A NOTE ON STRONG DIFFERENTIAL SUPERORDINATIONS USING A MULTIPLIER TRANSFORMATION AND RUSCHEWEYH OPERATOR

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Abstract. In the present paper we establish several strong differential superordinations regardind the new operator $IR_{\lambda,l}^{m}$ defined by convolution product of the extended multiplier transformation and the extended Ruscheweyh derivative, $IR_{\lambda,l}^{m}:\mathcal{A}_{n\zeta}^{*}\to\mathcal{A}_{n\zeta}^{*}$, $IR_{\lambda,l}^{m}f\left(z,\zeta\right)=\left(I\left(m,\lambda,l\right)*R^{m}\right)f\left(z,\zeta\right)$, $z\in U$, $\zeta\in\overline{U}$, where $R^{m}f(z,\zeta)$ denote the extended Ruscheweyh derivative, $I\left(m,\lambda,l\right)f\left(z,\zeta\right)$ is the extended multiplier transformation and $\mathcal{A}_{n\zeta}^{*}=\{f\in\mathcal{H}(U\times\overline{U}),\ f(z,\zeta)=z+a_{n+1}\left(\zeta\right)z^{n+1}+\dots,\ z\in U,\ \zeta\in\overline{U}\}$ is the class of normalized analytic functions.

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1. Introduction

Denote by U the unit disc of the complex plane $U=\{z\in\mathbb{C}:\ |z|<1\}$, $\overline{U}=\{z\in\mathbb{C}:\ |z|\leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U\times\overline{U})$ the class of analytic functions in $U\times\overline{U}$.

Let

$$\mathcal{A}_{n\zeta}^{*}=\{f\in\mathcal{H}(U\times\overline{U}),\ f(z,\zeta)=z+a_{n+1}\left(\zeta\right)z^{n+1}+\dots,\ z\in U,\ \zeta\in\overline{U}\},$$

where $a_{k}\left(\zeta\right)$ are holomorphic functions in $\overline{U}$ for $k\geq 2$, and

$$\mathcal{H}^{*}[a,n,\zeta]=\{f\in\mathcal{H}(U\times\overline{U}),f(z,\zeta)=a+a_{n}\left(\zeta\right)z^{n}+a_{n+1}\left(\zeta\right)z^{n+1}+\dots,z\in U,\zeta\in\overline{U}\},$$

for $a\in\mathbb{C}$, $n\in\mathbb{N}$, $a_{k}\left(\zeta\right)$ are holomorphic functions in $\overline{U}$ for $k\geq n$.

We also extend the known differential operators to the new class of analytic functions $\mathcal{A}_{n\zeta}^{*}$ introduced in [15].

Definition No. 1 *[7] For* $n \in \mathbb{N}$, $m \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, $f \in \mathcal{A}_{n\zeta}^*$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, *the operator* $I(m,\lambda,l) f(z,\zeta)$ *is defined by the following infinite series*

$$I(m,\lambda,l) f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1} \right)^m a_j(\zeta) z^j, \quad z \in U, \zeta \in \overline{U}.$$

Remark No. 1 *[7] It follows from the above definition that*

$$(l+1) I(m+1,\lambda,l) f(z,\zeta) = [l+1-\lambda] I(m,\lambda,l) f(z,\zeta) + \lambda z (I(m,\lambda,l) f(z,\zeta))'_z,$$

$z \in U, \zeta \in \overline{U}$.

Definition No. 2 *[4] For* $f \in \mathcal{A}_{n\zeta}^*$, $n, m \in \mathbb{N}$, *the operator* R^m *is defined by* $R^m : \mathcal{A}_{n\zeta}^* \to \mathcal{A}_{n\zeta}^*$,

$$\begin{aligned} R^0 f(z,\zeta) &= f(z,\zeta), \\ R^1 f(z,\zeta) &= z f'_z(z,\zeta), ..., \\ (m+1) R^{m+1} f(z,\zeta) &= z (R^m f(z,\zeta))'_z + m R^m f(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}. \end{aligned}$$

Remark No. 2 *[4] If* $f \in \mathcal{A}_{n\zeta}^*$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, *then* $R^m f(z,\zeta) = z + \sum_{j=n+1}^{\infty} C_{m+j-1}^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [14].

Definition No. 3 *[14] Let* $f(z,\zeta)$, $H(z,\zeta)$ *analytic in* $U \times \overline{U}$. *The function* $f(z,\zeta)$ *is said to be strongly superordinate to* $H(z,\zeta)$ *if there exists a function* w *analytic in* U, *with* $w(0) = 0$ *and* $|w(z)| < 1$, *such that* $H(z,\zeta) = f(w(z),\zeta)$, *for all* $\zeta \in \overline{U}$. *In such a case we write* $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.

Remark No. 3 *[14] (i) Since* $f(z,\zeta)$ *is analytic in* $U \times \overline{U}$, *for all* $\zeta \in \overline{U}$, *and univalent in* U, *for all* $\zeta \in \overline{U}$, *Definition 3 is equivalent to* $H(0,\zeta) = f(0,\zeta)$, *for all* $\zeta \in \overline{U}$, *and* $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.

(ii) If $H(z,\zeta) \equiv H(z)$ *and* $f(z,\zeta) \equiv f(z)$, *the strong superordination becomes the usual notion of superordination.*

Definition No. 4 *[9] We denote by* Q^* *the set of functions that are analytic and injective on* $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, *where* $E(f,\zeta) = \{y \in \partial U : \lim_{z \to y} f(z,\zeta) = \infty\}$, *and are such that* $f'_z(y,\zeta) \neq 0$ *for* $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. *The subclass of* Q^* *for which* $f(0,\zeta) = a$ *is denoted by* $Q^*(a)$.

We have need the following lemmas to study the strong differential superordinations.

Lemma No. 1 *[9] Let $h(z,\zeta)$ be a convex function with $h(0,\zeta) = a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with $Re\ \gamma \geq 0$. If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and*

$$h(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is convex and is the best subordinant.

Lemma No. 2 *[9] Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{\gamma} zq'_z(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, where $Re\ \gamma \geq 0$.*

If $p \in \mathcal{H}^*[a,n,\zeta] \cap Q^*$, $p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta)$ is univalent in $U \times \overline{U}$ and

$$q(z,\zeta) + \frac{1}{\gamma} zq'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{\gamma} zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t,\zeta) t^{\frac{\gamma}{n}-1} dt$, $z \in U$, $\zeta \in \overline{U}$. The function q is the best subordinant.

2. Main results

Definition No. 5 *[5] Let $\lambda, l \geq 0$ and $m \in \mathbb{N}$. Denote by $IR^m_{\lambda,l}$ the operator given by the Hadamard product (the convolution product) of the extended multiplier transformation $I(m,\lambda,l)$ and the extended Ruscheweyh operator R^m, $IR^m_{\lambda,l} : \mathcal{A}^*_{n\zeta} \to \mathcal{A}^*_{n\zeta}$,*

$$IR^m_{\lambda,l} f(z,\zeta) = (I(m,\lambda,l) * R^m) f(z,\zeta).$$

Remark No. 4 *[5] If $f \in \mathcal{A}^*_{n\zeta}$, $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j(\zeta) z^j$, then $IR^m_{\lambda,l} f(z,\zeta) = z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a^2_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.*

Remark No. 5 *For $l = 0$, $\lambda \geq 0$, we obtain the extended Hadamard product DR_λ^n ([6], [2], [12], [13]) of the extended generalized Sălăgean operator D_λ^n and the extended Ruscheweyh operator R^n.*

For $l = 0$ and $\lambda = 1$, we obtain the extended Hadamard product SR^n ([1], [3], [10], [11]) of the extended Sălăgean operator S^n and the extended Ruscheweyh operator R^n.

Theorem No. 1 *Let $h(z,\zeta)$ be a convex function in $U \times \overline{U}$ with $h(0,\zeta) = 1$. Let $m \in \mathbb{N}$, $\lambda, l \geq 0$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta)\, dt$, $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$, and suppose that $\left(IR_{\lambda,l}^m f(z,\zeta)\right)_z'$ is univalent in $U \times \overline{U}$, $\left(IR_{\lambda,l}^m F(z,\zeta)\right)_z' \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and*

$$h(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m f(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U}, \tag{1}$$

then

$$q(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is convex and it is the best subordinant.

Proof. We have

$$z^{c+1} F(z,\zeta) = (c+2) \int_0^z t^c f(t,\zeta)\, dt$$

and differentiating it, with respect to z, we obtain $(c+1) F(z,\zeta) + z F_z'(z,\zeta) = (c+2) f(z,\zeta)$ and

$$(c+1)\, IR_{\lambda,l}^m F(z,\zeta) + z \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z' = (c+2)\, IR_{\lambda,l}^m f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Differentiating the last relation with respect to z we have

$$\left(IR_{\lambda,l}^m F(z,\zeta)\right)_z' + \frac{1}{c+2} z \left(IR_{\lambda,l}^m F(z,\zeta)\right)_{z^2}'' = \left(IR_{\lambda,l}^m f(z,\zeta)\right)_z', \quad z \in U,\ \zeta \in \overline{U}. \tag{2}$$

Using (2), the strong differential superordination (1) becomes

$$h(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z' + \frac{1}{c+2} z \left(IR_{\lambda,l}^m F(z,\zeta)\right)_{z^2}''. \tag{3}$$

Denote

$$p(z,\zeta)=\left(IR_{\lambda,l}^{m}F(z,\zeta)\right)_{z}', \quad z\in U,\ \zeta\in\overline{U}. \tag{4}$$

Replacing (4) in (3) we obtain

$$h(z,\zeta)\prec\prec p(z,\zeta)+\frac{1}{c+2}zp_{z}'(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 1 for $\gamma=c+2$, we have

$q(z,\zeta)\prec\prec p(z,\zeta)$, $z\in U$, $\zeta\in\overline{U}$, i.e. $q(z,\zeta)\prec\prec\left(IR_{\lambda,l}^{m}F(z,\zeta)\right)_{z}'$, $z\in U$, $\zeta\in\overline{U}$,

where $q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary No. 1 *Let* $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$, *where* $\beta\in[0,1)$. *Let* $m\in\mathbb{N}$, $\lambda,l\geq 0$, $f(z,\zeta)\in\mathcal{A}_{n\zeta}^{*}$, $F(z,\zeta)=I_c(f)(z,\zeta)=\frac{c+2}{z^{c+1}}\int_0^z t^c f(t,\zeta)\,dt$, $z\in U$, $\zeta\in\overline{U}$, $Rec>-2$, *and suppose that* $\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)_{z}'$ *is univalent in* $U\times\overline{U}$, $\left(IR_{\lambda,l}^{m}F(z,\zeta)\right)_{z}'\in\mathcal{H}^{*}[1,n,\zeta]\cap Q^{*}$ *and*

$$h(z,\zeta)\prec\prec\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)_{z}', \quad z\in U,\ \zeta\in\overline{U}, \tag{5}$$

then

$$q(z,\zeta)\prec\prec\left(IR_{\lambda,l}^{m}F(z,\zeta)\right)_{z}', \qquad z\in U,\ \zeta\in\overline{U},$$

where q *is given by* $q(z,\zeta)=2\beta-\zeta+\frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}}\int_0^z\frac{t^{\frac{c+2}{n}-1}}{t+1}dt$, $z\in U$, $\zeta\in\overline{U}$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 1 and considering $p(z,\zeta)=\left(IR_{\lambda,l}^{m}F(z,\zeta)\right)_{z}'$, the strong differential superordination (5) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+\frac{1}{c+2}zp_{z}'(z,\zeta), \qquad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1 for $\gamma=c+2$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z h(t,\zeta)t^{\frac{c+2}{n}-1}dt=\frac{c+2}{nz^{\frac{c+2}{n}}}\int_0^z\frac{\zeta+(2\beta-\zeta)t}{1+t}t^{\frac{c+2}{n}-1}dt$$

$$= 2\beta - \zeta + \frac{2(c+2)(\zeta-\beta)}{nz^{\frac{c+2}{n}}} \int_0^z \frac{t^{\frac{c+2}{n}-1}}{t+1} dt \prec\prec \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z', \quad z \in U, \ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 2 *Let $q(z,\zeta)$ be a convex function in $U \times \overline{U}$ and let $h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q_z'(z,\zeta)$, where $z \in U$, $\zeta \in \overline{U}$, $Re c > -2$.*

Let $m \in \mathbb{N}$, $\lambda, l \geq 0$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$, $F(z,\zeta) = I_c(f)(z,\zeta) = \frac{c+2}{z^{c+1}} \int_0^z t^c f(t,\zeta) dt$, $z \in U$, $\zeta \in \overline{U}$, and suppose that $\left(IR_{\lambda,l}^m f(z,\zeta)\right)_z'$ is univalent in $U \times \overline{U}$, $\left(IR_{\lambda,l}^m F(z,\zeta)\right)_z' \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and

$$h(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m f(z,\zeta)\right)_z', \quad z \in U, \ \zeta \in \overline{U}, \tag{6}$$

then

$$q(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z', \quad z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant.

Proof. Following the same steps as in the proof of Theorem 1 and considering $p(z,\zeta) = \left(IR_{\lambda,l}^m F(z,\zeta)\right)'$, $z \in U$, $\zeta \in \overline{U}$, the strong differential superordination (6) becomes

$$h(z,\zeta) = q(z,\zeta) + \frac{1}{c+2} z q_z'(z,\zeta) \prec\prec p(z,\zeta) + \frac{1}{c+2} z p_z'(z,\zeta), \quad z \in U, \ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = c+2$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \ z \in U, \ \zeta \in \overline{U}, \text{ i.e. } q(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m F(z,\zeta)\right)_z', \ z \in U, \ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{c+2}{nz^{\frac{c+2}{n}}} \int_0^z h(t,\zeta) t^{\frac{c+2}{n}-1} dt$. The function q is the best subordinant.

Theorem No. 3 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta) = 1$. Let $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}_{n\zeta}^*$ and suppose that $\left(IR_{\lambda,l}^m f(z,\zeta)\right)_z'$ is univalent and $\frac{IR_{\lambda,l}^m f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(IR_{\lambda,l}^m f(z,\zeta)\right)_z', \quad z \in U, \ \zeta \in \overline{U}, \tag{7}$$

then

$$q(z,\zeta) \prec\prec \frac{IR^m_{\lambda,l}f\left(z,\zeta\right)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. *The function* q *is convex and it is the best subordinant.*

Proof. Consider $p\left(z,\zeta\right) = \frac{IR^m_{\lambda,l}f(z,\zeta)}{z} = \frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a^2_j(\zeta)z^j}{z} = 1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1}a^2_j\left(\zeta\right)z^{j-1}$. Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

We have $p\left(z,\zeta\right)+zp'_z\left(z,\zeta\right) = \left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$.

Then (7) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta)+zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec\prec \frac{IR^m_{\lambda,l}f\left(z,\zeta\right)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary No. 2 *Let* $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$, *where* $0 \le \beta < 1$. *Let* $\lambda, l \ge 0$, $m,n \in \mathbb{N}$, $f\left(z,\zeta\right) \in \mathcal{A}^*_{n\zeta}$ *and suppose that* $\left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z$ *is univalent and* $\frac{IR^m_{\lambda,l}f(z,\zeta)}{z} \in \mathcal{H}^*\left[1,n,\zeta\right] \cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{8}$$

then

$$q(z,\zeta) \prec\prec \frac{IR^m_{\lambda,l}f\left(z,\zeta\right)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where q *is given by* $q(z,\zeta) = 2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z \frac{t^{\frac{1}{n}-1}}{1+t}dt$, $z \in U$, $\zeta \in \overline{U}$. *The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 3 and considering $p(z,\zeta) = \frac{IR^m_{\lambda,l}f(z,\zeta)}{z}$, the strong differential superordination (8) becomes

$$h(z,\zeta) = \frac{\zeta + (2\beta - \zeta)z}{1+z} \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $\gamma = 1$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1} dt = \frac{1}{nz^{\frac{1}{n}}} \int_0^z t^{\frac{1}{n}-1} \frac{\zeta + (2\beta - \zeta)\, t}{1+t} dt$$

$$= 2\beta - \zeta + \frac{2(\zeta - \beta)}{nz^{\frac{1}{n}}} \int_0^z \frac{t^{\frac{1}{n}-1}}{1+t} dt \prec\prec \frac{IR^m_{\lambda,l}f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 4 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta)$. If $\lambda, l \geq 0$, $m, n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, suppose that $\left(IR^m_{\lambda,l}f(z,\zeta)\right)'_z$ is univalent, $\frac{IR^m_{\lambda,l}f(z,\zeta)}{z} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + zq'_z(z,\zeta) \prec\prec \left(IR^m_{\lambda,l}f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \qquad (9)$$

then

$$q(z,\zeta) \prec\prec \frac{IR^m_{\lambda,l}f(z,\zeta)}{z}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt$. The function q is the best subordinant.

Proof. Let $p(z,\zeta) = \frac{IR^m_{\lambda,l}f(z,\zeta)}{z} = \frac{z + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2(\zeta) z^j}{z} =$

$1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} a_j^2(\zeta)\, z^{j-1}$. Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta) + zp'_z(z,\zeta) = \left(IR^m_{\lambda,l}f(z,\zeta)\right)'_z$, $z \in U$, $\zeta \in \overline{U}$, and (9) becomes

$$q(z,\zeta) + zq'_z(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

Using Lemma 2 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.}$$

$$q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}} \int_0^z h(t,\zeta) t^{\frac{1}{n}-1} dt \prec\prec \frac{IR^m_{\lambda,l}f(z,\zeta)}{z}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Theorem No. 5 *Let $h(z,\zeta)$ be a convex function, $h(0,\zeta)=1$. Let $\lambda, l \geq 0$, $m,n \in \mathbb{N}$, $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$ and suppose that $\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z$ is univalent and $\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)} \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$. If*

$$h(z,\zeta) \prec\prec \left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z, \qquad z \in U,\ \zeta \in \overline{U}, \tag{10}$$

then

$$q(z,\zeta) \prec\prec \frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Proof. Consider $p(z,\zeta) = \frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)} = \frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C^{m+1}_{m+j}a^2_j(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C^{m}_{m+j-1}a^2_j(\zeta)z^j} = \frac{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C^{m+1}_{m+j}a^2_j(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C^{m}_{m+j-1}a^2_j(\zeta)z^{j-1}}$. Evidently $p \in \mathcal{H}^*[1,n,\zeta]$.

We have $p'_z(z,\zeta) = \frac{\left(IR^{m+1}_{\lambda,l}f(z,\zeta)\right)'_z}{IR^m_{\lambda,l}f(z,\zeta)} - p(z,\zeta)\cdot\frac{\left(IR^m_{\lambda,l}f(z,\zeta)\right)'_z}{IR^m_{\lambda,l}f(z,\zeta)}$. Then $p(z,\zeta)+zp'_z(z,\zeta) = \left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z$.

Then (10) becomes

$$h(z,\zeta) \prec\prec p(z,\zeta) + zp'_z(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $\gamma = 1$, we have

$$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U}, \quad \text{i.e.} \quad q(z,\zeta) \prec\prec \frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)},\ z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\, t^{\frac{1}{n}-1}dt$. The function q is convex and it is the best subordinant.

Corollary No. 3 *Let* $h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U\times\overline{U}$*, where* $0\leq\beta<1$*. Let* $\lambda,l\geq 0$*,* $m,n\in\mathbb{N}$*,* $f(z,\zeta)\in\mathcal{A}^*_{n\zeta}$ *and suppose that* $\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z$ *is univalent,* $\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$*. If*

$$h(z,\zeta)\prec\prec\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z,\qquad z\in U,\ \zeta\in\overline{U},\tag{11}$$

then

$$q(z,\zeta)\prec\prec\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)},\qquad z\in U,\ \zeta\in\overline{U},$$

where q *is given by* $q(z,\zeta)=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt$*,* $z\in U$*,* $\zeta\in\overline{U}$*. The function* q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 5 and considering $p(z,\zeta)=\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}$, the strong differential superordination (11) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z}\prec\prec p(z,\zeta)+zp'_z(z,\zeta),\quad z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1 for $\gamma=1$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt=\frac{1}{nz^{\frac{1}{n}}}\int_0^z t^{\frac{1}{n}-1}\frac{\zeta+(2\beta-\zeta)\,t}{1+t}dt$$

$$=2\beta-\zeta+\frac{2(\zeta-\beta)}{nz^{\frac{1}{n}}}\int_0^z\frac{t^{\frac{1}{n}-1}}{1+t}dt\prec\prec\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)},\quad z\in U,\ \zeta\in\overline{U}.$$

The function q is convex and it is the best subordinant.

Theorem No. 6 *Let* $q(z,\zeta)$ *be convex in* $U\times\overline{U}$ *and let* h *be defined by* $h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)$*. If* $\lambda,l\geq 0$*,* $m,n\in\mathbb{N}$*,* $f(z,\zeta)\in\mathcal{A}_{n\zeta}$*, suppose that* $\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z$ *is univalent,* $\frac{IR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$ *and satisfies the strong differential superordination*

$$h(z,\zeta)=q(z,\zeta)+zq'_z(z,\zeta)\prec\prec\left(\frac{zIR^{m+1}_{\lambda,l}f(z,\zeta)}{IR^m_{\lambda,l}f(z,\zeta)}\right)'_z,\qquad z\in U,\ \zeta\in\overline{U},\tag{12}$$

then

$$q(z,\zeta) \prec\prec \frac{IR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}, \qquad z\in U,\ \zeta\in\overline{U},$$

where $q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt$. *The function* q *is the best subordinant.*

Proof. Let $p(z,\zeta)=\frac{IR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}=\frac{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2(\zeta)z^j}{z+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2(\zeta)z^j}=$

$\frac{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m+1}C_{m+j}^{m+1}a_j^2(\zeta)z^{j-1}}{1+\sum_{j=n+1}^{\infty}\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{m}C_{m+j-1}^{m}a_j^2(\zeta)z^{j-1}}$. Evidently $p\in\mathcal{H}^*[1,n,\zeta]$.

Differentiating with respect to z, we obtain $p(z,\zeta)+zp_z'(z,\zeta)=\left(\frac{zIR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}\right)_z'$, $z\in U$, $\zeta\in\overline{U}$, and (12) becomes

$$q(z,\zeta)+zq_z'(z,\zeta)\prec\prec p(z,\zeta)+zp_z'(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}.$$

Using Lemma 2 for $\gamma=1$, we have

$$q(z,\zeta)\prec\prec p(z,\zeta), \quad z\in U,\ \zeta\in\overline{U}, \quad \text{i.e.}$$

$$q(z,\zeta)=\frac{1}{nz^{\frac{1}{n}}}\int_0^z h(t,\zeta)\,t^{\frac{1}{n}-1}dt \prec\prec \frac{IR_{\lambda,l}^{m+1}f(z,\zeta)}{IR_{\lambda,l}^{m}f(z,\zeta)}, \ \ z\in U,\ \zeta\in\overline{U},$$

and q is the best subordinant.

Theorem No. 7 *Let* $h(z,\zeta)$ *be a convex function,* $h(0,\zeta)=1$. *Let* $\lambda,l\geq 0$, $m,n\in\mathbb{N}$, $f(z,\zeta)\in\mathcal{A}_{n\zeta}^*$ *and suppose that* $\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\cdot\left[(m+1)IR_{\lambda,l}^{m+1}f(z,\zeta)-(m-2)IR_{\lambda,l}^{m}f(z,\zeta)\right]+\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z\frac{IR_{\lambda,l}^{m}f(t,\zeta)-t}{t^2}dt$ *is univalent and* $\left(IR_{\lambda,l}^{m}f(z,\zeta)\right)_z'\in\mathcal{H}^*[1,n,\zeta]\cap Q^*$. *If*

$$h(z,\zeta)\prec\prec\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1)IR_{\lambda,l}^{m+1}f(z,\zeta)-(m-2)IR_{\lambda,l}^{m}f(z,\zeta)\right]$$

$$+\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z\frac{IR_{\lambda,l}^{m}f(t,\zeta)-t}{t^2}dt, \tag{13}$$

$z \in U, \zeta \in \overline{U}$, *then*

$$q(z,\zeta) \prec\prec \left(IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h\left(t,\zeta\right) t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt$. *The function q is convex and it is the best subordinant.*

Proof. With notation

$p\left(z,\zeta\right) = \left(IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right)'_z = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^{m} j a_j^2\left(\zeta\right) z^{j-1}$ and $p\left(0,\zeta\right) = 1$, we obtain for $f(z,\zeta) = z + \sum_{j=n+1}^{\infty} a_j\left(\zeta\right) z^j$,

$p\left(z,\zeta\right) + zp'_z\left(z,\zeta\right) = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^{m} j a_j^2\left(\zeta\right) z^{j-1} +$
$\sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C_{m+j-1}^{m} j\left(j-1\right) a_j^2\left(\zeta\right) z^{j-1} =$
$\frac{1}{z}\left(\frac{m+1}{\lambda} IR_{\lambda,l}^{m+1} f\left(z,\zeta\right) - \frac{m-2}{\lambda} IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right)'_z +$
$\left(1 - \frac{m-1}{l+1} - \frac{2}{\lambda}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \int_0^z \frac{IR_{\lambda,l}^{m} f(t,\zeta)-t}{t^2} dt.$

Therefore $p\left(z,\zeta\right) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} zp'_z\left(z,\zeta\right) =$
$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[\left(m+1\right) IR_{\lambda,l}^{m+1} f\left(z,\zeta\right) - \left(m-2\right) IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right] +$
$\left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR_{\lambda,l}^{m} f(t,\zeta)-t}{t^2} dt.$

Then (13) becomes

$$h(z,\zeta) \prec\prec p\left(z,\zeta\right) + \frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right) - \left(l+1\right)} zp'_z\left(z,\zeta\right), \quad z \in U,\ \zeta \in \overline{U}.$$

By using Lemma 1 for $\gamma = 1 - \frac{m-1}{l+1} - \frac{1}{\lambda}$, we have

$q(z,\zeta) \prec\prec p(z,\zeta),\ z \in U,\ \zeta \in \overline{U}$, i.e. $q(z,\zeta) \prec\prec \left(IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right)'_z,\ z \in U,\ \zeta \in \overline{U}$,

where $q(z,\zeta) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h\left(t,\zeta\right) t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt$. The function q is convex and it is the best subordinant.

Corollary No. 4 *Let* $h(z,\zeta) = \frac{\zeta+(2\beta-\zeta)z}{1+z}$ *be a convex function in* $U \times \overline{U}$, *where* $0 \le \beta < 1$. *Let* $\lambda, l \ge 0$, $m, n \in \mathbb{N}$, $f\left(z,\zeta\right) \in \mathcal{A}_{n\zeta}^{*}$ *and suppose that*
$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \cdot \left[\left(m+1\right) IR_{\lambda,l}^{m+1} f\left(z,\zeta\right) - \left(m-2\right) IR_{\lambda,l}^{m} f\left(z,\zeta\right)\right] +$

$\left(1-\frac{l+1}{\lambda(l-m+2)-(l+1)}\right)-\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)}\int_0^z \frac{IR^m_{\lambda,l}f(t,\zeta)-t}{t^2}dt$ *is univalent,* $\left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z \in \mathcal{H}^*\left[1,n,\zeta\right]\cap Q^*$. *If*

$$h(z,\zeta) \prec\prec \frac{l+1}{\left[\lambda\left(l-m+2\right)-\left(l+1\right)\right]z}\left[\left(m+1\right)IR^{m+1}_{\lambda,l}f\left(z,\zeta\right)-\left(m-2\right)IR^m_{\lambda,l}f\left(z,\zeta\right)\right] \tag{14}$$

$$+\left(1-\frac{l+1}{\lambda\left(l-m+2\right)-\left(l+1\right)}\right)-\frac{2\left(l+1\right)\left(m-1\right)-2\lambda m}{\lambda\left(l-m+2\right)-\left(l+1\right)}\int_0^z \frac{IR^m_{\lambda,l}f\left(t,\zeta\right)-t}{t^2}dt,$$

$z \in U, \zeta \in \overline{U}$, *then*

$$q(z,\zeta) \prec\prec \left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z, \qquad z\in U,\ \zeta\in\overline{U},$$

where q *is given by*

$$q(z,\zeta)=2\beta-\zeta+2(\zeta-\beta)\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z \frac{t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}}{1+t}dt,\ z\in U,$$

$\zeta \in \overline{U}$.

The function q *is convex and it is the best subordinant.*

Proof. Following the same steps as in the proof of Theorem 7 and considering $p(z,\zeta)=\left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z$, the strong differential superordination (14) becomes

$$h(z,\zeta)=\frac{\zeta+(2\beta-\zeta)z}{1+z} \prec\prec p(z,\zeta)+\frac{\lambda\left(l+1\right)}{\lambda\left(l-m+2\right)-\left(l+1\right)}zp'_z(z,\zeta),\ z\in U,\ \zeta\in\overline{U}.$$

By using Lemma 1 for $\gamma=\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)}$, we have $q(z,\zeta)\prec\prec p(z,\zeta)$, i.e.

$$q(z,\zeta)=\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z h\left(t,\zeta\right)t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}dt=$$

$$\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}\frac{\zeta+\left(2\beta-\zeta\right)t}{1+t}dt=$$

$$2\beta-\zeta+2(\zeta-\beta)\frac{\lambda\left(l-m+2\right)-\left(l+1\right)}{\lambda\left(l+1\right)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}}\int_0^z\frac{t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}}}{1+t}dt\prec\prec\left(IR^m_{\lambda,l}f\left(z,\zeta\right)\right)'_z,$$

$z\in U, \zeta\in\overline{U}$.

The function q is convex and it is the best subordinant.

Theorem No. 8 *Let $q(z,\zeta)$ be convex in $U \times \overline{U}$ and let h be defined by $h(z,\zeta) = q(z,\zeta) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} z \cdot q'_z(z,\zeta)$, $\lambda, l \geq 0$, $m, n \in \mathbb{N}$. If $f(z,\zeta) \in \mathcal{A}^*_{n\zeta}$, suppose that $\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1) IR^{m+1}_{\lambda,l} f(z,\zeta) - (m-2) IR^m_{\lambda,l} f(z,\zeta)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \cdot \int_0^z \frac{IR^m_{\lambda,l} f(t,\zeta)-t}{t^2} dt$ is univalent, $\left(IR^m_{\lambda,l} f(z,\zeta)\right)'_z \in \mathcal{H}^*[1,n,\zeta] \cap Q^*$ and satisfies the strong differential superordination*

$$h(z,\zeta) = q(z,\zeta) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} z q'_z(z,\zeta) \prec\prec \frac{l+1}{[\lambda(l-m+2)-(l+1)]z} \cdot \tag{15}$$

$$\left[(m+1) IR^{m+1}_{\lambda,l} f(z,\zeta) - (m-2) IR^m_{\lambda,l} f(z,\zeta)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) -$$

$$\frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t,\zeta)-t}{t^2} dt, \qquad z \in U,\ \zeta \in \overline{U},$$

then

$$q(z,\zeta) \prec\prec \left(IR^m_{\lambda,l} f(z,\zeta)\right)'_z, \qquad z \in U,\ \zeta \in \overline{U},$$

where $q(z,\zeta) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h(t,\zeta) t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt$. The function q is the best subordinant.

Proof. Let

$p(z,\zeta) = \left(IR^m_{\lambda,l} f(z,\zeta)\right)'_z = 1 + \sum_{j=n+1}^{\infty} \left(\frac{1+\lambda(j-1)+l}{l+1}\right)^m C^m_{m+j-1} j a_j^2(\zeta) z^{j-1}$.

Differentiating with respect to z, we obtain
$p(z,\zeta) + z p'_z(z,\zeta) = \frac{1}{z}\left(\frac{m+1}{\lambda} IR^{m+1}_{\lambda,l} f(z,\zeta) - \frac{m-2}{\lambda} IR^m_{\lambda,l} f(z,\zeta)\right) + \frac{\lambda(m-1)-(l+1)}{\lambda(l+1)}\left(IR^m_{\lambda,l} f(z,\zeta)\right)'_z + \left(1 - \frac{m-1}{l+1} - \frac{2}{\lambda}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t,\zeta)-t}{t^2} dt$

and $p(z,\zeta) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} z p'_z(z,\zeta) =$
$\frac{l+1}{[\lambda(l-m+2)-(l+1)]z}\left[(m+1) IR^{m+1}_{\lambda,l} f(z,\zeta) - (m-2) IR^m_{\lambda,l} f(z,\zeta)\right] + \left(1 - \frac{l+1}{\lambda(l-m+2)-(l+1)}\right) - \frac{2(l+1)(m-1)-2\lambda m}{\lambda(l-m+2)-(l+1)} \int_0^z \frac{IR^m_{\lambda,l} f(t,\zeta)-t}{t^2} dt$, $z \in U$, $\zeta \in \overline{U}$, and (15) becomes

$$q(z,\zeta) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} z q'_z(z,\zeta) \prec\prec p(z,\zeta) + \frac{\lambda(l+1)}{\lambda(l-m+2)-(l+1)} z p'_z(z,\zeta),$$

$z \in U, \zeta \in \overline{U}$.

Using Lemma 2 for $\gamma = 1 - \frac{m-1}{l+1} - \frac{1}{\lambda}$, we have $q(z,\zeta) \prec\prec p(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$, i.e.

$$q(z,\zeta) = \frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)nz^{\frac{\lambda(l-m+2)-(l+1)}{\lambda(l+1)n}}} \int_0^z h(t,\zeta)\, t^{\frac{\lambda(l-m-nl-n+2)-(l+1)}{\lambda(l+1)n}} dt \prec\prec \left(IR^m_{\lambda,l} f(z,\zeta)\right)'_z,$$

$z \in U$, $\zeta \in \overline{U}$, and q is the best subordinant.

References

[1] A. Alb Lupaş, *Certain strong differential subordinations using Sălăgean and Ruscheweyh operators*, Advances in Applied Mathematical Analysis, Volume 6, Number 1 (2011), 27–34.

[2] A. Alb Lupaş, *A note on strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator*, submitted 2010.

[3] A. Alb Lupaş, *A note on strong differential subordinations using Sălăgean and Ruscheweyh operators*, Libertas Mathematica, submitted 2010.

[4] A. Alb Lupaş, G.I. Oros, Gh. Oros, *On special strong differential subordinations using Sălăgean and Ruscheweyh operators*, Journal of Computational Analysis and Applications, Vol. 14, 2012 (to appear).

[5] A. Alb Lupaş, *Certain strong differential subordinations using a multiplier transformation and Ruscheweyh operator*, International Journal of Open Problems in Complex Analysis, Vol. 3, No. 1, 2011, 1-8.

[6] A. Alb Lupaş, *Certain strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh operator*, submitted 2010.

[7] A. Alb Lupaş, *On special strong differential subordinations using multiplier transformation*, Applied Mathematics Letter, submitted 2011.

[8] A. Alb Lupaş, *On special strong differential subordinations using a generalized Sălăgean operator and Ruscheweyh derivative*, Journal of Concrete and Applicable Mathematics, 2012 (to appear).

[9] A. Alb Lupaş, *On special strong differential superordinations using Sălăgean and Ruscheweyh operators*, submitted 2011.

[10] A. Alb Lupaş, *Certain strong differential superordinations using Sălăgean and Ruscheweyh operators*, Acta Universitatis Apulensis, submitted 2011.

[11] A. Alb Lupaş, *A note on strong differential superordinations using Sălăgean and Ruscheweyh operators*, Journal of Applied Functional Analysis, 2012 (to appear).

[12] A. Alb Lupaş, *Certain strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator*, Journal of Applied Functional Analysis, 2012 (to appear).

[13] A. Alb Lupaş, *A note on strong differential superordinations using a generalized Sălăgean operator and Ruscheweyh operator*, submitted GFTA 2011.

[14] G.I. Oros, *Strong differential superordination*, Acta Universitatis Apulensis, Nr. 19, 2009, 101-106.

[15] G.I. Oros, *On a new strong differential subordination*, (to appear).

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