Acta Universitatis Apulensis No. 38/2014
ISSN: 1582-5329 pp. 55-66

ON THE APPROXIMATE SOLUTIONS OF THE PEXIDERIZED GOŁĄB-SCHINZEL FUNCTIONAL EQUATION

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Abstract. We give the Ulam-Găvruta stability of the pexiderized Gołąb-Schinzel functional equation $f(x + y\sigma \circ f(x)) = g(x)h(y)$ and the Ulam-Găvruta super-stability of functional equation $h(x + y\sigma \circ h(x)) = h'(x)h(y)$ under the condition that $\lim_{t\to 0} h(tx)$ exists and is non-zero.

2000 *Mathematics Subject Classification*: Primary 39B82, 39B52, Secondary 39B32, 39B42.

Keywords: Gołąb-Schinzel equation, Ulam-Găvruta stability, Ulam-Găvruta superstability.

1. Introduction

Let E be a $\mathbb{K}$-vector space where $\mathbb{K}$ is a field of real or complex numbers. The Gołąb-Schinzel function equation

$$f(x + yf(x)) = f(x)f(y) \tag{1}$$

for all $x, y \in E$, is considered one of the most intensively studied equations in terms of solving equation and in terms of its stability. It was introduced and studied for the first time by Golab-Schinzel [23] around 1959. It play a prominent role in the theory of functional equations (see, [1, 2, 3, 5, 11, 29, 35]).

For more details concerning studies done on equations of that type, the interested reader should refer to J. Brzedek ([8, 9, 10, 11, 12, 13]), N. Brillouit-Belluot and J. Brzedek [5, 6, 7], J. Chudziak ([17, 18, 20]), J. Chudziak and J. Tabor [19], E. Jablonska ([27, 28, 29]), A.Murenko ([31, 32, 33]), A. Charifi, B. Bouikhalane, S. Kabbaj, J. M. Rassias ([14, 15]).

In [4, 13, 14, 16, 18, 19, 22], the stability problem was considered under various classes of functions for following equations

$$f(x + y) = f(x)f(y), \tag{2}$$

$$f(x + yf(x)) = \lambda f(x)f(y),\ \lambda \in \mathbb{K} \setminus \{0\}, \tag{3}$$

$$f(0)f(x + y\sigma \circ f(x)) = f(x)f(y), \tag{4}$$

$$f(x + yf^n(x)) = \lambda f(x)f(y),\ \lambda \in \mathbb{K} \setminus \{0\},\ n \in \mathbb{N}, \tag{5}$$

$$f(x + yf(x)) = g(x)h(y), \tag{6}$$

for all $x, y \in E$, where $f, g, h : E \to \mathbb{K}$, and $\sigma : \mathbb{K} \to \mathbb{K}$ are functions.

The purpose of the present paper is to study the stability of the pexiderized Gołąb-Schinzel equation

$$f(x + y\sigma \circ f(x)) = g(x)h(y),\ x, y \in E \tag{7}$$

in the class of function which satisfy conditions (C1) and (C3) defined in the following section.

Of course, equations (1), (3), (4), (5) and (6) are particular cases of (7). Unlike equations (2), (3), (4), in general equations (5) and (6) are not super-stable ([14], Remark 3.4). The present paper is a continuation of a previous work by A. Charifi et al. [14].

It is organized as follows: in the second section after this introduction we give some notations and conditions that will be used throughout the rest. In the third section, we obtain preliminary results of the proof of main results. In the fourth section we investigate the stability of equation (7), in the fifth section we derive the Hyers-Ulam-Găvruta super-stability of equation (4). In the final section, we give some applications of this work.

2. Notations

Throughout this paper we use the following notations.
• $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$.
• E is a $\mathbb{K}$ -vector space.
• For a mapping $h : E \to \mathbb{K}$ we note:

$$S_h^0 = \{x \in E : h(x) \neq 0\} \text{ and if } 0 \in S_h^0,\ h'(x) = \frac{h(x)}{h(0)}.$$

The following conditions will be used later.
• (C1) For all $x \in E$, $\lim_{t\to 0} h(tx)$ exists and is non-zero.
• (C2) $\varphi : E^2 \to \mathbb{R}_+$ be a function satisfying $\lim_{t\to 0} \varphi(x, ty)$ exists for all $x, y \in E$.
• (C3) There exists $a \in S_f^0$ such that $f(a) = g(a)h(0)$.

3. Preliminary results

In this section we establish some useful results for the proof of main theorems.

Lemma 1. *Let $\varphi : E^2 \to \mathbb{R}_+$ be a function bounded with respect to the first projection and $f, g, h : E \to \mathbb{K}$; $\sigma : \mathbb{K} \to \mathbb{K}$ be functions satisfying the inequality:*

$$\mid f(x + y\sigma \circ f(x)) - g(x)h(y) \mid \leq \varphi(x, y),\ x, y \in E. \tag{8}$$

Then we have:
(I) If f is unbounded, then for every sequence $(x_n)_{n\in\mathbb{N}}$ in E,

$$\lim_{n\to+\infty} |f(x_n)| = +\infty \ \textit{ if only if } \lim_{n\to+\infty} |g(x_n)| = +\infty.$$

(II) The following proprieties are equivalents.

(i) *f is unbounded.*

(ii) *g is unbounded and $h(0) \neq 0$.*

(iii) *There exist a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\lim_{n\to+\infty} |f(x_n)| = \lim_{n\to+\infty} |g(x_n)| = +\infty$ and $f(x_n)g(x_n) \neq 0,\ n \in \mathbb{N}$.*

(III) If h and f are unbounded then

(i) $\sigma \circ f|_{S_g^0} \neq 0$,

(ii) *there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\lim_{n\to+\infty} |f(x_n)| = +\infty$ and $\sigma \circ f(x_n) \neq 0$, for all $n \in \mathbb{N}$.*

Proof. (I) and (II) follow immediately from the fact that, φ is bounded with respect to x, and taking $y = 0$ in (8), we get

$$\mid f(x) - g(x)h(0) \mid \leq \varphi(x, 0),\ x \in E. \tag{9}$$

(III) The functions h and f are unbounded. By (II) iii), there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\lim_{n\to+\infty} \mid f(x_n) \mid = \lim_{n\to+\infty} \mid g(x_n) \mid = +\infty$ and $f(x_n)g(x_n) \neq 0,\ n \in \mathbb{N}$. Suppose that for all sub-sequence $(x_n)_{n\in\mathbb{N}}$, $\sigma \circ f(x_n) = 0,\ n \in \mathbb{N}$. In view of (8) we get

$$\mid f(x_n) - g(x_n)h(y) \mid \leq \varphi(x_n, y),\ n \in \mathbb{N},\ y \in E.$$

By passing to the limit, h is a constant. This yields a contradiction. Then, i) and ii) are fulfilled.

Theorem 2. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f, g, h : E \to \mathbb{K}$, $\varphi : E^2 \to \mathbb{R}+$ be functions satisfying (8). Assume that φ is bounded with respect to the first projection, f, h are unbounded, conditions (C1) and (C2) hold. Then*

$$h'(x + y\sigma \circ h'(x)) = h'(x)h'(y) \tag{10}$$

and

$$\sigma \circ h'(x + y\sigma \circ h'(x)) = \sigma \circ h'(x)\sigma \circ h'(y) \tag{11}$$

for all $x, y \in E$.

Proof. Since f is unbounded, by Lemma 1 (II), iii) there exist a sequence $(x_n)_{n\in\mathbb{N}}$ in E such that $\lim_{n\to+\infty} \mid f(x_n) \mid = \lim_{n\to+\infty} \mid g(x_n) \mid = +\infty$ and $f(x_n)g(x_n) \neq 0$. Now, for $x, y \in E$ and $n \in \mathbb{N}$ we consider

$$a_n = x_n + x\sigma \circ f(x_n),\ b_n = a_n + y\sigma \circ f(a_n) \ \text{ and } \ c_n = x_n + (x + y\sigma \circ h'(x))\sigma \circ f(x_n).$$

Without loss of generality, we can assume that either $b_n = c_n,\ n \in \mathbb{N}$ or $b_n \neq c_n,\ n \in \mathbb{N}$. The proof of the theorem is the same for both cases, the only difference is that in the case $b_n = c_n,\ n \in \mathbb{N}$, we have $h(\frac{c_n - b_n}{\sigma \circ f(c_n)}) = h(0)$. That is why we are going to demonstrate the theorem just in the case $b_n \neq c_n,\ n \in \mathbb{N}$.
1) Assume that $x \in S_h^0$, in view of (8) and the choice of (x_n), we get

$$\lim_{n\to+\infty} \frac{f(c_n)}{g(x_n)} = h(x + y\sigma \circ h'(x)) \text{ and } \lim_{n\to+\infty} \frac{f(a_n)}{g(x_n)} = h(x).$$

First case: $y \in S_h^0$. According to Lemma 1 (II), iii) and (III), ii) we can assume that

$$f(x_n)f(a_n)f(b_n)f(c_n)\sigma \circ f(x_n)\sigma \circ f(a_n)\sigma \circ f(b_n)\sigma \circ f(c_n)g(x_n)g(a_n)g(b_n) \neq 0,\ n \in \mathbb{N}.$$

Thus, by Lemma 1 (I), (II) we can write

$$\lim_{n\to+\infty} \frac{f(b_n)}{g(x_n)} = \lim_{n\to+\infty} \frac{f(b_n)}{g(a_n)} \frac{g(a_n)}{f(a_n)} \frac{f(a_n)}{g(x_n)} = h(x)h'(y).$$

Putting $\alpha_n = \frac{c_n - b_n}{\sigma \circ f(b_n)}$ which can be written in the form

$$\begin{aligned} \alpha_n &= y\frac{\sigma \circ h'(x)\sigma \circ f(x_n) - \sigma \circ f(a_n)}{\sigma \circ f(b_n)} \\ &= y[\sigma \circ h'(x) - \frac{\sigma \circ f(a_n)}{\sigma \circ f(x_n)}] \times \frac{\sigma \circ f(x_n)}{\sigma \circ f(b_n)}. \end{aligned}$$

Thus, we can easily seen that

$$\lim_{n\to+\infty} \frac{\sigma \circ h'(x)\sigma \circ f(x_n) - \sigma \circ f(a_n)}{\sigma \circ f(b_n)} = 0.$$

From (8), we get

$$| f(c_n) - g(b_n)h(\alpha_n) |\leq \varphi(b_n, \alpha_n),\ n \in \mathbb{N}.$$

Thus, $\lim_{n\to+\infty} \frac{f(c_n)}{g(x_n)} = \lim_{n\to+\infty} \frac{g(b_n)}{g(x_n)} h(\alpha_n)$

$$\begin{aligned}
\lim_{n\to+\infty} \frac{f(c_n)}{g(x_n)} &= \lim_{n\to+\infty} \frac{g(b_n)}{g(x_n)} h(\alpha_n)\\
&= \lim_{n\to+\infty} \frac{g(b_n)}{g(x_n)} l(y)\\
&= \lim_{n\to+\infty} \frac{g(b_n)}{f(b_n)} \frac{f(b_n)}{g(a_n)} \frac{g(a_n)}{f(a_n)} \frac{f(b_n)}{g(x_n)} l(y),
\end{aligned}$$

which gives $h(x + y\sigma \circ h'(x)) = h'(x)h'(y)l(y)$. So, taking $x = 0$, we deduce that $l(y) = h(0)$ and consequently $h(x + y\sigma \circ h'(x)) = h(x)h'(y)$.
Second Case: $y \notin S_h^0$. We are going to show that the previous equality is valid. In fact, suppose that $h(x + y\sigma \circ h'(x)) \neq h(x)h'(y)$. By (8), we have

$$| f(b_n) - g(c_n)h(-\alpha_n \frac{\sigma \circ f(b_n)}{\sigma \circ f(c_n)}) |\leq \varphi(c_n, -\alpha_n \frac{\sigma \circ f(b_n)}{\sigma \circ f(c_n)}),\ n \in \mathbb{N}.$$

Thus,

$$\begin{aligned}
\lim_{n\to+\infty} \frac{f(b_n)}{g(x_n)} &= \lim_{n\to+\infty} \frac{g(c_n)}{g(x_n)} h(-\alpha_n \frac{\sigma \circ f(b_n)}{\sigma \circ f(c_n)})\\
&= \lim_{n\to+\infty} \frac{g(c_n)}{g(x_n)} l(y)\\
&= \lim_{n\to+\infty} \frac{g(c_n)}{f(c_n)} \frac{f(c_n)}{g(x_n)} l(y),
\end{aligned}$$

So, we obtain $h'(x+y\sigma\circ h'(x))l(y) = h(x)h'(y)$. Whence, we get $h(x+y\sigma\circ h'(x)) = 0$, which yields a contradiction.
2) Assume now that $x \notin S_h^0$. In this case, clearly that the sought equality is verified, $h'(x + y\sigma \circ h'(x)) = h'(x)h'(y)$. So, this gives (10) and Consequently by the fact that σ is a multiplication application, we get (11). This completes the proof of theorem.

Corollary 3. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f, h : E \to \mathbb{K}$, $\varphi : E^2 \to \mathbb{R}+$ be functions satisfying*

$$| f(x + y\sigma \circ f(x)) - f(x)h(y) |\leq \varphi(x, y). \tag{12}$$

Assume that f, h are unbounded, φ is bounded with respect to the first projection, conditions (C1) and (C2) hold. Then

$$h(x + y\sigma \circ h(x)) = h(x)h(y) \tag{13}$$

and

$$\sigma \circ h(x + y\sigma \circ h(x)) = \sigma \circ h(x)\sigma \circ h(y)$$

for all $x, y \in E$.

Proof. Just take $y = 0$ in (12), we get that $h(0) = 1$ and consequently Theorem 2 gives the result.

4. STABILITY OF EQ (7)

In this part we investigate the stability of equation (7).

Theorem 4. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f, g, h : E \to \mathbb{K}$, $\varphi : E^2 \to \mathbb{R}+$ be functions satisfying (8). Assume that f, h are unbounded, φ is bounded with respect to the first projection, conditions (C1), (C2) and (C3) hold. Then there exists a unique pair of functions $F, G : E \to \mathbb{K}$ such that $F(a) = f(a)$,*

$$F(x + y\sigma \circ F(x)) = G(x)h(y),\ x, y \in E, \tag{14}$$

$$\mid F(x) - f(x) \mid \leq \varphi(a, \frac{x-a}{\sigma \circ f(a)}),\ x \in E, \tag{15}$$

and

$$\mid G(x) - g(x) \mid \leq \frac{1}{\mid h(0) \mid}\{\varphi(a, \frac{x-a}{\sigma \circ f(a)}) + \varphi(x, 0)\},\ x \in E. \tag{16}$$

Proof. By $(C3)$, $g(a) \neq 0$. Considering in (8), $x = a$ and $y = \frac{z-a}{\sigma \circ f(a)}$, $z \in E$, we obtain

$$\mid f(z) - g(a)h(\frac{z-a}{\sigma \circ f(a)}) \mid \leq \varphi(a, \frac{z-a}{\sigma \circ f(a)}),\ z \in E.$$

Thus, taking

$$F(x) = g(a)h(\frac{x-a}{\sigma \circ f(a)}),$$

we get (15), secondly taking $y = 0$ in (8) we obtain

$$\begin{aligned}
\mid F(x) - g(x)h(0) \mid &= \mid F(x) - f(x) + f(x) - g(x)h(0) \mid \\
&\leq \mid F(x) - f(x) \mid + \mid f(x) - g(x)h(0) \mid \\
&\leq \varphi(a, \frac{x-a}{\sigma \circ f(a)}) + \varphi(x, 0).
\end{aligned}$$

Thus, taking $G(x) = \frac{1}{h(0)}g(a)h(\frac{x-a}{\sigma\circ f(a)})$ we get (16). Furthermore, from Theorem 2, we have

$$h'(x + y\sigma \circ h'(x)) = h'(x)h'(y),$$
$$\sigma \circ h'(x + y\sigma \circ h'(x)) = \sigma \circ h'(x)\sigma \circ h'(y)$$

and

$$\begin{aligned}
F(x + y\sigma \circ F(x)) &= g(a)h(\frac{x + y\sigma \circ F(x) - a}{\sigma \circ f(a)}) \\
&= g(a)h(\frac{x-a}{\sigma \circ f(a)} + y\sigma(\frac{F(x)}{f(a)})) \\
&= g(a)h(\frac{x-a}{\sigma \circ f(a)} + y\sigma(\frac{g(a)h(\frac{x-a}{\sigma\circ f(a)})}{f(a)})) \\
&= g(a)h(\frac{x-a}{\sigma \circ f(a)})h'(y) \\
&= G(x)h(y),
\end{aligned}$$

for all $x, y \in E$.
The uniqueness is given by the fact that $F(x) = G(a)h(\frac{x-a}{\sigma\circ f(a)})$ and $G(a) = g(a) = \frac{f(a)}{h(0)}$. Indeed, suppose that there exist other functions $F', G' : E \to \mathbb{K}$ such that

$$F'(x + y\sigma \circ F'(x)) = G'(x)h(y),\ x, y \in E$$

and $F'(a) = G'(a)h(0) = f(a)$ we get

$$\begin{aligned}
F'(x) &= G'(a)h(\frac{x-a}{\sigma \circ F'(a)}) \\
&= g(a)h(\frac{x-a}{\sigma \circ f(a)}) \\
&= F(x)
\end{aligned}$$

for all $x \in E$. Thus $F' = F$ and $G' = G$.

Remark 1. *The condition (C3) is not necessary when, φ is also bounded with respect the second projection ([14], Theorem 3.2) or when $g = \lambda f$.*

Corollary 5. *Let δ be a positive number, $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f, g, h : E \to \mathbb{K}$, $\phi : E \to \mathbb{R}+$ be functions satisfying the following inequality*

$$|f(x + y\sigma \circ f(y)) - g(x)h(y)| \leq \delta + \phi(y),\ x, y \in E.$$

Assume that f, h are unbounded, conditions (C1), (C2) and (C3) hold. Then there exists a unique pair of functions $F, G : E \to \mathbb{K}$ such that $F(a) = f(a)$,

$$F(x + y\sigma \circ F(x)) = G(x)h(y), \; x, y \in E, \tag{17}$$

$$\mid F(x) - f(x) \mid \leq \delta + \phi(\frac{x-a}{\sigma \circ f(a)}), \; x \in E \tag{18}$$

and

$$\mid G(x) - g(x) \mid \leq \frac{\delta}{h(0)} + \frac{1}{\mid h(0) \mid}\{\phi(\frac{x-a}{\sigma \circ f(a)}) + \phi(0)\}, \; x \in E. \tag{19}$$

5. Super-stability of eq (4)

Theorem 6. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f : E \to \mathbb{K}$, $\varphi : E^2 \to \mathbb{R}+$ be functions satisfying the following inequality*

$$|f(x + y\sigma \circ f(x) - f'(x)f(y)| \leq \varphi(x, y), \; x, y \in E. \tag{20}$$

Assume that f is unbounded, φ is bounded with respect to x, conditions (C1) and (C2) hold. Then

$$f(x + y\sigma \circ f(x)) = f'(x)f(y), \; x, y \in E. \tag{21}$$

Proof. By Theorem 2 and the condition (20) we get

$$|f'(x)f(y) - f'(x)f(y\sigma \circ f(0))| \leq \varphi(x, y), x, y \in E.$$

Indeed, we have

$$\begin{aligned} & |f'(x)[f(y) - f(y\sigma \circ f(0))]| \\ = \; & |f'(x)f(y) - f'(x)f(y\sigma \circ f(0))| \\ = \; & |f'(x)f(y) - f(x + y\sigma \circ f(x)) + f(x + y\sigma \circ f(x)) - f'(x)f(y\sigma \circ f(0))| \\ \leq \; & \varphi(x, y), \; x, y \in E. \end{aligned}$$

Thus, $f(y) = f(y\sigma \circ f(0)), \; y \in E$ which gives the sought result.

Corollary 7. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function and $f : E \to \mathbb{K}$, $\phi : E \to \mathbb{R}+$ be functions satisfying the following inequality*

$$|f(x + y\sigma \circ f(x) - f'(x)f(y)| \leq \delta + \phi(y), \; x, y \in E. \tag{22}$$

Assume that f is unbounded, conditions (C1) and (C2) hold. Then

$$f(x + y\sigma \circ f(x)) = f'(x)f(y), \; x, y \in E. \tag{23}$$

6. Applications

Corollary 8. *Let δ be a positive number and let $f : E \to \mathbb{K}$, $\varphi : E^2 \to \mathbb{R}+$ be functions satisfying*

$$|f(x + yf(x)) - f(x)f'(y)| \leq \delta + \varphi(x, y), \quad x, y \in E. \tag{24}$$

Assume that φ is bounded with respect to the first projection, f is unbounded, conditions (C1) and (C2) hold. Then we have

$$f(x + yf(x)) = f(x)f'(y), \quad x, y \in E.$$

Corollary 9. *Let δ be a positive number, λ a non null scalar, $n \in \mathbb{Q}_+\backslash\{0\}$ and let $f : E \to \mathbb{K}$, $\phi : E \to \mathbb{R}_+$ be functions satisfying*

$$|f(x + yf^n(x)) - \lambda f(x)f(y)| \leq \delta + \phi(y), \quad x, y \in E. \tag{25}$$

Assume that f is unbounded, conditions (C1) and (C2) hold. Then we have

$$f(x + yf^n(x)) = \lambda f(x)f(y), \quad x, y \in E.$$

Corollary 10. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function, δ a positive number, λ a non null scalar, $f : E \to \mathbb{K}$ and $\phi : E \to \mathbb{R}_+$ be functions satisfying*

$$|f(x + y\sigma \circ f(x)) - \lambda f(x)f(y)| \leq \delta + \phi(y), \quad x, y \in E. \tag{26}$$

Assume that f is unbounded, conditions (C1) and (C2) hold. Then we have

$$f(x + y\sigma \circ f(x)) = \lambda f(x)f(y), \quad x, y \in E.$$

Corollary 11. *Let $\sigma : \mathbb{K} \to \mathbb{K}$ be a multiplicative continuous function, δ a positive number, λ a non null scalar and $f : E \to \mathbb{K}$, $\phi : E \to \mathbb{R}_+$ be functions satisfying*

$$|f(x + y\sigma \circ f(x) + z\sigma \circ f(x)\sigma \circ f(y)) - \lambda f(x)f(y)f(z)| \leq \delta + \phi(y), \quad x, y, z \in E. \tag{27}$$

Assume that f is unbounded, conditions (C1) and (C2) hold. Then we have

$$f(x + y\sigma \circ f(x) + z\sigma \circ f(x)\sigma \circ f(y)) = \lambda f(x)f(y)f(z), \quad x, y, z \in E.$$

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