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ON IDEAL ANALOGUE OF ASYMPTOTICALLY LACUNARY STATISTICAL EQUIVALENCE OF SEQUENCES

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Abstract. For an admissible ideal $\mathcal{I} \subseteq P(\mathbb{N})$ and a lacunary sequence $\theta = (k_r)$, the aim of the present work is to introduce certain new notions of asymptotically $\mathcal{I}-$ lacunary statistically equivalent, asymptotically $\mathcal{I}-$statistically equivalent, and asymptotically $\mathcal{I} - N_\theta-$equivalent sequences of multiple L which are natural combination of notions of asymptotically equivalent, lacunary statistical convergence and $N_\theta-$convergence of sequences of numbers. We study some connections between these notions.

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1. Introduction

Fast [3] and Schoenberg [16], independently introduced the notion of statistical convergence which have been extensively discussed in Number theory, Ergodic Theory and Fourier analysis. Later, the idea was further investigated from the sequence space point of view and linked with summability theory by Connor [1], Fridy [4], Maddox [8], Šalát [15], Tripathy [17] and many others. In recent years, many generalizations of statistical convergence have been appeared in literature and found useful in the study of strong integral summability theory. One among these is presented by Fridy *et al*. [5] with use of lacunary sequences and called it lacunary statistical convergence. Another one is $\mathcal{I}-$convergence which is introduced by Kostyrko *et al*. [6] with the help of an admissible idea $\mathcal{I}$ of subsets of $\mathbb{N}$, the set of positive integers. Subsequently, ideal convergence have been discussed in some spaces such as random-2-normed spaces [10], probabilistic normed spaces [11] and intuitionistic fuzzy normed spaces [12]. Quite recently, Das *et al*. [2] unified these two approaches to introduce new concepts- $\mathcal{I}-$ statistical convergence, $\mathcal{I}-$lacunary statistical convergence and investigated some of its consequences. On the other side Marouf [9] introduced asymptotically equivalent sequences, asymptotic regular matrices and studied their relationships in summability theory. However, Patterson [13] extended these concepts by presenting an asymptotically statistical equivalent analogue of these definitions and natural regularity conditions for nonnegative summability matrices. In addition to the above extensions, Patterson and Savaş incorporated lacunary sequences into these notions in [14]. Kumar and Sharma [7] studied asymptotically generalized statistical equivalent sequences using deals. This paper extends the definitions presented in [14] in term of a generalized notion $\mathcal{I}-$lacunary statistically convergence and establish the natural inclusion theorems with respect to this new kind of equivalency.

2. Preliminaries

In this section, we recall some definitions and results which form the base for present study. We begin with the following definitions.

Definition 1. *[9] Two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically equivalent of multiple L provided that*

$$\lim_{k\to\infty}\left(\frac{x_k}{y_k}\right) = L,$$

(denoted by $x \sim y$) and simply asymptotically equivalent if $L = 1$.

Definition 2. *[3] A number sequence $x = (x_k)$ is said to be statistically convergent to a number L provided that for every $\epsilon > 0$,*

$$\lim_{n\to\infty}\frac{1}{n}\left|\ \{k \leq n\ :\ |x_k - L|\ \geq\ \epsilon\}\ \right| = 0,$$

where vertical bars denote the cardinality of the enclosed set. In this case we write $S - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S)$.

Patterson [13] presented a natural combination of Definitions 1 and 2 to introduce asymptotically statistically equivalence as follows.

Definition 3. *[13] The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically statistically equivalent of multiple L provided that for every $\epsilon > 0$, we have*

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k \geq n :\ \left|\left(\frac{x_k}{y_k}\right) - L\right|\ \geq\ \epsilon\right\}\right| = 0,$$

(denoted by $x \sim^{S^L} y$) and simply asymptotically statistical equivalent if $L = 1$.

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, \ \ k_r]$, and the ratio $\left(\frac{k_r}{k_{r-1}}\right)$ will be abbreviated by q_r. For the lacunary sequence $\theta = (k_r)$, the sequence space N_θ is defined by

$$N_\theta = \left\{ x = (x_k) : \ \exists\ L \in \mathbb{R}, \quad \lim_{r\to\infty} \left(\frac{1}{h_r} \sum_{k \in I_r} |x_k - L| \right) \ = 0 \right\}.$$

Fridy and Orhan [5] introduced a concept of convergence related to statistical convergence with the help of lacunary sequence as follows.

Definition 4. *[5] Let $\theta = (k_r)$ be a lacunary sequence. The number sequence $x = (x_k)$ is said to be $S_\theta-$convergent to a number L provided that for every $\epsilon > 0$,*

$$\lim_{r\to\infty} \quad \frac{1}{h_r} \left| \{ k \in I_r : |x_k - L| \ \geq \ \epsilon \} \right| = 0.$$

In this case, we write $S_\theta - \lim_{k\to\infty} x_k = L$ or $x_k \to L(S_\theta)$ as $k \to \infty$.

Patterson and Savaş [14] combined Definitions 1 and 4 to introduce new concept of asymptotically lacunary statistically equivalence as follows.

Definition 5. *[14] Let $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically lacunary statistically equivalent of multiple L provided that for each $\epsilon > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r} \left| \ \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \ \right| = 0,$$

(denoted by $x \sim^{S_\theta^L} y$) and simply asymptotically lacunary statistically equivalent if L=1.

Finally, we recall the terminology of $\mathcal{I}-$convergence which is one of the main notions that we need in the sequel.

For any non-empty set X, let $P(X)$ denotes the power set.

A family $\mathcal{I} \subseteq P(X)$ is said to be an ideal in X if (i) $\emptyset \in \mathcal{I}$; (ii) $A, \ B \in \mathcal{I}$ imply $A \cup B \in \mathcal{I}$; (iii) $A \in \mathcal{I}, \ B \subset A$ imply $B \in \mathcal{I}$.

A non-empty family $\mathcal{F} \subseteq P(X)$ is said to be a filter in X if (i) $\emptyset \notin \mathcal{F}$; (ii) $A, \ B \in \mathcal{F}$ imply $A \cap B \in \mathcal{F}$; (iii) $A \in \mathcal{F}, \ B \supset A$ imply $B \in \mathcal{F}$.

An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq \{\emptyset\}$ and $X \notin \mathcal{I}$. A nontrivial ideal $\mathcal{I}$ is called admissible if it contains all the singleton sets. If $\mathcal{I}$ is a non-trivial ideal on

X, then $\mathcal{F} = \mathcal{F}(\mathcal{I}) = \{X - A : A \in \mathcal{I}\}$ is a filter on X and conversely. The filter $\mathcal{F}(\mathcal{I})$ is called the filter associated with the ideal $\mathcal{I}$.

For further study, we take X=$\mathbb{N}$ and use the notation A^C to denote the complement of any set A.

Definition 6. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal. A number sequence $x = (x_k)$ is said to be $\mathcal{I}-$convergent to a number L provided that for each $\epsilon > 0$, $A(\epsilon) = \{k \in \mathbb{N} : |x_k - L| \geq \epsilon\} \in \mathcal{I}$. In this case we write $\mathcal{I} - \lim_{k\to\infty} x_k = L$.*

Definition 7. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal. A sequences $x = (x_k)$ is said to be $\mathcal{I}-$statistically convergent to L provided that for each $\epsilon > 0$, and every $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} |\{k \leq n : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - S - \lim_{k\to\infty} x_k = L$ or $x_k \to L(\mathcal{I} - S)$ as $k \to \infty$.

Definition 8. *[2] Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal. A sequences $x = (x_k)$ is said to be $\mathcal{I}-$lacunary statistically convergent to L provided that for each $\epsilon > 0$, and every $\delta > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r} |\{k \in I_r : |x_k - L| \geq \epsilon\}| \geq \delta\right\} \in \mathcal{I}.$$

In this case we write $\mathcal{I} - S_\theta - \lim_{k\to\infty} x_k = L$ or $x_k \to L(\mathcal{I} - S_\theta)$ as $k \to \infty$.

3. Main Results

Following the above definitions and results, we aim in this section to introduce some new notions of asymptotically equivalence with the use of ideals and lacunary sequences and obtain some analogous results from the new definitions point of views.

Definition 9. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}-$statistically equivalent of multiple L provided that for each $\epsilon > 0$, and every $\delta > 0$,*

$$\left\{n \in \mathbb{N} : \frac{1}{n} \left| k \leq n : \left| \left(\frac{x_k}{y_k}\right) - L\right| \geq \epsilon\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

(denoted by $x \sim^{S^L(\mathcal{I})} y$) and simply asymptotically $\mathcal{I}-$statistically equivalent if L = 1.

In addition, let $S^L(\mathcal{I})$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \sim^{S^L(\mathcal{I})} y$.

Definition 10. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I} - N_\theta-$equivalent of multiple L provided that for every $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \ \geq \delta \right\} \in \mathcal{I},$$

(denoted by $x \sim^{N_\theta^L(\mathcal{I})} y$) and simply asymptotically $\mathcal{I} - N_\theta-$ equivalent if L=1.

In addition, let $N_\theta^L(\mathcal{I})$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \sim^{N_\theta^L(\mathcal{I})} y$.

Definition 11. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\theta = (k_r)$ be a lacunary sequence. The two non-negative sequences $x = (x_k)$ and $y = (y_k)$ are said to be asymptotically $\mathcal{I}-$ lacunary statistically equivalent of multiple L provided that for each $\epsilon > 0$, and every $\delta > 0$,*

$$\left\{ r \in \mathbb{N} : \ \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \right| \ \geq \delta \right\} \in \mathcal{I},$$

(denoted by $x \sim^{S_\theta^L(\mathcal{I})} y$) and simply asymptotically $\mathcal{I}-$ lacunary statistically equivalent if L=1.

In addition, let $S_\theta^L(\mathcal{I})$ denote the set of all sequences $x = (x_k)$ and $y = (y_k)$ such that $x \sim^{S_\theta^L(\mathcal{I})} y$.

Theorem 1. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\theta = (k_r)$ be a lacunary sequence. Then*
(i) $x \sim^{N_\theta^L(\mathcal{I})} y$ implies $x \sim^{S_\theta^L(\mathcal{I})} y$, and $N_\theta^L(\mathcal{I})$ is a proper subset of $S_\theta^L(\mathcal{I})$;
(ii) If $x = (x_k)$, $y = (y_k) \in \ell_\infty$ and $x \sim^{S_\theta^L(\mathcal{I})} y$, then $x \sim^{N_\theta^L(\mathcal{I})} y$ and
(iii) $S_\theta^L(\mathcal{I}) \cap \ell_\infty = N_\theta^L(\mathcal{I}) \cap \ell_\infty$,
where ℓ_∞ denotes the set of bounded sequences.

Proof. (i) Suppose $x = (x_k)$ and $y = (y_k)$ be such that $x \sim^{N_\theta^L(\mathcal{I})} y$. We will show that $x \sim^{S_\theta^L(\mathcal{I})} y$. Since, for any $\epsilon > 0$

$$\sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \sum_{k \in I_r \text{ and } \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \epsilon} \left| \left(\frac{x_k}{y_k} \right) - L \right|$$

$$\geq \ \epsilon \ . \left| \ \left\{ k \in I_r : \ \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \ \right|,$$

$$\text{or} \qquad \frac{1}{\epsilon.h_r}\sum_{k\in I_r}\left|\left(\frac{x_k}{y_k}\right)-L\right| \;\geq\; \frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-L\right| \;\geq\; \epsilon\right\}\right|,$$

it follows for any $\delta > 0$,

$$\frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-L\right| \;\geq\; \epsilon\right\}\right| \geq \delta \ \text{ implies } \ \frac{1}{h_r}\sum_{k\in I_r}\left|\left(\frac{x_k}{y_k}\right)-L\right| \geq \epsilon\ \delta.$$

$$\text{Thus,} \qquad \left\{r\in\mathbb{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-L\right| \;\geq\; \epsilon\right\}\right| \geq \delta\right\} \subset$$

$$\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\sum_{k\in I_r}\left|\left(\frac{x_k}{y_k}\right)-L\right| \geq \epsilon\ \delta\right\}.$$

Since $x \sim^{N_\theta^L(\mathcal{I})} y$, so the later set belongs to $\mathcal{I}$ which immediately implies

$$\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-L\right| \;=\; \epsilon\right\}\right| = \delta\right\} \in \mathcal{I}.$$

This shows that $x \sim^{S_\theta^L(\mathcal{I})} y$.

In order to establish the inclusion $N_\theta^L(\mathcal{I}) \subseteq S_\theta^L(\mathcal{I})$ is proper, let $\theta = (k_r)$ be a lacunary sequence and $x = (x_k)$, $y = (y_k)$ be defined as follows.

Let x_k to be 1, 2, $\cdots$, $\left[\sqrt{h_r}\right]$ at first $\left[\sqrt{h_r}\right]$ integers in I_r and $x_k = 0$ otherwise; $y_k = 1$ for all k. Note that the sequence $x = (x_k)$ is not bounded. Further, for any $\epsilon > 0$, the fact

$$\frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-0\right| \;\geq\; \epsilon\right\}\right| \leq \frac{\left[\sqrt{h_r}\right]}{h_r} \quad \text{and} \quad \left(\frac{\left[\sqrt{h_r}\right]}{h_r}\right) \to 0 \quad \text{as } r\to\infty \tag{1}$$

immediately implies that

$$\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-0\right| \geq \epsilon\right\}\right| \geq \delta\right\} \subseteq \left\{r\in\mathbb{N}:\ \frac{\left[\sqrt{h_r}\right]}{h_r} \geq \delta\right\}.$$

By virtue of last half of (1), the set on the right side is a finite set and so belongs to $\mathcal{I}$. Consequently, we have

$$\left\{r\in\mathbb{N}:\ \frac{1}{h_r}\left|\left\{k\in I_r:\ \left|\left(\frac{x_k}{y_k}\right)-0\right| \geq \epsilon\right\}\right| \geq \delta\right\} \in \mathcal{I},$$

and therefore $x \sim^{S_\theta^0(\mathcal{I})} y$. On the other hand, we shall show that $x \sim^{N_\theta^0(\mathcal{I})} y$ is not satisfied. Suppose that $x \sim^{N_\theta^0(\mathcal{I})} y$. Then for every $\delta > 0$, we have

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - 0 \right| \geq \delta \right\} \in \mathcal{I}. \tag{2}$$

Now, $$\frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - 0 \right| = \frac{1}{h_r} \left(\frac{\left[\sqrt{h_r}\right] \left(\left[\sqrt{h_r}\right] + 1 \right)}{2} \right) \to \frac{1}{2} \quad \text{as} \quad r \to \infty.$$

It follows for the particular choice $\delta = \frac{1}{4}$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - 0 \right| \geq \frac{1}{4} \right\} = \left\{ r \in \mathbb{N} : \left(\frac{\left[\sqrt{h_r}\right] \left(\left[\sqrt{h_r}\right] + 1 \right)}{h_r} \right) \geq \frac{1}{2} \right\}$$

$$= \{m, \ m+1, \ \cdots\}$$

for some $m \in \mathbb{N}$ which belongs to $\mathcal{F}(\mathcal{I})$ as $\mathcal{I}$ is admissible. This contradicts (2) for the choice $\delta = \frac{1}{4}$ and therefore $x \sim^{N_\theta^0(\mathcal{I})} y$ is not satisfied.

(ii) Suppose $x = (x_k)$, $y = (y_k) \in \ell_\infty$ such that $x \sim^{S_\theta^L(\mathcal{I})} y$. We shall prove that $x \sim^{N_\theta^L(\mathcal{I})} y$. We may suppose that there exists a number $M > 0$ such that

$$\left| \left(\frac{x_k}{y_k} \right) - L \right| \leq M \quad \text{for all } k \in \mathbb{N}.$$

Given $\epsilon > 0$, we have

$$\frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - L \right| = \frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \frac{\epsilon}{2}}} \left| \left(\frac{x_k}{y_k} \right) - L \right| +$$

$$\frac{1}{h_r} \sum_{\substack{k \in I_r \\ \left| \left(\frac{x_k}{y_k} \right) - L \right| < \frac{\epsilon}{2}}} \left| \left(\frac{x_k}{y_k} \right) - L \right|$$

$$\leq \frac{M}{h_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \frac{\epsilon}{2} \right\} \right| + \frac{\epsilon}{2}.$$

Thus, if we denote the sets

$$B(\epsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \sum_{k \in I_r} \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \epsilon \right\} \text{ and}$$

$$A(\epsilon) = \left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \frac{\epsilon}{2} \right\} \right| \geq \frac{\epsilon}{2M} \right\},$$

then $B(\epsilon) \subset A(\epsilon)$ and by virtue of $x \sim^{S_\theta^L(\mathcal{I})} y$, we have $B(\epsilon) \in \mathcal{I}$. This shows that $x \sim^{N_\theta^L(\mathcal{I})} y$.

(iii) This is an immediate consequence of (i) and (ii).

Theorem 2. *Let $\mathcal{I} \subseteq P(\mathbb{N})$ be a non-trivial ideal and $\theta = (k_r)$ be a lacunary sequence with* $\liminf_r q_r > 1$, *then $x \sim^{S^L(\mathcal{I})} y$ implies $x \sim^{S_\theta^L(\mathcal{I})} y$.*

Proof. Suppose that $\liminf_r q_r > 1$; then there exists a $\gamma > 0$ such that $q_r \geq 1 + \gamma$ for sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geq \frac{\gamma}{1+\gamma}.$$

If $x \sim^{S^L(\mathcal{I})} y$, then for every $\epsilon > 0$ and for sufficiently large r, we have

$$\frac{1}{k_r} \left| \left\{ k \leq k_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \right| = \frac{1}{k_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \right|$$

$$= \left(\frac{\gamma}{1+\gamma} \right) \ \cdot \ \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \right|.$$

and therefore for any $\delta > 0$,

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r} \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \geq \epsilon \right\} \right| \geq \delta \right\} \subseteq$$

$$\left\{ r \in \mathbb{N} : \frac{1}{k_r} \left| \left\{ k \leq k_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| = \epsilon \right\} \right| \geq \frac{\gamma\delta}{(1+\gamma)} \right\} \in \mathcal{I}.$$

This shows that $x \sim^{S_\theta^L(\mathcal{I})} y$.

Theorem 3. *Let $\mathcal{I} = \mathcal{I}_{fin} = \{A \subset \mathbb{N} :$ A is finite set$\}$ be a non-trivial ideal and $\theta = (k_r)$ be a lacunary sequence with* $\limsup_r q_r < \infty$, *then $x \sim^{S_\theta^L(\mathcal{I})} y$ implies $x \sim^{S^L(\mathcal{I})} y$.*

Proof. If $\limsup_r q_r < \infty$, then there is an $H > 0$ such that $q_r < H$ for all r. Suppose that $x \sim^{S_\theta^L(\mathcal{I})} y$, and let

$$N_r = \left| \left\{ k \in I_r : \left| \left(\frac{x_k}{y_k} \right) - L \right| \ \geq \ \epsilon \right\} \right|$$

Since $x \sim^{S_\theta^L(\mathcal{I})} y$, it follows that for every $\epsilon > 0$ and $\delta > 0$

$$\left\{r \in \mathbb{N} : \ \frac{1}{h_r}\left|\left\{k \in I_r : \ \left|\left(\frac{x_k}{y_k}\right) - L\right| \ \geq \ \epsilon\right\}\right| \geq \delta\right\} = \left\{r \in \mathbb{N} : \ \frac{N_r}{h_r} \geq \delta\right\} \in \mathcal{I},$$

and therefore is a finite set. So we can choose a positive integer $r_0 \in \mathbb{N}$ such that

$$\frac{N_r}{h_r} < \delta \ \text{ for all } r > \ r_0. \tag{3}$$

Now let $M = \max\{N_r : 1 \leq r \leq r_0\}$ and n be any integer satisfying $k_{r-1} < n \leq k_r$, then we have

$$\begin{aligned}
\frac{1}{n}\left|\left\{k \leq n : \ \left|\left(\frac{x_k}{y_k}\right) - L\right| \ \geq \ \epsilon\right\}\right| &\leq \frac{1}{k_{r-1}}\left|\left\{k \leq k_r : \ \left|\left(\frac{x_k}{y_k}\right) - L\right| \ \geq \ \epsilon\right\}\right| \\
&= \frac{1}{k_{r-1}}\{N_1 + N_2 + \cdots + N_{r_0} + N_{r_0+1} + \cdots + N_r\} \\
&\leq \left(\frac{M}{k_{r-1}}\right) r_0 + \ \frac{1}{k_{r-1}}\left\{h_{r_0+1}\ \left(\frac{N_{r_0+1}}{h_{r_0+1}}\right) + \cdots + h_r\ \left(\frac{N_r}{h_r}\right)\right\} \\
&\leq \left(\frac{M}{k_{r-1}}\right) r_0 + \ \frac{1}{k_{r-1}}\left(\sup_{r > \ r_0}\left(\frac{N_r}{h_r}\right)\right)\{h_{r_0+1} \ + \cdots + h_r \ \} \\
&\leq \left(\frac{M}{k_{r-1}}\right) r_0 + \ \delta\left(\frac{k_r - k_{r_0}}{k_{r-1}}\right) \qquad \text{using (3)} \\
&\leq \left(\frac{M}{k_{r-1}}\right) r_0 + \ \delta q_r \\
&\leq \left(\frac{M}{k_{r-1}}\right) r_0 + \ \delta H.
\end{aligned}$$

Since $k_{r-1} \to \infty$ as $n \to \infty$, it follows that $\frac{1}{n}\left|\left\{k \leq n : \ \left|\left(\frac{x_k}{y_k}\right) - L\right| \ \geq \ \epsilon\right\}\right| \to 0$ and, consequently, for any $\delta_1 > 0$, the set

$$\left\{n \in \mathbb{N} : \ \frac{1}{n}\left|\left\{k \leq n : \ \left|\left(\frac{x_k}{y_k}\right) - L\right| \ \geq \ \epsilon\right\}\right| \geq \delta_1\right\} \in \mathcal{I}$$

as it is an finite set. This shows that $x \sim^{S^L(\mathcal{I})} y$.

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