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A CLASSIFICATION OF THREE DIMENSIONAL $N(K)$ CONTACT METRIC MANIFOLDS

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Abstract. The object of the present paper is to study three dimensional $N(k)$-contact metric manifolds satisfying certain curvature conditions.

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1. Introduction

Let M be a contact Riemannian manifold and (ϕ, ξ, η, g) its contact structure. If the characteristic vector field ξ is a Killing vector, then M is called a K-contact manifold. Further if the curvature tensor R satisfies $R(X,Y)\xi = \eta(Y)X - \eta(X)Y$, then M is called a Sasakian manifold. Again, if the characteristic vector field $\xi \in k$-nullity distribution, then the manifold is said to be an $N(k)$-contact metric manifold. For an $N(k)$-contact metric manifold of dimension $2n+1 > 3$, Tanno [14] (resp. Okumura [12]) proved that if M is Einstein manifold (resp. $\nabla S = 0$) then M is Einstein-Sasakian. Again in [9], D. Perrone proved that if M satisfies *a*) $R(X,\xi).S = 0$ and *b*) $R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y]$, for some function k on M, then either M is locally isometric to the Riemannian product $E^{n+1}(0) \times S^n(4)$ or M is an Einstein-Sasakian manifold. Three dimensional $N(k)$-contact manifolds have been studied by D. E. Blair, Th. Koufogiorgos and R. Sharma in [4]. In [16], Z. Olszak studied η-parallel Ricci tensor on Sasakian manifolds. Motivated by these works in this paper we study three dimensional $N(k)$-contact metric manifolds with $R(X,Y).S = 0$ and this manifold satisfying certain curvature conditions.

The paper is organized as follows: After preliminaries, in section 3 we study a three dimensional $N(k)$-contact metric manifold satisfying $R(X,Y).S = 0$ and obtain some equivalent conditions. Also some important corollaries are stated here. Section 4 is devoted to study cyclic parallel Ricci tensor on three dimensional $N(k)$-contact manifolds and we prove that the manifold is either Sasakian or of constant curvature. In section 5, we find a necessary and sufficient condition of three dimensional $N(k)$-contact manifolds to have η-parallel Ricci tensor. Also in this section we establish a relation between η-parallelity and cyclic parallelity of Ricci tensor of this manifold. Finally, in Section 6 we construct two non-trivial examples of three dimensional $N(k)$-contact metric manifolds.

2. Preliminaries

A $(2n+1)$-dimensional manifold $M^{(2n+1)}$ is said to admit an almost contact metric structure if it admits a tensor field ϕ of type $(1,1)$, a vector field ξ and a 1-form η satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0 \text{ and } \eta \circ \phi = 0. \tag{1}$$

An almost contact metric structure is said to be normal if the induced almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

is integrable, where X is tangent to M, t is the coordinate of $\mathbb{R}$ and f is a smooth function on $M \times \mathbb{R}$. Let g be a compatible Riemannian metric with almost contact structure (ϕ, ξ, η), i.e.,

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y). \tag{2}$$

Then M becomes an almost contact metric manifold equipped with an almost contact metric structure (ϕ, ξ, η, g). From (1) it can be easily seen that

$$g(X, \phi Y) = -g(\phi X, Y), \qquad g(X, \xi) = \eta(X), \tag{3}$$

for all vector fields $X, Y \in \chi(M)$. An almost contact metric structure becomes a contact metric structure if

$$g(X, \phi Y) = d\eta(X,Y), \tag{4}$$

for all vector fields $X, Y \in \chi(M)$. The 1-form η is then called a contact form and ξ is its characteristic vector field. We define a $(1,1)$ tensor field h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds$ denotes the Lie derivative. Then h is symmetric and satisfies $h\phi = -\phi h$. We have $Tr.h = Tr.\phi h = 0$ and $h\xi = 0$. Also

$$\nabla_X \xi = -\phi X - \phi h X \tag{5}$$

holds in a contact metric manifold. A normal contact metric manifold is a Sasakian manifold. An almost contact metric manifold is Sasakian if and only if

$$(\nabla_X \phi)(Y) = g(X, Y)\xi - \eta(Y)X, \qquad X, Y \in \chi(M), \tag{6}$$

where ∇ is the Levi-Civita connection of the Riemannian metric g. A contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$ for which ξ is a Killing vector field is said to be a K-contact manifold. A Sasakian manifold is K-contact but not conversely. However a 3-dimensional K-contact manifold is Sasakian [10]. It is well known that the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X, Y)\xi = 0$ [5]. On the other hand on a Sasakian manifold the following relation holds:

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y. \tag{7}$$

As a generalization of both $R(X, Y)\xi = 0$ and the Sasakian case: D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou [6] introduced the (k, μ)- nullity distribution on a contact metric manifold and gave several reasons for studying it. The (k, μ)-nullity distribution $N(k, \mu)$ ([6],[2]) of a contact metric manifold M is defined by

$$\begin{aligned} N(k, \mu) \quad : \quad & p \longrightarrow N_p(k, \mu) \\ & = \{W \in T_pM : R(X, Y)W = (kI + \mu h)(g(Y, W)X - g(X, W)Y)\}, \end{aligned}$$

for all $X, Y \in \chi(M)$, where $(k, \mu) \in \mathbb{R}^2$. A contact metric manifold M^{2n+1} with $\xi \in N(k, \mu)$ is called a (k, μ)-contact manifold. In particular on a (k, μ)-contact manifold, we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY]. \tag{8}$$

On a (k, μ)-contact manifold $k \leq 1$. If $k = 1$, the structure is Sasakian ($h = 0$ and μ is indeterminant) and if $k < 1$, then the (k, μ)-nullity condition determines the curvature of M^{2n+1} completely [6]. Infact, for a (k, μ)-manifold, the condition of being Sasakian, a K-contact manifold , $k = 1$ and $h = 0$ are all equivalent.

The k-nullity distribution $N(k)$ of a Riemannian manifold M is defined by [14]

$$N(k) : p \longrightarrow N_p(k) = \{Z \in T_pM : R(X, Y)Z = k[g(Y, Z)X - g(X, Z)Y]\},$$

k being a constant. If the characteristic vector field $\xi \in N(k)$, then we call the manifold an $N(k)$-contact metric manifold [7]. If $k = 1$, then the manifold is Sasakian and if $k = 0$, then the manifold is locally isometric to the product $E^{n+1}(0) \times S^n(4)$ for $n > 1$ and flat for $n = 1$ [5]. In a (k, μ)-contact manifold if $\mu = 0$, then the

manifold becomes an $N(k)$-contact manifold.
In [3], $N(k)$-contact metric manifold were studied in details. For more details we refer to ([4],[8]).

In $N(k)$-contact metric manifold the following relations hold:

$$h^2 = (k-1)\phi^2, \quad k \leq 1, \tag{9}$$

$$(\nabla_X \phi)(Y) = g(X + hX, Y)\xi - \eta(Y)(X + hX), \tag{10}$$

$$R(\xi, X)Y = k[g(X,Y)\xi - \eta(Y)X], \tag{11}$$

$$S(X, \xi) = 2nk\eta(X), \tag{12}$$

$$\begin{aligned} S(X,Y) &= 2(n-1)g(X,Y) + 2(n-1)g(hX,Y) \\ &\quad +[2(1-n) + 2nk]\eta(X)\eta(Y), \quad n \geq 1, \end{aligned} \tag{13}$$

$$r = 2n(2n - 2 + k), \tag{14}$$

$$S(\phi X, \phi Y) = S(X,Y) - 2nk\eta(X)\eta(Y) - 4(n-1)g(hX,Y), \tag{15}$$

$$(\nabla_X \eta)(Y) = g(X + hX, \phi Y), \tag{16}$$

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{17}$$

$$\eta(R(X,Y)Z) = k[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{18}$$

where $X, Y, Z \in \chi(M)$, R is the Riemannian curvature tensor and S is the Ricci tensor of the manifold.

In a three dimensional Riemannian manifold we have,

$$\begin{aligned} R(X,Y)Z &= g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X \\ &\quad -S(X,Z)Y - \frac{r}{2}[g(Y,Z)X - g(X,Z)Y], \end{aligned} \tag{19}$$

where Q is the Ricci operator and r is the scalar curvature of the manifold. Putting $Z = \xi$ in (19) and using (12) and (17) yields

$$\begin{aligned} k[\eta(Y)X - \eta(X)Y] &= \eta(Y)QX - \eta(X)QY \\ &\quad +(2k - \frac{r}{2})[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{20}$$

Replacing Y by ξ in (20), we obtain

$$QX = [\frac{r}{2} - k]X + [3k - \frac{r}{2}]\eta(X)\xi. \tag{21}$$

From (21) we have a three dimensional $N(k)$-contact metric manifold is an η-Einstein manifold.

Now we prove a Lemma here:

Lemma 2.1. A 3-dimensional N(k)-contact metric manifold is a manifold of constant curvature if and only if the scalar curvature r=6k.

Proof: Using (21) in (19) we obtain

$$\begin{aligned} R(X,Y)Z &= \frac{r-4k}{2}[g(Y,Z)X - g(X,Z)Y] \\ &+ \frac{6k-r}{2}[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y]. \end{aligned} \tag{22}$$

It is clear from (22) that the manifold is of constant curvature if and only if the scalar curvature $r = 6k$. This proves the lemma.

3. Ricci semisymmetric $N(k)$-contact metric manifolds of dimension three

Definition 3.1. A Riemannian manifold is said to be Ricci semisymmetric if $R(X,Y).S = 0$, where $R(X,Y)$ is treated as a derivation of the tensor algebra for any tangent vector X,Y; R denotes the curvature tensor and S is the Ricci tensor of type $(0,2)$ of the manifold.

In this section we consider Ricci semisymmetric $N(k)$-contact metric manifolds of dimension three. Then we have

$$(R(X,Y).S)(U,V) = 0. \tag{23}$$

Putting $X = \xi$ in (23), we obtain

$$(R(\xi,Y).S)(U,V) = 0. \tag{24}$$

i.e.,

$$S(R(\xi,Y)U,V) + S(U,R(\xi,Y)V) = 0. \tag{25}$$

Using (11) and (12) in (25) yields

$$k[2kg(Y,U)\eta(V) - S(Y,V)\eta(U) + 2kg(Y,V)\eta(U) - S(U,Y)\eta(V)] = 0. \tag{26}$$

Therefore from (26) we have

$$2kg(Y,U)\eta(V) - S(Y,V)\eta(U) + 2kg(Y,V)\eta(U) - S(U,Y)\eta(V) = 0, \tag{27}$$

provided $k \neq 0$. Now, in the relation (27), putting $Y = U = e_i$, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space and summing up for $i = 1$ to 3, we obtain

$$\begin{aligned} 2k\sum_{i=1}^{3} g(e_i,e_i)\eta(V) \quad &- \quad \sum_{i=1}^{3}\eta(e_i)S(e_i,V) + 2k\sum_{i=1}^{3} g(e_i,V)\eta(e_i) \\ &- \quad \sum_{i=1}^{3}\eta(V)S(e_i,e_i) = 0. \end{aligned} \tag{28}$$

i.e.,

$$(6k - r)\eta(V) = 0. \tag{29}$$

Since V is an arbitary vector field therefore $\eta(V) \neq 0$, in general. Hence we have from (29), $r = 6k$. Using the value of r in (21) yields

$$QX = 2kX \tag{30}$$

i.e.,

$$S(X,Y) = 2kg(X,Y). \tag{31}$$

Thus the manifold under consideration is an Einstein manifold. It is easy to show that for an Einstein $N(k)$-contact metric manifold $\nabla S = 0$.

Consversely, if Ricci tensor S is parall then $R.S = 0$. Hence we state the folowing:

Theorem 3.1. In a three dimensional $N(k)$-contact metric manifold with $k \neq 0$ the following conditions are equivalent:
i) $R.S = 0$,
ii) the manifold is an Einstein manifold,
iii) $\nabla S = 0$.

From (19) it follows that in a three dimensional Riemannian manifold, semisymmetry ($R.R = 0$) and Ricci-semisymmetry ($R.S = 0$) are equivalent. Therefore in view of Lemma 2.1 we state the following:

Corollary 3.1. In a three dimensional $N(k)$-contact metric manifold with $k \neq 0$ the following conditions are equivalent:
i) $R.R = 0$,
ii) the manifold is of constant curvature,
iii) $\nabla R = 0$.

In [14], the authors proved that for an Einstein $N(k)$-contact metric manifold $k = 1$ and hence the manifold is Sasakian. Thus we state the following corollary:

Corollary 3.2. A Ricci-semisymmetric three dimensional $N(k)$-contact metric manifold with is an Einstein-Sasakian manifold, provided $k \neq 0$.

Again in [4], Blair, Koufogiorgos and Sharma proved that a 3-dimensional $N(k)$-contact metric manifold is locally ϕ-symmetric if and only if the scalar curvature of the manifold is constant. Therefore in view of Theorem 3.1 we state the following:

Corollary 3.3. A Ricci semisymmetric $N(k)$-contact metric manifold of dimension three is locally ϕ-symmetric, provided $k \neq 0$.

4. Three dimensional $N(k)$-contact metric manifolds satisfying cyclic parallel Ricci tensor

A. Gray [1] introduced two classes of Riemannian manifold determined by covariant derivative of Ricci tensor. The class A consisting of all Riemannian manifolds whose Ricci tensor S is a Codazzi tensor, i.e.,

$$(\nabla_X S)(Y, Z) = (\nabla_Y S)(X, Z).$$

The class B consisting of all Riemannian manifolds whose Ricci tensor is cyclic parallel, i.e.,

$$(\nabla_X S)(Y, Z) + (\nabla_Y S)(Z, X) + (\nabla_Z S)(X, Y) = 0. \tag{32}$$

A Riemannian manifold is said to satisfy cyclic parallel Ricci tensor if the Ricci tensor is non-zero and satisfies the condition (32). It is known [15] that Cartan hypersurface are manifolds with non-parallel Ricci tensor satisfying the condition (32). In this section we consider a three dimensional $N(k)$-contact metric manifolds satisfying cyclic parallel Ricci tensor.

Putting $Y = Z = e_i$ in (32) and taking summation over $i = 1$ to 3, where $\{e_i\}$, $i = 1, 2, 3$, is an orthonormal basis of the tangent space, we obtain

$$(\nabla_X S)(e_i, e_i) + 2(\nabla_{e_i} S)(X, e_i) = 0. \tag{33}$$

Now,

$$(\nabla_X S)(e_i, e_i) = \nabla_X S(e_i, e_i) - 2S(\nabla_X e_i, e_i). \tag{34}$$

We know that the scalar curvature $r = \sum_i S(e_i, e_i)$. Also, in local coordinates $\nabla_X e_i = X^j \Gamma^h_{ji} e_h$, where Γ^h_{ji} are the Christoffel symbols. Since $\{e_i\}$ are orthonormal basis, the metric tensor $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta and hence the Cristoffel symbols are zero. Therefore, $\nabla_X e_i = 0$. Hence from (34) it follows that

$$(\nabla_X S)(e_i, e_i) = dr(X). \tag{35}$$

Again, we know that

$$\begin{aligned}(divQ)(X) &= tr(Z \to (\nabla_Z Q)(X)) \\ &= \sum_i g((\nabla_{e_i} Q)(X), e_i).\end{aligned}$$

But it is known [13] that $(divQ)(X) = \frac{1}{2} dr(X)$.
Hence

$$(\nabla_Z S)(X, e_i) = \frac{1}{2} dr(X). \tag{36}$$

Using (35) and (36) in (33), we obtain $dr(X) = 0$, for all $X \in \chi(M)$.

From (21) we have

$$S(X, Y) = (\frac{r}{2} - k) g(X, Y) + (3k - \frac{r}{2}) \eta(X) \eta(Y). \tag{37}$$

Differentiating (37) covariantly with respect to Z, we obtain

$$\begin{aligned}(\nabla_Z S)(X, Y) &= \frac{1}{2} dr(Z) g(X, Y) - \frac{1}{2} dr(Z) \eta(X) \eta(Y) \\ &\quad + (3k - \frac{r}{2})[(\nabla_Z \eta)(X) \eta(Y) + \eta(X)(\nabla_Z \eta)(Y)].\end{aligned} \tag{38}$$

Using $dr(Z) = 0$ in (38) yields

$$(\nabla_Z S)(X, Y) = (3k - \frac{r}{2})[(\nabla_Z \eta)(X) \eta(Y) + \eta(X)(\nabla_Z \eta)(Y)]. \tag{39}$$

Using (39) in (32), we get

$$\begin{aligned}(3k - \frac{r}{2})[(\nabla_X \eta)(Y) \eta(Z) &+ (\nabla_X \eta)(Z) \eta(Y) + (\nabla_Y \eta)(X) \eta(Z) \\ &+ (\nabla_Y \eta)(Z) \eta(X) + (\nabla_Z \eta)(X) \eta(Y) \\ &+ (\nabla_Z \eta)(Y) \eta(X)] = 0.\end{aligned} \tag{40}$$

Hence either $r = 6k$ or

$$\begin{aligned}(\nabla_X \eta)(Y)\eta(Z) &+ (\nabla_X \eta)(Z)\eta(Y) + (\nabla_Y \eta)(X)\eta(Z) \\ &+ (\nabla_Y \eta)(Z)\eta(X) + (\nabla_Z \eta)(X)\eta(Y) \\ &+ (\nabla_Z \eta)(Y)\eta(X) = 0.\end{aligned} \tag{41}$$

Using (16) and (3) in (41), we obtain

$$2[g(X, \phi hZ)\eta(Y) + g(Y, \phi hX)\eta(Z) + g(Z, \phi hY)\eta(X)] = 0. \tag{42}$$

Putting $X = \xi$ in (42) and using $h\xi = 0$ and $\phi\xi = 0$, we get

$$g(Z, \phi hY) = 0. \tag{43}$$

i.e.,

$$g(\phi Z, hY) = 0. \tag{44}$$

Putting hY instead of Y in (44) and using (9), we have

$$(1 - k)g(\phi X, Y) = 0. \tag{45}$$

Since $g(\phi X, Y) \neq 0$, hence (45) gives $k = 1$. Again, we know that an $N(k)$-contact metric manifold with $k = 1$ is a Sasakian manifold. Hence in view of the above discussions we state the following:

Proposition 4.1. A three dimensional $N(k)$-contact metric manifold with cyclic parallel Ricci tensor is either a Sasakian manifold or the scalar curvature $r = 6k$ =constant.

Again, we know that a three dimensional $N(k)$-contact metric manifold is of constant curvature if and only if $r = 6k$. Therefore we have the following:

Theorem 4.1. A three dimensional $N(k)$-contact metric manifold with cyclic parallel Ricci tensor is either a Sasakian manifold or a manifold of constant curvature.

5. Three dimensional $N(k)$-contact metric manifold with η-parallel Ricci tensor

Definition 5.1. A three dimensional $N(k)$-contact manifold is said to have a η-parallel Ricci tensor if the Ricci tensor satisfies

$$(\nabla_X S)(\phi Y, \phi Z) = 0, \tag{46}$$

for $X, Y, Z \in \chi(M)$

In this section we consider a three dimensional $N(k)$-contact manifold with η-parallel Ricci tensor.

From (46) we obtain

$$\nabla_X S(\phi Y, \phi Z) - S((\nabla_X \phi)(Y), \phi Z) - S(\phi Y, (\nabla_X \phi)(Z)) = 0. \tag{47}$$

Using (10), (15) and (16) in (47) yields

$$\begin{aligned}(\nabla_X S)(Y, Z) &= 2k[g(X + hX, \phi Y)\eta(Z) + g(X + hX, \phi Z)\eta(Y)] \\ &\quad -[S(X + hX, \phi Z)\eta(Y) + S(X + hX, \phi Y)\eta(Z)].\end{aligned} \tag{48}$$

Again, if the manifold satisfies the condition (48) then it is clear that the Ricci tensor of the manifold is η-parallel. Therefore we state the following:

Theorem 5.1. A necessary and sufficient condition for η-parallelity of the Ricci tensor in a three dimensional $N(k)$-contact metric manifold is that the relation (48) holds.

Using (48) we have

$$\begin{aligned}&(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) \\ =\ & 2k[g(X + hX, \phi Y)\eta(Z) + g(X + hX, \phi Z)\eta(Y) \\ &+g(Y + hY, \phi Z)\eta(X) + g(Y + hY, \phi X)\eta(Z) \\ &+g(Z + hZ, \phi X)\eta(Y) + g(Z + hZ, \phi Y)\eta(X)] \\ &-[S(X + hX, \phi Y)\eta(Z) + S(X + hX, \phi Z)\eta(Y) \\ &+S(Y + hY, \phi Z)\eta(X) + S(Y + hY, \phi X)\eta(Z) \\ &+S(Z + hZ, \phi X)\eta(Y) + S(Z + hZ, \phi Y)\eta(X)].\end{aligned} \tag{49}$$

Using (3) in (49), we obtain

$$\begin{aligned}&(\nabla_X S)(Y, Z) + (\nabla_Y S)(X, Z) + (\nabla_Z S)(X, Y) \\ =\ & 4k[g(hX, \phi Y)\eta(Z) + g(hZ, \phi X)\eta(Y) \\ &+g(hY, \phi Z)\eta(X)] - \eta(Z)[S(hX, \phi Y) \\ &+S(hY, \phi X)] - \eta(Y)[S(hX, \phi Z) \\ &+S(hZ, \phi X)] - \eta(X)[S(hY, \phi Z) + S(hZ, \phi Y)].\end{aligned} \tag{50}$$

Replacing X and Y by hX and ϕY respectively in (37), we have

$$S(hX, \phi Y) = (\frac{r}{2} - k)g(hX, \phi Y). \tag{51}$$

Using (51) in (50) yields

$$\begin{aligned} & (\nabla_X S)(Y,Z) + (\nabla_Y S)(X,Z) + (\nabla_Z S)(X,Y) \\ = \; & (6k-r)[g(hX,\phi Y)\eta(Z) + g(hZ,\phi X)\eta(Y) \\ & + g(hY,\phi Z)\eta(X)]. \end{aligned} \tag{52}$$

From (52), it is clear that the Ricci tensor of the manifold is cyclic parallel if $r = 6k$ or

$$g(hX,\phi Y)\eta(Z) + g(hZ,\phi X)\eta(Y) + g(hY,\phi Z)\eta(X) = 0. \tag{53}$$

Replacing X by ξ in (53), we obtain

$$g(hY,\phi Z) = 0. \tag{54}$$

Putting hX instead of X in (54) and using (9) yields

$$(1-k)g(Y,\phi Z) = 0. \tag{55}$$

Since $g(Y,\phi Z) \neq 0$, therefore we must have $k = 1$, i.e., the manifold is Sasakian. From (52) and (55) we state the following:

Proposition 5.1. If a three dimensional $N(k)$-contact metric manifold satisfies η-parallel Ricci tensor as well as cyclic parallel Ricci tensor, then either the scalar curvature is constant or the manifold is Sasakian.

Again, in view of Lemma 2.1 we state the following:

Theorem 5.2. If a three dimensional $N(k)$-contact metric manifold satisfies η-parallel Ricci tensor as well as cyclic parallel Ricci tensor, then either the manifold is of constant curvature or Sasakian.

6. Example

In this section we construct two examples of three dimensional $N(k)$-contact metric manifold.

Example 1:

We consider 3-dimensional manifold $M = \{(x,y,z) \in \mathbb{R}^3\}$, where (x,y,z) are the standard coordinate in $\mathbb{R}^3$. Let e_1, e_2, e_3 are three vector fields in $\mathbb{R}^3$ which satisfies

$$[e_1,e_2] = (1+\lambda)e_3, \quad [e_2,e_3] = 2e_1 \quad \text{and} \quad [e_3,e_1] = (1-\lambda)e_2,$$

λ being a real number.
Let g be the Riemannian metric defined by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_1, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1.$$

Let η be the 1-form defined by

$$\eta(U) = g(U, e_1),$$

for any $U \in \chi(M)$. Let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = 0, \ \ \phi e_2 = e_3, \ \ \phi e_3 = -e_2.$$

Using the linearity of ϕ and g we have

$$\eta(e_1) = 1,$$

$$\phi^2(U) = -U + \eta(U)e_1$$

and

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

for any $U, W \in \chi(M)$. Moreover

$$he_1 = 0, \quad he_2 = \lambda e_2 \ \text{ and } \ he_3 = -\lambda e_3.$$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formulae,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula we calculate the following:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \quad \nabla_{e_1} e_2 = 0, \quad \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = -(1+\lambda)e_3, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_2} e_3 = (1+\lambda)e_1, \\ &\nabla_{e_3} e_1 = (1-\lambda)e_2, \quad \nabla_{e_3} e_2 = -(1-\lambda)e_1, \quad \nabla_{e_3} e_3 = 0. \end{aligned}$$

In view of the above relations we have

$$\nabla_X \xi = -\phi X - \phi h X, \quad \text{for} \quad e_1 = \xi$$

Therefore the manifold is a contact metric manifold with the contact structure (ϕ, ξ, η, g).

Now, we find the curvature tensors of the manifold as follows:

$$R(e_1, e_2)e_2 = (1-\lambda^2)e_1, \quad R(e_3, e_2)e_2 = -(1-\lambda^2)e_3,$$
$$R(e_1, e_3)e_3 = (1-\lambda^2)e_1, \quad R(e_2, e_3)e_3 = -(1-\lambda^2)e_2,$$
$$R(e_2, e_3)e_1 = 0, \quad R(e_1, e_2)e_1 = -(1-\lambda^2)e_2, \quad R(e_3, e_1)e_1 = (1-\lambda^2)e_3.$$

In view of the expressions of the curvature tensors we conclude that the manifold is a $N(1-\lambda^2)$-contact metric manifold.

Example 2: In [11], J. Milner gave a complete classification of three dimensional manifolds admitting the Lie algebra structure

$$[e_2, e_3] = c_1 e_1, \ \ [e_3, e_1] = c_2 e_2, \ \ [e_1, e_2] = c_3 e_3. \tag{56}$$

As in the case of the given example of [6], let us consider η be the dual 1-form to the vector field e_1. Using (56), we get

$$d\eta(e_2, e_3) = -d\eta(e_3, e_2) = \frac{c_1}{2} \neq 0$$

and $d\eta(e_i, e_j) = 0$ for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that η is a contact form and e_1 is the characteristic vector field. Defining a Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, then, because we must have $d\eta(e_i, e_j) = g(e_i, \phi e_j)$, ϕ has the same metric as $d\eta$ with respect to the basis e_i. Moreover, for g to be an associated metric, we must have $\phi^2 = -I + \eta \otimes e_1$. So for (ϕ, e_1, η, g) to be a contact metric structure we must have $c_1 = 2$. The unique Riemannian connection ∇ coresponding to g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) - g(X, [Y, Z]) \\ &- g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned}$$

So, using $c_1 = 2$ we easily obtain

$$\nabla_{e_1} e_1 = 0, \ \ \nabla_{e_2} e_2 = 0, \ \ \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2)e_3, \ \ \nabla_{e_2} e_1 = \frac{1}{2}(c_2 - c_3 - 2)e_3,$$

$$\nabla_{e_1} e_3 = -\frac{1}{2}(c_2 + c_3 - 2)e_2, \ \ \nabla_{e_3} e_1 = \frac{1}{2}(2 + c_2 - c_3)e_2.$$

But we also know that

$$\nabla_{e_2} e_1 = -\phi e_2 - \phi h e_2.$$

Comparing two values of $\nabla_{e_2} e_1$ and using $\phi e_1 = 0$, $\phi e_3 = -e_2$ we conclude that

$$he_2 = \frac{c_3 - c_2}{2} e_2.$$

And hence

$$he_3 = -\frac{c_3 - c_2}{2} e_3.$$

Thus e_i are eigenvectors of h with corresponding eigenvalues $(0, \lambda, -\lambda)$, where $\lambda = \frac{c_3 - c_2}{2} e_2$. Moreover by direct calculation we have

$$R(e_2, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_2 + [2 - c_2 - c_3]he_2,$$

$$R(e_3, e_1)e_1 = [1 - \frac{(c_3 - c_2)^2}{4}]e_3 + [2 - c_2 - c_3]he_3.$$

$$R(e_2, e_3)e_1 = 0.$$

Putting $k = 1 - \frac{(c_3 - c_2)^2}{4}$ and $\mu = 2 - c_2 - c_3$ we conclude, from these relations that e_1 belongs to the (k, μ)-nullity distribution, for any c_2, c_3. In particular, if we consider the values of c_2, c_3 in such a way that $c_2 + c_3 = 2$ then it is seen that $\mu = 0$ and the manifold becomes an $N(k)$-contact metric manifold.

References

[1] A. Gray, *Einstein-like manifolds which are not Einstein*, Geom. Dedicata, 7(1978), 259 – 280.

[2] B. J. Papantoniou, *Contact Riemannian manifolds satisfying* $R(\xi, X).R = 0$ *and* $\xi \in (k, \mu)$*-nullity distribution*, Yokohama Math. J., 40(1993), 149 – 161.

[3] C. Baikoussis, D. E. Blair and Th. Koufogiorgos, *A decomposition of the curvature tensor of a contact manifold satisfying* $R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y)$, Mathematics Technical Report, University of Ioanniana, 1992.

[4] D. E. Blair, Th. Koufogiorgos and R. Sharma, *A classification of* 3*-dimensional contact metric manifolds with* $Q\phi = \phi Q$, Kodai Math. J., 13(1990), 391 – 401.

[5] D. E. Blair, *Two remarks on contact metric structures*, Tohoku Math. J., 29(1977), 319 – 324.

[6] D. E. Blair, Th. Koufogiorgos and B. J. Papantoniou, *Contact metric manifolds satisfying a nullity condition*, Israel J. Math., 91(1995), 189 – 214.

[7] D. E. Blair, J. S. Kim and M. M. Tripathi, *On the concircular curvature tensor of a contact metric manifold*, J. Korean Math. Soc., 42(5)(2005), 883 – 992.

[8] D. E. Blair and H. Chen, *A classification of* 3*-dimensional contact metric manifolds with* $Q\phi = \phi Q$, II, Bulletin of the Institute of Mathematics Academia Sinica, 20(1992), 379 – 383.

[9] D. Perrone, *Contact Riemannian manifolds satisfying* $R(X, \xi).R = 0$, Yokohama Math. J., 39(1992), 141 – 149.

[10] J-B. Jun and U. K. Kim, *On* 3*-dimensional almost contact metric manifolds*, Kyungpook Math. J., 34(1994), 293 – 301.

[11] J. Milnor, *Curvature of left invariant metrics on Lie groups*, Advances in mathematics, 21(1976), 293-329.

[12] M. Okumura, *Some remarks on space with certain contact structures*, Tohoku Math. J., 14(1962), 135 – 145.

[13] P. Peterson, *Riemannian Geometry*, Springer, p-33.

[14] S. Tanno, *Ricci curvature of contact Riemannian manifolds*, Tohoku Math. J., 40(1988), 441 – 448.

[15] U. H. Kim and H. Nakagawa, *A characterization of the Cartan hypersurface in a sphere*, Tohoku Math. J., 39(1987), 27 – 40.

[16] Z. Olszak, *Certain property of the Ricci tensor on Sasakian manifolds*, Colloquium Mathematicum, XL(1979), 235 – 237.

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