 0$ *and* $0 < r < 1$ *such that* $B(x, r, t) \subset A$. *Then,* τ *is a topology on* X *induced by the fuzzy metric* M.
3) A sequence $\{x_n\}$ *in* X *converges to* x *if and only if for any* $0 < \epsilon < 1$ *and* $t > 0$*, there exists* $n_0 \in N$ *such that for all* $n \geq n_0$*,* $M(x_n, x, t) > 1 - \epsilon$*; i.e.,* $M(x_n, x_m, t) \rightarrow 1 as\ n \rightarrow 1$ *for all* $t > 0$.
4) A sequence $\{x_n\}$ *in* X *is called a Cauchy sequence if and only if for any* $0 < \epsilon < 1$ *and* $t > 0$*, there exists* $n_0 \in N$ *such that for all* $n, m \geq n_0$*,* $M(x_n, x_m, t) > 1 - \epsilon$*; i.e.,* $M(x_n, x_m, t) \rightarrow 1$ *as* $n, m \rightarrow 1$ *for all* $t > 0$.
5) A fuzzy metric space (X, M, t) *in which every Cauchy sequence is convergent is said to be complete.*

Definition 4. *A subset* A *of* X *is said to be F-bounded if there exists* $t > 0$ *and* $0 < r < 1$ *such that* $M(x, y, t) > 1 - r$ *for all* $x, y \in A$.

Lemma 1. *[6] Let* $(X, M, *)$ *be a fuzzy metric space. Then,* $M(x, y, t)$ *is non-decreasing with respect to* t*, for all* x, y *in* X.

Lemma 2. *[6] Let* $(X, M, *)$ *be a fuzzy metric space. Then,* M *is a continuous function on* $X^2 \times (0, \infty)$.

Theorem 3. *[3] Let* (X, d) *and* (Y, ρ) *be complete metric spaces, let* T *be a continuous mappings of* X *into* Y *and let* S *be a mappings of* Y *into* X *satisfying the inequalities*

$$d(STx, STx') \le c \max\left\{d(x, x'), d(x, STx), d(x', STx'), \rho(Tx, Tx')\right\},$$

$$\rho(TSy, TSy') \le c \max\left\{\rho(y, y'), \rho(y, TSy), \rho(y', TSy'), d(Sy, Sy')\right\},$$

for all x, x' in X and y, y' in Y, where $0 \le c \le 1$.

Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Theorem 4. *[4] Let (X, d) and (Y, ρ) be complete metric spaces, let A, B be mappings of X into Y, and let S, T be mappings of Y into X satisfying the inequalities*

$$d(SAx, TBx') \le c \max\left\{d(x, x'), d(x, SAx), d(x', TBx'), \rho(Ax, Bx')\right\},$$

$$\rho(BSy, ATy') \le c \max\left\{\rho(y, y'), \rho(y, BSy), \rho(y', ATy'), d(Sy, Ty')\right\},$$

for all x, x' in X and y, y' in Y, where $0 \le c \le 1$. If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

The following lemma will be useful in the proof of theorem 6 and theorem 7.

Lemma 5. *[2] Let $\{x_n\}$ be a sequence in a fuzzy metric spaces $(X, M, *)$ with $M(x, y, t) \to 1$ as $t \to \infty$ for all $x, y \in X$. If there exists a number $k \in (0, 1)$ such that*

$$M(x_{n+1}, x_n, kt) \ge M(x_n, x_{n-1}, t),$$

for all $t > 0$ and $n = 1, 2, 3, ...$ Then $\{x_n\}$ is a cauchy sequence in X.

2. Main results

Theorem 6. *Let (X, M_1, θ_1) and (Y, M_2, θ_2) be complete fuzzy metric spaces with $M_1(x, x', t) \to 1$ as $t \to \infty$ for all $x, x' \in X$ and $M_2(y, y', t) \to 1$ as $t \to \infty$ for all $y, y' \in Y$. Let $T : X \to Y, S : Y \to X$ be mappings satisfying:*

$$M_1\left(STx, STx', kt\right) \ge \min\left\{M_1(x, x', t), M_1(x, STx, t), M_1(x', STx', t), M_2(Tx, Tx', t)\right\}, \tag{1}$$

$$M_2\left(TSy, TSy', kt\right) \ge \min\left\{M_2(y, y', t), M_2(y, TSy, t), M_2(y', TSy', t), M_1(Sy, Sy', t)\right\}, \tag{2}$$

for all $x, x' \in X$, $y, y' \in Y$ and for all $t > 0$, where $0 < k < 1$. Then ST has a unique fixed point z in X and TS has a unique fixed point w in Y. Further, $Tz = w$ and $Sw = z$.

Proof. Let x be an arbitrary point in X. We define the sequences $\{x_n\}$ and $\{y_n\}$ in X and Y respectively by:

$$Sy_n = x_n, Tx_{n-1} = y_n,$$

for n=1, 2, ... Putting $x = x_n$ and $y = y_n$ for all n. Applying inequality (1),we get

$$M_1(x_{n+1}, x_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_1(x_n, x_{n+1}, t), M_1(x_{n-1}, x_n, t), M_2(y_{n+1}, y_n, t)\}, \tag{3}$$

Using inequality (2), we have

$$M_2(y_{n+1}, y_n, kt) \geq \min\{M_2(y_n, y_{n-1}, t), M_2(y_n, y_{n+1}, t), M_2(y_{n-1}, y_n, t), M_1(x_n, x_{n-1}, t)\}, \tag{4}$$

involve, respectively

$$M_1(x_{n+1}, x_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_{n+1}, y_n, t)\}, \tag{5}$$

$$M_2(y_{n+1}, y_n, kt) \geq \min\{M_2(y_n, y_{n-1}, t), M_1(x_n, x_{n-1}, t)\}, \tag{6}$$

using inequalty (1) again, it follows that

$$M_1(x_{n-1}, x_n, kt) \geq \min\{M_1(x_{n-2}, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\}. \tag{7}$$

Similary, using inequality (2), we get

$$M_2(y_{n+1}, y_n, kt) \geq \min\{M_2(y_n, y_{n-1}, t), M_1(x_n, x_{n-1}, t)\}, \tag{8}$$

and

$$M_2(y_n, y_{n-1}, kt) \geq \min\{M_2(y_{n-1}, y_{n-2}, t), M_1(x_{n-1}, x_{n-2}, t)\}. \tag{9}$$

Using inequalities (5) and (8), we have

$$M_1(x_{n+1}, x_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\}, \tag{10}$$

and similary, from inequalities (7) and (9), we get

$$M_1(x_{n+1}, x_n, kt) \geq \min\{M_1(x_n, x_{n-1}, t), M_2(y_n, y_{n-1}, t)\}. \tag{11}$$

It now follows inequalities (8),(9),(10) and (11) that

$$M_1(x_{n+1}, x_n, kt) \geq M_2(y_n, y_{n-1}, t), \tag{12}$$

$$M_2(y_{n+1}, y_n, kt) \geq M_1(x_n, x_{n-1}, t). \tag{13}$$

Using (12) and (13) we have for n=1, 2, ...

$$M_1\left(x_{n+1},x_n,t\right)\geq M_1(x_n,x_{n-1},\frac{t}{k^2}),$$

$$M_2\left(y_{n+1},y_n,t\right)\geq M_2(y_n,y_{n-1},\frac{t}{k^2}).$$

From Lemma 1, it follows that $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in X and Y respectively. Hence $\{x_n\}$ converges to z in X and $\{y_n\}$ converges to w in Y. Now suppose that T is continuous, then

$$\lim Tx_{n-1}=Tz=\lim y_n=w,$$

and so $Tz=w$. Applying inequalitiy (1), we have

$$\begin{aligned} M_1 \quad & (STz,STx_{n-1},kt)\geq \\ & \min\left\{M_1(z,x_{n-1},t),M_1(z,STz,t),M_1(x_{n-1},STx_{n-1},t),M_2(Tz,Tx_{n-1},t)\right\}, \end{aligned}$$

letting n tend to infinity, we have

$$M_1\left(Sw,z,kt\right)\geq \min\left\{1,M_1(z,Sw,t),1\right\},$$

so $Sw=z$. In the same manner we can show that $Tz=w$. Finally we show that the fixed point is unique. Suppose that ST has a second fixed point z' in X. Then, using inequality (1), we have

$$M_1\left(z,z',kt\right)\geq \min\left\{M_1(z,z',t),M_2(Tz,Tz',t)\right\}. \tag{14}$$

Next, using inequality (2), we have

$$M_2\left(Tz,Tz',kt\right)\geq \min\left\{M_2(Tz,Tz',t),M_2(Tz,Tz,t),M_2(Tz',Tz',t),M_1(z,z',t)\right\}. \tag{15}$$

It now follows easily from inequalities (14) and (15) that

$$M_1\left(z,z',kt\right)\geq M_2(Tz,Tz',t)$$

and

$$M_2\left(Tz,Tz',kt\right)\geq M_1(z,z',t).$$

Hence

$$M_1\left(z,z',t\right)\geq M_1(z,z',\frac{t}{k^2}),$$

and so $z=z'$. The uniqueness of w follows in a similar manner.

Theorem 7. *Let (X, M_1, θ_1) and (Y, M_2, θ_2) be complete fuzzy metric spaces with $M_1(x, x', t) \to 1$ as $t \to \infty$ for all $x, x' \in X$ and $M_2(y, y', t) \to 1$ as $t \to \infty$ for all $y, y' \in Y$. Let $A, B : X \to Y$, $S, T : Y \to X$ be mappings satisfying:*

$$M_1\left(SAx, TBx', kt\right) \geq \min\left\{M_1(x, x', t), M_1(x, SAx, t), M_1(x', TBx', t), M_2(Ax, Bx', t)\right\}, \tag{16}$$

$$M_2\left(BSy, ATy', kt\right) \geq \min\left\{M_2(y, y', t), M_2(y, BSy, t), M_2(y', ATy', t), M_1(Sy, Ty', t)\right\}, \tag{17}$$

for all $x, x' \in X$, $y, y' \in Y$ and for all $t > 0$, where $0 < k < 1$.If one of the mappings A, B, S and T is continuous then SA and TB have a common fixed point z in X and BS and AT have a common fixed point w in Y. Further, $Az = Bz = w$ and $Sw = Tw = z$.

Proof. Let x be an arbitrary point in X. We define the sequences $\{x_n\}$, and $\{y_n\}$ in X and Y respectively by:

$$Sy_{2n-1} = x_{2n-1}, Bx_{2n-1} = y_{2n}, Ty_{2n} = x_{2n}, Ax_{2n} = y_{2n+1},$$

for n=1, 2, ... Putting $x = x_{2n}$ and $y = y_{2n}$ in(16), we get

$$M_1\left(x_{2n+1}, x_{2n}, kt\right) \geq \min\left\{M_1(x_{2n}, x_{2n-1}, t), M_1(x_{2n}, x_{2n+1}, t), M_1(x_{2n-1}, x_{2n}, t), M_2(y_{2n+1}, y_{2n}, t)\right\}, \tag{18}$$

Using inequality (17), we have

$$M_2\left(y_{2n+1}, y_{2n}, kt\right) \geq \min\left\{M_2(y_{2n}, y_{2n-1}, t), M_2(y_{2n}, y_{2n+1}, t), M_2(y_{2n-1}, y_{2n}, t), M_1(x_{2n+1}, x_{2n}, t)\right\}. \tag{19}$$

Therfore

$$M_1\left(x_{2n+1}, x_{2n}, kt\right) \geq \min\left\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n+1}, y_{2n}, t)\right\}, \tag{20}$$

$$M_2\left(y_{2n+1}, y_{2n}, kt\right) \geq \min\left\{M_2(y_{2n}, y_{2n-1}, t), M_1(x_{2n+1}, x_{2n}, t)\right\}. \tag{21}$$

Applying the inequality (16) again, it follows that

$$M_1\left(x_{2n-1}, x_{2n}, kt\right) \geq \min\left\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t)\right\}. \tag{22}$$

Similary, using inequality (17), we get

$$M_2\left(y_{2n+1}, y_{2n}, kt\right) \geq \min\left\{M_2(y_{2n}, y_{2n-1}, t), M_1(x_{2n-1}, x_{2n}, t)\right\}, \tag{23}$$

and

$$M_2\left(y_{2n}, y_{2n-1}, kt\right) \geq \min\left\{M_2(y_{2n-1}, y_{2n-2}, t), M_1(x_{2n-1}, x_{2n-2}, t)\right\}. \tag{24}$$

Using inequalities (20) and (23) that

$$M_1\left(x_{2n+1}, x_{2n}, kt\right) \geq \min\left\{M_1(x_{2n}, x_{2n-1}, t), M_2(y_{2n}, y_{2n-1}, t)\right\}, \tag{25}$$

and in a similar manner, from inequalities (22) and (24), we obtain

$$M_1\left(x_{n+1}, x_{2n}, kt\right) \geq \min\left\{M_1(x_{2n-2}, x_{2n-1}, t), M_2(y_{2n-1}, y_{2n-2}, t)\right\}. \tag{26}$$

It now follows inequalities (23), (24), (25) and (26) that

$$M_1\left(x_{n+1}, x_n, kt\right) \geq M_2(y_n, y_{n-1}, t), \tag{27}$$

and

$$M_2\left(y_{n+1}, y_n, kt\right) \geq M_1(x_n, x_{n-1}, t). \tag{28}$$

Using (27) and (28) we have for n=1, 2, ...

$$M_1\left(x_{n+1}, x_n, t\right) \geq M_1(x_n, x_{n-1}, \frac{t}{k^2}),$$

and

$$M_2\left(y_{n+1}, y_n, t\right) \geq M_2(y_n, y_{n-1}, \frac{t}{k^2}).$$

From lemma 1, it follows that $\{x_n\}$ and $\{y_n\}$ are cauchy sequences in X and Y respectively. Hence $\{x_n\}$ converges to z in X and $\{y_n\}$ converges to w in Y. If A is continuous, then

$$\lim Ax_{2n} = Az = \lim y_{2n+1} = w,$$

and so $Az = w$. Using inequalitiy (16), we have

$$M_1\left(SAz, TBx_{2n-1}, kt\right) \geq$$
$$\min\left\{M_1(z, x_{2n-1}, t), M_1(z, SAz, t), M_1(x_{2n-1}, TBx_{2n-1}, t), M_2(Az, y_{2n}, t)\right\},$$

$$M_1\left(Sw, x_{2n}, kt\right) \geq \min\left\{M_1(z, x_{2n-1}, t), M_1(z, Sw, t), M_1(x_{2n-1}, x_{2n}, t), M_2(w, y_{2n}, t)\right\},$$

letting n tend to infinity, we get

$$M_1\left(Sw, z, kt\right) \geq \min\left\{1, M_1(z, Sw, t), 1\right\},$$

and so $Sw = z$. Similary we can show that $Az = w$. Now, $SAz = Sw = z$ and $ASw = Az = w$ The same result hold also if one of the mappings B, S, T is continuous. To prove the uniqueness of z, suppose that SA has a second fixed point z' in X.

Then, using inequality (16), we have

$$M_1\left(SAz, TBz', kt\right) \geq \min\left\{M_1(z,z',t), M_1(z,SAz,t), M_1(z',TBz',t), M_2(Az,Bz',t)\right\}. \quad (29)$$

Next, using inequality (17), we have

$$M_2\left(Bz, Az', kt\right) \geq \min\left\{M_2(y,y',t), M_2(y,Bz,t), M_2(y',Az',t), M_1(z,z',t)\right\}. \quad (30)$$

$$M_2\left(Az, Bz', kt\right) \geq \min\left\{M_2(Az,Bz',t), M_2(Az,Bz,t), M_2(Az',Az',t), M_1(z,z',t)\right\}. \quad (31)$$

It now follows easily from inequalities (29) and (30) that

$$M_1\left(z, z', kt\right) \geq \min\left\{1, M_1(z,z',t), M_2(Az,Bz',t)\right\},$$

and

$$M_1\left(Az, Bz', kt\right) \geq \min\left\{1, M_1(z,z',t), M_2(Az,Bz',t)\right\}.$$

Then

$$M_1\left(z, z', kt\right) \geq M_2(Az,Bz',t).$$

Similarly, we have

$$M_2\left(Az, Bz', kt\right) \geq M_1(z,z',t).$$

Hence

$$M_1\left(z, z', t\right) \geq M_1(z,z',\frac{t}{k^2}),$$

and so $z = z'$. The uniqueness of w follows in a similar manner.

The following example illustrate the theorem 6 theorem 7.

Example 2. *Let $X = Y = [0,1]$ and $M_1(x,y,t) = M_2(x,y,t) = \frac{t}{t+|x-y|}$. For all $x \in X$ and for all $t > 0$, let A, B be mappings of X into Y define by:*

$$Ax = Bx = \begin{cases} \frac{x}{2} & \text{if} \quad x \in \left(0, \frac{1}{2}\right] \\ \frac{1}{2} & \text{if} \quad x = 0 \end{cases},$$

and for all $y \in Y$ and for all $t > 0$, let S, T be mappings of Y into X define by: $Sy = Ty = \frac{1}{2}$. In this example, the ineqality (16) is satisfied since the value of the left hand side of inequalitiy is 1 and the inequality (17) is satisfied. Clearly, $SA(\frac{1}{2}) = TB(\frac{1}{2}) = \frac{1}{2}$, $BS(\frac{1}{4}) = AT(\frac{1}{4}) = \frac{1}{4}$, $A\left(\frac{1}{2}\right) = B\left(\frac{1}{2}\right) = \frac{1}{4}$ and $S\left(\frac{1}{4}\right) = T\left(\frac{1}{4}\right) = \frac{1}{2}$.

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