SOME KUMMER-TYPE CONVERGENCE CRITERIA

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Abstract. Positive terms series are an important part of the mathematical analysis. The study of the convergence of series of this interested many mathematicians, these type of in particular convergent series having many applications in various domains. In this paper we present some convergence criteria based on the Kummer convergence criterion.

Keywords: *numerical series, positive terms series, divergence.*

Definition 1. *A series* $\sum_{n=1}^{\infty} a_n$ *is called a positive terms series if* a_n *is a positive for every* $n = 1,2,...,\infty$.

Theorem 1.(Kummer). *Let* $a_1, a_2, ..., a_n, ...$ *be a sequence of positive numbers. If there is a fixed* $k \geq 0$ *such that*

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} > k$$

for every n, then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

If

$$a_n \frac{u_n}{u_{n+1}} - a_{n+1} \leq k$$

for every n and the series $\sum_{n=1}^{\infty} \frac{1}{a_n}$ *is divergent then also the series* $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Proof. In the first case we have

$$u_{n+1} < \frac{1}{k}\left(a_n u_n - a_{n+1} u_{n+1}\right)$$

hence

$$u_2 < \frac{1}{k}\left(a_1 u_1 - a_2 u_2\right)$$

$$u_3 < \frac{1}{k}\left(a_2 u_2 - a_3 u_3\right)$$

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$$u_{n+1} < \frac{1}{k}\left(a_n u_n - a_{n+1}u_{n+1}\right)$$

and

$$S_{n+1} = u_1 + u_2 + \ldots + u_{n+1} < \frac{1}{k}\left(a_1 u_1 - a_{n+1}u_{n+1}\right) + u_1 < u_1 + \frac{1}{k}a_1 u_1 .$$

The sequence S_n is convergent since it is increasing and upper bounded.
In the second case we have

$$a_n u_n - a_{n+1}u_{n+1} \leq 0$$

or

$$\frac{a_n}{a_{n+1}} \leq \frac{u_{n+1}}{u_n} \Leftrightarrow \frac{\frac{1}{a_{n+1}}}{\frac{1}{a_n}} \leq \frac{u_{n+1}}{u_n} .$$

Using the comparison criterion and the divergence of series

$$\sum_{n=1}^{\infty}\frac{1}{a_n} ,$$

we obtain that the series $\sum_{n=1}^{\infty} u_n$ is divergent as well.

Remark.
1)In the application one calculates

$$\lim_{n\to\infty}\left(a_n \frac{u_n}{u_{n+1}} - a_{n+1}\right) = \lambda .$$

a)If $\lambda > 0$ the series $\sum_{n=1}^{\infty} u_n$ is convergent.

b)If $\lambda < 0$ the series $\sum_{n=1}^{\infty} u_n$ is divergent.

2)If $a_n = 1$ we obtain the d'Alembert criterion.
3)If $a_n = n$ we obtain the Duhamel criterion.

Corrolary 1. *Let us consider the sequence* $a_n = n^p$ *where* $p \in [0,1]$ *and* $\sum_{n=1}^{\infty} u_n$ *is a positive terms series. If there is a fixed* $k > 0$ *such that*

$$n^p \frac{u_n}{u_{n+1}} - (n+1)^p > k$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

If

$$n^p \frac{u_n}{u_{n+1}} - (n+1)^p \le 0$$

then the series $\sum_{n=1}^{\infty} u_n$ *is divergent.*

The proof of this corollary is based on the proof on the previous theorem with the sequence a_n chosen to be $a_n = n^p$.

Remark.

1)If $p = 0$ we obtain the d'Alembert criterion.
2)If $p = 1$ we obtain the Duhamel criterion we note that this corollary is a move general form of Raabe-Duhamel criterion where can be obtained from if we set $p = 1$.

Remark. In application one evaluates

$$\lim_{n\to\infty}\left(n^p \frac{u_n}{u_{n+1}} - (n+1)^p \right) = k .$$

a)If $k > 0$ the series $\sum_{n=1}^{\infty} u_n$ is convergent.

b)If $k < 0$ the series $\sum_{n=1}^{\infty} u_n$ is divergent.

Corollary 2. *Let us consider the sequence*

$$a_n = \ln n,\ n > 0$$

such that $a_n > 0$.

If $\sum_{n=1}^{\infty} u_n$ *is a positive terms series and there is a* $k > 0$ *such that*

i) $\ln n\left(\frac{u_n}{u_{n+1}}\right) - \ln(n+1) > k$, *then the series* $\sum_{n=1}^{\infty} u_n$ *is convergent.*

ii) $\frac{u_n}{u_{n+1}} \le \frac{\ln(n+1)}{\ln n}$, *then the series* $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Remark. If we evaluate

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} \le \lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = 1,$$

we obtain the divergence condition from the raport criterion.

Corollary 3. *Let us consider the sequence*

$$a_n = \frac{1}{n^2} > 0, \forall n \in N^*.$$

If $\sum_{n=1}^{\infty} u_n$ *is a positive terms series and there is a fixed* $k > 0$ *such that:*

i)*If*

$$\frac{1}{n^2}\frac{u_n}{u_{n+1}} - \frac{1}{(n+1)^2} > k,$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

ii)*If*

$$\frac{u_n}{u_{n+1}} \le \frac{n^2}{(n+1)^2}, \forall n \in N^*,$$

then the series $\sum_{n=1}^{\infty} u_n$ *is divergent.*

Corollary 4. *Let us consider the sequence*

$$a_n = \frac{1}{n!} > 0, \forall n \in N^*.$$

If $\sum_{n=1}^{\infty} u_n$ *is a positive terms series and there is a fixed* $k > 0$ *such that:*

i)*If*

$$\frac{1}{n!}\frac{u_n}{u_{n+1}} - \frac{1}{(n+1)!} > k,$$

then the series $\sum_{n=1}^{\infty} u_n$ *is convergent.*

ii)*If*

$$\frac{u_n}{u_{n+1}} \le \frac{1}{n+1}, \forall n \in N^*,$$

then the series $\sum_{n=1}^{\infty} u_n$ *is divergent.*

These convergence criteria can be used to investigate the convergence of some positive terms series for which classical convergence criteria cannot be applied. Some they are based on the Kummer criterion they can also be called Kummer-type criteria.

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