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ON NILPOTENT GALOIS GROUPS AND THE SCOPE OF THE NORM LIMITATION THEOREM IN ONE-DIMENSIONAL ABSTRACT LOCAL CLASS FIELD THEORY

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Abstract. Let E be a primarily quasilocal field, $E_{\rm sep}$ a separable closure of E, R a finite extension of E in $E_{\rm sep}$, $R_{\rm ab}$ the maximal abelian subextension of E in R, and M the minimal Galois extension of E in $E_{\rm sep}$ including R. The main result of this paper shows that the norm groups $N(R/E)$ and $N(R_{\rm ab}/E)$ are equal, if the Galois group $G(M/E)$ is nilpotent. It proves that this is not necessarily true, if $G(M/E)$ is isomorphic to any given nonnilpotent finite group G.

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1. Introduction

This paper is devoted to the study of norm groups of finite separable extensions of primarily quasilocal fields (briefly, PQL-fields), i.e. of p-quasilocal fields with respect to every prime number p. It has been proved in [7] that such a field E admits one-dimensional local p-class field theory, provided that the p-component $\mathrm{Br}(E)_p$ of the Brauer group $\mathrm{Br}(E)$ is nontrivial. This theory shows that finite abelian p-extensions of E are subject to exact analogues to the local reciprocity law and the local Hasse symbol (cf. [25, Ch. 6, Theorem 8], [16, Ch. 2, 1.3] and [7, Theorems 2.1 and 2.2]), which leads to a satisfactory description of the norm group of any abelian finite extension of E.

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The present paper proves that if R/E is a finite separable extension and M is a normal closure of R over E, then the nilpotency of the Galois group $G(M/E)$ is a minimal sufficient condition for the validity of the norm group equality $N(R/E) = N(R_{\rm ab}/E)$, where $R_{\rm ab}$ is the maximal abelian subextension of E in R. Our main result has been used elsewhere for describing the norm groups of finite separable extensions of Henselian discrete valued fields whose finite extensions are strictly PQL, and of formally real quasilocal fields (see [10] and the references there).

The basic notions of (one-dimensional) local class field theory, used in the sequel, are defined in Section 2. For each field E, we denote by $P(E)$ the set of those prime numbers p, for which there exists at least one cyclic extension of E of degree p. Clearly, a prime number p lies in $P(E)$ if and only if E does not equal its maximal p-extension $E(p)$ in a separable closure $E_{\rm sep}$ of E. Let us note that a field E is said to be p-quasilocal, if it satisfies some of the following two conditions: (i) ${\rm Br}(E)_p = \{0\}$ or $p \notin P(E)$; (ii) cyclic extensions of E of degree p embed as E-subalgebras in each central division E-algebra of Schur index p. When this occurs, we say that E is strictly p-quasilocal, provided that ${\rm Br}(E)_p \neq \{0\}$ or $p \notin P(E)$. The field E is called strictly primarily quasilocal, if it is strictly p-quasilocal, for every prime p; it is said to be quasilocal, if its finite extensions are PQL-fields. It has been proved in [7] that strictly PQL-fields admit local class field theory. As to the converse, it holds in each of the following special cases: (i) E contains a primitive p-th root of unity, for each prime $p \geq 5$ not equal to ${\rm char}(E)$; (ii) E is an algebraic extension of a global field E_0. It should also be noted that all presently known fields with local class field theory are strictly PQL (see Proposition 2.5 and Remark 2.6).

The description of norm groups of finite extensions of strictly PQL-fields is a major objective of local class field theory. When E is a local field, this is achieved by describing closed subgroups of finite index in the multiplicative group E^* of E (see, for example, [12, Ch. IV, (6.2)]), since the norm limitation theorem (cf. [13, Ch. 6, Theorem 8]) yields $N(R/E) = N(R_{\rm ab}/E)$ whenever R/E is a finite extension. As a part of a generalization of the whole theory, the theorem has been extended by Moriya [21] to the case where R is separable over E and E possesses a Henselian discrete valuation with a quasifinite residue field $\widehat{E}$ (see [26; 31] and [12, Ch. V] as well). The purpose of this paper is to shed light on the scope of validity of the norm limitation theorem by proving the following assertions:

Theorem 1.1. *Let E be a* PQL*-field, M/E a finite Galois extension with a nilpotent Galois group $G(M/E)$, and R an intermediate field of M/E. Then $N(R/E) = N(R_{\rm ab}/E)$.*

Theorem 1.2. *For each nonnilpotent finite group G, there exists a strictly* PQL*-field $E(G)$ and a Galois extension $M(G)$ of $E(G)$, for which the following is true:*

(i) *$E(G)$ is an algebraic extension of the field $\mathbb{Q}$ of rational numbers;*

(ii) *The Galois group $G(M(G)/E(G))$ is isomorphic to G, and $N(M(G)/E(G))$ is a proper subgroup of $N(M(G)_{\rm ab}/E(G))$.*

The proof of Theorem 1.2 relies on the fact that every algebraic strictly PQL-extension E of a global field E_0 possesses a certain characteristic system $\{v(p) : p \in P(E)\}$ of nontrivial absolute values of E (see Proposition 2.7). This has been established in [9] and used there for proving that if R/E is a finite extension, then $N(R/E) = N(\Phi(R)/E)$, for some finite abelian extension $\Phi(R)$ of E in $E_{\rm sep}$, uniquely determined by the local behaviour of R/E at $v(p)$, when p runs through the set of elements of $P(E)$ dividing the degree $[M : E]$ of the normal closure M of R in $E_{\rm sep}$ over E. It should also be pointed out that the conclusion of Theorem 1.1 remains true without restrictions on $G(M/E)$, under the hypothesis that E is a quasilocal field and the natural Brauer group homomorphism $\mathrm{Br}(E) \to \mathrm{Br}(L)$ is surjective, for every finite extension L of E. This condition has been found in [10], and its role for the present research is clarified there by providing a series of examples of a field E with the following two properties: (i) finite extensions of E are strictly PQL-fields; (ii) the Galois group $G(E_{\rm sep}/E)$ is not pronilpotent; (iii) if R/E is a finite extension, then $N(R/E) \neq N(R_1/E)$, for any finite abelian extension R_1/E, unless the normal closure of R in $E_{\rm sep}$ over E satisfies the condition of Theorem 1.1.

Theorems 1.1 and 1.2 are proved in Sections 3 and 4, and Section 2 includes preliminaries needed for our considerations. Throughout the paper, algebras are understood to be associative with a unit, simple algebras are supposed to be finite-dimensional over their centres, Brauer groups of fields are viewed as additively presented, absolute values are assumed to be nontrivial, and Galois groups are regarded as profinite with respect to the Krull topology. For each algebra A, we consider only subalgebras of A containing its unit, and denote by A^* the multiplicative group of A. As usual, a field E is said to be formally real, if -1 is not presentable as a finite sum of squares of elements of E. Our basic terminology and notation concerning valuation theory, simple algebras and Brauer groups is standard (such as can be found, for example, in [17; 14] and [22]), as well as those concerning profinite groups, Galois cohomology, field extensions and Galois theory (see, for example, [27; 23] and [17]).

2. Preliminaries

Let E be a field, E_{sep} a separable closure of E, $\mathrm{Nr}(E)$ the set of norm groups of finite extensions of E, and $\Omega(E)$ the set of finite abelian extensions of E in E_{sep}. We say that E admits (one-dimensional) local class field theory, if the mapping π of $\Omega(E)$ into $\mathrm{Nr}(E)$ defined by the rule $\pi(F) = N(F/E)$: $F \in \Omega(E)$, is injective and satisfies the following two conditions, for each pair $(M_1, M_2) \in \Omega(E) \times \Omega(E)$:

The norm group of the compositum M_1M_2 is equal to the intersection $N(M_1/E) \cap N(M_2/E)$ and $N((M_1 \cap M_2)/E)$ equals the inner group product $N(M_1/E)N(M_2/E)$.

We call E a field with one-dimensional local p-class field theory, for some prime number p, if the restriction of π on the set of abelian finite p-extensions of E in E_{sep} has the same properties. Our research is based on the fact that the description of norm groups of finite Galois extensions with nilpotent Galois groups reduces to the special case of p-extensions. This possibility can be seen from the following two lemmas.

Lemma 2.1. *Let E be a field and L an extension of E presentable as a compositum of extensions L_1 and L_2 of E of relatively prime degrees. Then $N(L/E) = N(L_1/E) \cap N(L_2/E)$, $N(L_1/E) = E^* \cap N(L/L_2)$, and there is a group isomorphism $E^*/N(L/E) \cong (E^*/N(L_1/E)) \times (E^*/N(L_2/E))$.*

Proof. The inclusion $N(L/E) \subseteq N(L_i/E)$: $i = 1, 2$ follows at once from the transitivity of norm mappings in towers of field extensions of finite degrees (cf. [17, Ch. VIII, Sect. 5]). Conversely, let $c \in N(L_i/E)$ and $[L_i : E] = m_i$: $i = 1, 2$. As g.c.d.$(m_1, m_2) = 1$, this implies consecutively that $[L : L_2] = m_1$, $c^{m_1} \in N(L/E)$, $[L : L_1] = m_2$, $c^{m_2} \in N(L/E)$ and $c \in N(L/E)$, and so proves the equality $N(L/E) = N(L_1/E) \cap N(L_2/E)$. Since $E^*/N(L_i/E)$ is a group of exponent dividing m_i: $i = 1, 2$, it is also clear that $N(L_1/E)N(L_2/E) = E^*$. These observations prove the concluding assertion of the lemma. Our argument also shows that $N_{L_2}^{L}(\lambda_1) = N_E^{L_1}(\lambda_1)$: $\lambda_1 \in L_1$, whence $N(L_1/E) \subseteq$

$E^*\cap N(L/L_2)$. Considering now an element $s \in E^*\cap N(L/L_2)$, one obtains that $s^{m_2} \in N(L/E)$. This means that $s^{m_2} \in N(L_1/E)$, and since $s^{m_1} \in N(L_1/E)$, finally yields $s \in N(L_1/E)$, which completes the proof of Lemma 2.1.

LEMMA 2.2. *Let E be a field, M a finite Galois extension of E with a nilpotent Galois group $G(M/E)$, R an intermediate field of M/E not equal to E, $P(R/E)$ the set of prime numbers dividing $[R : E]$, M_p the maximal p-extension of E in M, and R_p the intersection $R \cap M_p$, for each $p \in P(R/E)$. Then the following is true:*

(i) *R is equal to the compositum of the fields $R_p : p \in P(R/E)$, and $[R : E] = \prod_{p\in P(R/E)}[R_p : E]$;*

(ii) *The norm group $N(R/E)$ equals the intersection $\cap_{p\in P(R/E)}N(R_p/E)$ and the quotient group $E^*/N(R/E)$ is isomorphic to the direct product of the groups $E^*/N(R_p/E)$: $p \in P(R/E)$.*

Proof. Statement (i) follows from Galois theory and the Burnside-Wielandt characterization of nilpotent finite groups (cf. [15, Ch. 6, Sect. 2]). Proceeding by induction on the number s of the elements of $P(R/E)$, and taking into account that if $s \geq 2$, then R_p and the compositum R'_p of the fields $R_{p'}$: $p' \in (P(R/E) \setminus \{p\})$, are of relatively prime degrees over E, one deduces Lemma 2.2 (ii) from Lemma 2.1.

It is clear from Lemma 2.2 that a field E admits local class field theory if and only if it is a field with local p-class field theory, for every $p \in P(E)$. The following lemma, proved in [8, Sect. 4], shows that the group $\mathrm{Br}(E)_p$ is necessarily nontrivial, if E admits local p-class field theory, for a given $p \in P(E)$.

LEMMA 2.3. *Let E be a field, such that $\mathrm{Br}(E)_p = \{0\}$, for some prime number p. Then $\mathrm{Br}(E_1)_p = \{0\}$ and $N(E_1/E) = E^*$, for every finite extension E_1 of E in $E(p)$.*

The main result of [8] used in the present paper can be stated as follows:

PROPOSITION 2.4. *Let E be a strictly p-quasilocal field, for some $p \in P(E)$. Assume also that R is an extension of E in $E(p)$, and D is a central division E-algebra of p-primary dimension. Then R is a p-quasilocal field and the following statements are true:*

(i) *D is a cyclic E-algebra and $\mathrm{ind}(D) = \exp(D)$;*

(ii) *$\mathrm{Br}(R)_p$ is a divisible group unless $p = 2$, $R = E$ and E is a formally real field; in the noted exceptional case, $\mathrm{Br}(E)_2$ is of order 2 and $E(2)/E$ is a* *quadratic extension;*

(iii) *The natural homomorphism of* $\mathrm{Br}(E)$ *into* $\mathrm{Br}(R)$ *maps* $\mathrm{Br}(E)_p$ *surjectively on* $\mathrm{Br}(R)_p$*; in particular, every* E*-automorphism of the field* R *is extendable to a ring automorphism on each central division* R*-algebra of* p*-primary dimension;*

(iv) R *is embeddable in* D *as an* E*-subalgebra if and only if the degree* $[R : E]$ *divides* $\mathrm{ind}(D)$*;* R *is a splitting field of* D *if and only if* $[R : E]$ *is infinite or divisible by* $\mathrm{ind}(D)$.

The place of strictly PQL-fields in one-dimensional local class field theory is determined by the following result of [7]:

Proposition 2.5. *Strictly* PQL*-fields admit local class field theory.*

Conversely, a field E *admitting local class field theory and satisfying the condition* $\mathrm{Br}(E) \neq \{0\}$ *is strictly* PQL*, provided that every central division* E*-algebra of prime exponent* p *is similar to a tensor product of cyclic division* E*-algebras of Schur index* p.

Remark 2.6. The question of whether central division algebras of prime exponent p over an arbitrary field E are necessarily similar to tensor products of cyclic E-algebras of index p is open. It is known that the answer is affirmative in each of the following special two special cases: (i) if E contains a primitive p-th root of unity or $p = \mathrm{char}(E)$ (cf. [19, (16.1)] and [1, Ch. VII, Theorem 30]); (ii) if E is an algebraic extension of a global field (cf. [2, Ch. 10, Corollary to Theorem 5]). Also, it has been proved in [7] that finite extensions of a field E admit local class field theory if and only if these extensions are strictly PQL-fields.

Our next result characterizes fields with local class field theory and with proper maximal abelian extensions, in the class of algebraic extensions of global fields:

Proposition 2.7. *Let* E_0 *be a global field,* $\overline{E}_0$ *an algebraic closure of* E_0*, and* E *an extension of* E_0 *in* $\overline{E}_0$*, such that* $P(E) \neq \phi$*. Then the following conditions are equivalent:*

(i) E *admits local class field theory;*

(ii) *For each* $p \in P(E)$*,* $\mathrm{Br}(E)_p \neq \{0\}$ *and there exists an absolute value* $v(p)$ *of* E*, such that the tensor product* $E(p) \otimes_E E_{v(p)}$ *is a field, where* $E_{v(p)}$ *is the completion of* E *with respect to the topology induced by* $v(p)$.

When these conditions are in force, the following statements are true, for every $p \in P(E)$*:*

The absolute value $v(p)$ is uniquely determined, up-to an equivalence, the natural homomorphism of $\mathrm{Br}(E)$ *into* $\mathrm{Br}(E_{v(p)})$ *maps* $\mathrm{Br}(E)_p$ *bijectively on* $\mathrm{Br}(E_{v(p)})_p$*, and $E(p) \otimes_E E_{v(p)}$ is isomorphic as an $E_{v(p)}$-algebra to the maximal p-extension $E_{v(p)}(p)$ of $E_{v(p)}$.*

Definition. *Let E be a strictly PQL-extension of a global field E_0, such that $P(E) \neq \phi$. By a characteristic system of E, we mean a system $V(E) = \{v(p) : p \in P(E)\}$ of absolute values of E, determined in accordance with Proposition 2.7 (ii).*

Note finally that if E is an algebraic extension of a global field E_0, R is a finite extension of E in E_{sep}, then the group $N_{loc}(R/E)$ of local norms of R/E consists of the elements of E^* lying in the norm groups $N(R_{v'}/E_v)$, whenever v is an absolute value of E, and v' is a prolongation of v on R. It has been proved in [9] that if E is a strictly PQL-field with $P(E) \neq \phi$, and R' is the normal closure of R in E_{sep} over E, then $N_{loc}(R/E) \subseteq N(R/E)$, and both groups are fully determined by the local behaviour of R/E at the subset of $V(E)$, indexed by the divisors of $[R' : E]$ in $P(E)$. In this paper, we shall need this result only in the special case where $R = R'$, i.e. R/E is a Galois extension.

Proposition 2.8. *Assume that E_0 is a global field, E is an algebraic strictly* PQL*-extension of E_0 with $P(E) \neq \phi$, and $V(E) = \{v(p) : p \in P(E)\}$ is a characteristic system of E. Also, let M be a finite Galois extension of E, and $P(M/E)$ the set of prime numbers dividing $[M : E]$. Then there exists a finite abelian extension $\widetilde{M}$ of E satisfying the following conditions:*

(i) *The norm groups $N(\widetilde{M}/E)$, $N(M/E)$ and $N_{loc}(M/E)$ are equal;*

(ii) *The degree $[\widetilde{M} : E]$ divides $[M : E]$; in particular, $\widetilde{M} = E$, provided that $E(p) = E$, $\forall p \in P(M/E)$;*

(iii) *For each prime number p dividing $[\widetilde{M} : E]$, the maximal p-extension $\widetilde{M}_p$ of E in $\widetilde{M}$ has the property that $\widetilde{M}_p \otimes_E E_{v(p)}$ is $E_{v(p)}$-isomorphic to the maximal abelian p-extension of $E_{v(p)}$ in the completion $M_{v(p)'}$, where $v(p)'$ is an absolute value of M extending $v(p)$.*

The field $\widetilde{M}$ is uniquely determined by M, up-to an E-isomorphism.

It is worth mentioning that if M is an algebraic extension of a global field E_0, and E is a subfield of M, such that $E_0 \subseteq E$ and M/E is a finite Galois extension, then $N(M/E) \subseteq N_{loc}(M/E)$. Identifying M with its E-isomorphic copy in $E_{v,sep}$, for a fixed absolute value v of E, one deduces this from the fact that the Galois groups $G(M_{v'}/E_v)$ and $G(M/(M \cap E_v))$ are canonically isomorphic, $N(M/E) \subseteq N(M/(M \cap E_v))$, and $N^{M}_{(M\cap E_v)}(\mu) = N^{M_{v'}}_{E_v}(\mu)$, in case $\mu \in M^*$ and v' is a prolongation of v on M. Moreover, it follows from Tate's description of $N_{loc}(M/E)/N(M/E)$ [3, Ch. VII, Sect. 11.4] (see also [23, Sect. 6.3]), in the special case where E_0 is an algebraic number field and $E = E_0$, that M/E can be chosen so that $N(M/E) \neq N_{loc}(M/E)$.

3. Norm groups of intermediate fields of finite Galois extensions with nilpotent Galois groups

The purpose of this Section is to prove Theorem 1.1. Clearly, our assertion can be deduced from Galois theory, Lemma 2.2 and the following result:

Theorem 3.1. *Let E be a p-quasilocal field, M/E a finite p-extension of E, and R an intermediate field of M/E. Then $N(R/E) = N(R_{\rm ab}/E)$.*

Proof. In view of Lemma 2.3 and Proposition 2.4 (ii), one may consider only the special case in which $p \in P(E)$ and $\mathrm{Br}(E)_p$ is an infinite group. Suppose first that R is a Galois extension of E. It follows from Galois theory that then the maximal abelian extension in R of any normal extension of E in R is itself normal over E and contains $R_{\rm ab}$ as a subfield. Since the intermediate fields of R/E are p-quasilocal fields, these observations show that it is sufficient to prove the equality $N(R/E) = N(R_{\rm ab}/E)$, under the hypothesis that $G(R/E)$ is a Miller-Moreno group, i.e. a nonabelian group with abelian proper subgroups. For convenience of the reader, we begin the consideration of this case with the following elementary lemma:

Lemma 3.2. *Assume that P is a Miller-Moreno p-group. Then the following is true:*

(i) *The commutator subgroup $[P, P]$ of P is of order p, the centre $Z(P)$ of P equals the Frattini subgroup $\Phi(P)$, and the group $P/\Phi(P)$ is elementary abelian of order p^2;*

(ii) *A subgroup H of P is normal in P if and only if $[P, P] \subseteq H$ or $H \subseteq \Phi(P)$;*

(iii) *The quotient group of P by its normal subgroup H_0 is cyclic if and only if H_0 is not included in $\Phi(P)$; in particular, this occurs in the special case of $H_0 = P_0.[P, P]$, where P_0 is a subgroup of P that is not is not normal in P;*

(iv) *If P is not isomorphic to the quaternion group of order* 8*, then it possesses a subgroup P_0 with the property required by (iii).*

Proof. It is well-known that $\Phi(P)$ is a normal subgroup of P including $[P,P]$, and such that $P/\Phi(P)$ is an elementary abelian p-group of rank $r \geq 1$; this implies the normality of the subgroups of P including $\Phi(P)$. Recall further that $r \geq 2$, since, otherwise, P must possess exactly one maximal subgroup, and therefore, must be nontrivial and cyclic, in contradiction with the noncommutativity of P. On the other hand, it follows from the noted properties of $\Phi(P)$ that if k is a natural number less than r and S is a subset of P with k elements, then the subgroup $P(S)$ of P generated by the union $\Phi(P) \cup S$ is of order dividing $|\Phi(P)|.p^k$; in particular, $P(S)$ is a proper subgroup of P. Since proper subgroups of P are abelian, these observations show that $\Phi(P) \subseteq Z(P)$ and $r = 2$. At the same time, the noncommutativity of P ensures that $P/Z(P)$ is a noncyclic group, whence it becomes clear that $\Phi(P) = Z(P)$. Let h be an element of $P \setminus \Phi(P)$. Then there exists an element $g \in P$, such that the system of co-sets $\{h\Phi(P), g\Phi(P)\}$ generates the group $P/\Phi(P)$. Using the fact that $[P,P] \subseteq \Phi(P) = Z(P)$ and $h^p \in Z(P)$, one obtains by direct calculations that each element of $[P,P]$ is a power of the commutator $h^{-1}g^{-1}hg := (h,g)$, and also, that $(h,g)^p = 1$. This completes the proof of Lemma 3.2 (i), and thereby, implies Lemma 3.2 (ii). The latter part of Lemma 3.2 (iii) follows from the former one and Lemma 3.2 (ii). As $\Phi(P)$ consists of all non-generators of P (cf. [15, Ch. 1, Theorem 2]), the systems $\{h, g\}$ and $\{h.[P,P], g.[P,P]\}$ generate the groups P and $P/[P,P]$, respectively. Therefore, the quotient group $P/[P,P]\langle h \rangle)$ is cyclic, which proves the former part of Lemma 3.2 (iii). The concluding assertion of the lemma can be obtained from the classification of Miller-Moreno p-groups [20] (cf. also [24, Theorem 444]), namely, the fact that if P is not isomorphic to the quaternion group of order 8, then it has one of the following presentations:

$P_1 = \langle g_1, h_1, z : g_1^{p^m} = h_1^{p^n} = z^p = 1, g_1z = zg_1, h_1z = zh_1, h_1g_1h_1^{-1} = g_1z \rangle$, $m \geq n \geq 1$ (P_1 is of order P^{m+n+1});

$P_2 = \langle g_2, h_2 : g_2^{p^m} = h_2^{p^n} = 1, h_2g_2h_2^{-1} = g_2^{1+p^{m-1}} \rangle$, $m \geq 2, n \geq 1, p^{m+n} > 8$ (P_2 is of order p^{m+n}).

Clearly, the subgroup of P_i generated by h_i is not normal, for any index $i \leq 2$ and any admissible pair (m, n), so Lemma 3.2 is proved.

We continue with the proof of Theorem 3.1 (under the hypothesis that $\mathrm{Br}(E)_p$ is infinite). By Proposition 2.4 (ii) and (iv), this means that $\mathrm{Br}(\Lambda)_p \neq$

0, for every finite extension Λ of E in $E(p)$. The proposition also ensures that Λ is strictly p-quasilocal, whence, by [7, Theorem 2.1], it admits local p-class field theory. Suppose first that $G(R/E)$ is not isomorphic to the quaternion group $\mathbb{Q}_8$. It follows from Galois theory and Lemma 3.2 that then the extension R/E possesses an intermediate field L for which the following is true:

(3.1) (i) $R = LR_{\rm ab}$ and L is not normal over E;

(ii) The intersection $L \cap R_{\rm ab} := F$ is a cyclic (proper) extension of E of degree $[L : E]/p$.

It is clear from (3.1) (ii) that L/F is a cyclic extension of degree p. Let σ and ψ be generators of the Galois groups $G(L/F)$ and $G(F/E)$, respectively. Fix an element ω of F^*, denote by Δ the cyclic F-algebra $(L/F, \sigma, \omega)$, and by $\bar{\psi}$ some embedding of L in $E(p)$ as an E-subalgebra, inducing ψ on F. By Proposition 2.4 (iii), ψ is extendable to an automorphism $\tilde{\psi}$ of Δ as an algebra over E. Observing also that $\bar{\psi}(L) \neq L$ and arguing as in the proof of [5, Lemma 3.2], one concludes that there exists an F-isomorphism $\Delta \cong (L/F, \sigma, \psi(\beta)\beta^{-1})$, for some $\beta \in F^*$. Hence, by [22, Sect. 15.1, Proposition b], $\omega\beta\psi(\beta)^{-1}$ is an element of the norm group $N(L/F)$. In view of [7, Lemma 3.2], this means that $\omega \in N(R_{\rm ab}/F)$ if and only if $\omega\beta\psi(\beta)^{-1} \in (N(L/F) \cap N(R_{\rm ab}/F))$. Since $F \neq E$ and $G(R/E)$ is a Miller-Moreno group, R/F is an abelian extension, so it follows from (3.1) (i) and the availability of a local p-class field theory on F that $N(R/F) = N(L/F) \cap N(R_{\rm ab}/F)$. As $N_E^F(\omega) = N_E^F(\omega\beta\psi(\beta)^{-1})$, our argument and the transitivity of norm mappings in towers of finite extensions prove that $N(R/E) = N(R_{\rm ab}/E)$.

Assume now that $p = 2$ and $G(R/E)$ is a quaternion group of order 8. It this case, by Galois theory, $R_{\rm ab}$ is presentable as a compositum of two different quadratic extensions E_1 and E_2 of E; one also sees that R/E_1 and R/E_2 are cyclic extensions of degree 4. Let ψ_1 be an E_1-automorphism of R of order 4, σ_1 an E-automorphism of E_1 of order 2, and γ an element of $R_{\rm ab}^*$ not lying in $N(R/R_{\rm ab})$. The field E_1 is 2-quasilocal, which implies the existence of a central division E_1-algebra D of index 4, such that $D \otimes_{E_1} R_{\rm ab}$ is similar to the cyclic $R_{\rm ab}$-algebra $(R/R_{\rm ab}, \psi_1^2, \gamma)$. Using again Proposition 2.4, one concludes that D is isomorphic to the cyclic E_1-algebra $(R/E_1, \psi_1, \rho)$, for some $\rho \in E_1^*$. Therefore, there exists an $R_{\rm ab}$-isomorphism $(R/R_{\rm ab}, \psi_1^2, \gamma) \cong (R/R_{\rm ab}, \psi_1^2, \rho)$, and by [22, Sect. 15.1, Proposition b], $\gamma\rho^{-1} \in N(R/R_{\rm ab})$. By Proposition 2.4 (iii), the normality of R over E, and the Skolem-Noether theorem (cf. [22, Sect. 12.6]), σ_1 is extendable to an automorphism $\tilde{\sigma}_1$ of

D as an algebra over E, such that $\tilde{\sigma}_1(R) = R$. In addition, our assumption on $G(R/E)$ indicates that $(\tilde{\sigma}_1\psi_1)(r) = (\psi_1^3\tilde{\sigma}_1)(r)$, for each $r \in R^*$. It is now easy to see that D is isomorphic to the cyclic E_1-algebras $(R/E_1, \psi_1^3, \sigma_1(\rho))$ and $(R/E_1, \psi_1^3, \rho^3)$. Hence, by [22, Sect. 15.1, Proposition b], $\rho^3\sigma_1(\rho)^{-1}$ lies in $N(R/E_1)$. Taking also into account that $\gamma\rho^{-1} \in N(R/R_{\rm ab})$ and $N_{E_1}^{R_{\rm ab}}(\gamma\rho^{-1}) = \rho^{-2}N_{E_1}^{R_{\rm ab}}(\gamma)$, one concludes that $N_{E_1}^{R_{\rm ab}}(\gamma).(\rho\sigma_1(\rho)^{-1}) \in N(R/E_1)$. This result shows that $N_E^{R_{\rm ab}}(\gamma) \in N(R/E)$, which completes the proof of Theorem 3.1 in the special case where R/E is a Galois extension.

Suppose finally that R is an arbitrary extension of E in $E(p)$ of degree p^m, for some $m \in \mathbb{N}$, and denote by R_0 the maximal normal extension E in R. Proceeding by induction on m and taking into account that R_0 is strictly p-quasilocal and $R_0 \neq E$, one obtains that now it suffices to prove Theorem 3.1 in the special case where $R \neq R_0$ and R is abelian over R_0, and assuming that the conclusion of the theorem is valid for each intermediate field of R/E not equal to R. Our inductive hypothesis indicates that then there exists an embedding ψ of R in $E(p)$ as an E-subalgebra, such that $\psi(R) \neq R$. It is easily verified that $R_0 = \psi(R_0)$ and $\psi(R)/R_0$ is an abelian extension; hence, R and $\psi(R)$ are abelian extensions of the intersection $R \cap \psi(R) := R_1$. Observing also that R_1 is a p-quasilocal field, one gets from [7, Theorem 2.1] that $R_1^* = N(R/R_1)N(\psi(R)/R_1)$. This, combined with the transitivity of norm mappings in towers of finite extensions, and with the fact that $\psi(N_{R_0}^R(\lambda)) = N_{R_0}^{\psi(R)}(\psi(\lambda))$, for each $\lambda \in R^*$, implies that $N(R/E) = N(R_1/E)$. Since $R_1 \neq R, R_{\rm ab} \subseteq R_0 \subseteq R_1$, the proof of Theorem 3.1 can be accomplished by applying the obtained result and the inductive hypothesis.

The concluding result of this Section is of interest because it is not known whether Henselian discrete valued fields with local class field theory are strictly PQL.

Corollary 3.3. *Let (E, v) be a Henselian discrete valued field with local class field theory, M/E a finite Galois extension with a nilpotent Galois group, and R an intermediate field of M/E. Then $N(R/E) = N(R_{\rm ab}/E)$.*

Proof. It suffices to consider the special case of a proper p-extension M/E. Then $p \in P(E)$, and by [6, Theorem 2.1], $\widehat{E}(p)/\widehat{E}$ is a $\mathbb{Z}_p$-extension, where $\widehat{E}$ is the residue field of (E, v). If $p \neq \mathrm{char}(\widehat{E})$ and $\widehat{E}$ does not contain a primitive p-th root of unity, this means that $E(p)/E$ is a $\mathbb{Z}_p$-extension with finite subextensions inertial over E (see, for example, [4, Lemma 1.1]), so our assertion becomes trivial. In view of Theorem 3.1, it remains to be seen that E is p-quasilocal in the following special cases: (i) $\widehat{E}$ contains a primitive p-th root of unity; (ii) $\mathrm{char}(\widehat{E}) = p$. If $\mathrm{char}(\widehat{E}) = p$ and $\mathrm{char}(E) = 0$, this result has been obtained in [6, Sect. 2], and if $\mathrm{char}(E) = p$, it is contained in [7, Proposition 2.4]. The Henselian property of v implies in case (i) the existence of a primitive p-th root of unity in E, (ii) which reduces our assertion to a special case of [7, Proposition 2.4].

4. On nonnilpotent finite Galois extensions of strictly PQL-fields algebraic over $\mathbb{Q}$

Our objective in this Section is to prove Theorem 1.2 by applying the general properties of algebraic strictly PQL-extensions of global fields established in [9]. Proposition 2.7 and [11, Sect. 2, Theorem 4] indicate that if E is an algebraic extension of a global field E_0, F is an intermediate field of E/E_0, and for each $p \in P(E)$, $w(p)$ is the absolute value of F induced by $v(p)$, then the groups $\mathrm{Br}(F)_p$ and $\mathrm{Br}(F_{w(p)})_p$ are nontrivial. Therefore, our research concentrates as in [9] on the study of the following class of fields:

Definition 4.1. *Let E_0 be a global field, $\overline{E}_0$ an algebraic closure of E_0, F an extension of E_0 in $\overline{E}_0$, P a nonempty set of prime numbers for which $Br(F)_p \neq \{0\}$, and $\{w(p) : p \in P\}$ a system of absolute values of F, such that $Br(F_{w(p)})_p \neq \{0\}$, $p \in P$. Denote by $\Omega(F, P, W)$ the set of intermediate fields E of $\overline{E}/F$ with the following properties:*

(i) E admits local class field theory and $P(E) = P$;

(ii) The characteristic system $\{v(p) : p \in P\}$ of E can be chosen so that $v(p)$ is a prolongation of $w(p)$, for each $p \in P$.

The existence of the system $\{w(p) : p \in P\}$ appearing in Definition 4.1 follows from global class field theory (cf. [30, Ch. XIII, Sect. 3] and [9, Proposition 1.2]), and some of the main results of [9] about the set $\Omega(F, P, W)$ can be stated as follows:

Proposition 4.2. *With assumptions and notations being as above, $\Omega(F, P, W)$ is a nonempty set, for which the following assertions hold true:*

(i) *Every field $E \in \Omega(P, W; F)$ possesses a unique subfield $R(E)$ that is a minimal element of $\Omega(P, W; F)$ (with respect to inclusion);*

(ii) *If* E *is a minimal element of* $\Omega(F, P, W)$, $p \in P$ *and* $F_{w(p)}$ *is the closure of* F *in* $E_{v(p)}$, *then the degrees of the finite extensions of* $F_{w(p)}$ *in* $E_{v(p)}$ *are not divisible by* p.

Proposition 4.2 plays a crucial role in the proof of the following precise form of Theorem 1.2.

PROPOSITION 4.3. *Let* G *be a nonnilpotent finite group,* $\overline{P}$ *the set of all prime numbers,* $w(p)$ *the normalized* p*-adic absolute value of the field* $\mathbb{Q}$ *of rational numbers, for each* $p \in \overline{P}$, *and* $W = \{w(p) : p \in \overline{P}\}$. *Then there is a field* $E \in \Omega(\mathbb{Q}, \overline{P}, W)$ *possessing a Galois extension* M *in* $\overline{\mathbb{Q}}$, *such that* $G(M/E)$ *is isomorphic to* G *and* $N(M/E) \neq N(M_{\mathrm{ab}}/E)$.

Proof. Our argument relies on several observations described by the following four lemmas.

LEMMA 4.4. *Let* E *be an algebraic strictly* PQL-*extension of* $\mathbb{Q}$ *with* $P(E) \neq \phi$, *and let* $\{v(p) : p \in P(E)\}$ *be a characteristic system of* E. *Assume also that* M/E *is a finite Galois extension such that* $G(M/E)$ *is nonnilpotent, each prime* p *dividing* $[M : E]$ *lies in* $P(E)$ *and* $M_{v(p)'}/E_{v(p)}$ *is a* p*-extension with* $G(M_{v(p)'}/E_{v(p)})$ *isomorphic to the Sylow* p*-subgroups of* $G(M/E)$, *where* $v(p)'$ *is an arbitrary absolute value of* M *extending* $v(p)$. *Then* $N(M/E) \neq N(M_{\mathrm{ab}}/E)$.

Proof. By the Burnside-Wielandt theorem, the assumption that G is nonnilpotent means that it possesses a maximal subgroup H that is not normal. Let p be a prime number dividing the index $|G : H|$, A_p the maximal p-extension of E in M, H_p a Sylow p-subgroup of H, G_p a Sylow p-subgroup of $G(M/E)$ including H_p, and K, K_p and M_p the intermediate fields of M/E corresponding by Galois theory to H, H_p and G_p, respectively. It follows from Galois theory and the normality of maximal subgroups of finite p-groups that $A_p \cap K = E$, which indicates that $(A_pK)/K$ is a p-extension with $G((A_pK)/K)$ isomorphic to $G(A_p/E)$. The extensions $(A_pK_p)/K_p$ and $(A_pM_p)/M_p$ have the same property, since the choice of K, K_p and M_p guarantees that the degrees $[K_p : K]$ and $[M_p : E]$ are not divisible by p. This implies that $[(A_pK_p) : K_p] = [(A_pM_p) : M_p] = [A_p : E]$ and $[(A_pK_p) : M_p] = [(A_pM_p) : M_p].[K_p : M_p]$. Thus it turns out that $A_pM_p \cap K_p = M_p$, which means that G_p is of greater rank as a p-group than $G(A_p/E)$, and because of Proposition 2.8, proves Lemma 4.4.

LEMMA 4.5. *Let* F *be an algebraic number field,* $\overline{P}$ *the set of all prime* *numbers, $W = \{w(p) : p \in \overline{P}\}$ a system of absolute values of F fixed as in Definition 4.1, M_0/F a finite Galois extension, and for each prime p dividing $[M_0 : F]$, let $M_{0,w(p)'}/F_{w(p)}$ be a normal extension with a Galois group isomorphic to the Sylow p-subgroups of $G(M_0/F)$, where $w(p)'$ is an arbitrary prolongation of $w(p)$ on M_0. Assume also that E is a minimal element of $\Omega(F, \overline{P}, W)$, $V(E) = \{v(p) : p \in \overline{P}\}$ is a characteristic system of of E, and $M = M_0E$. Then M/E is a Galois extension satisfying the conditions of Lemma 4.4, and the Galois groups $G(M/E)$ and $G(M_0/F)$ are canonically isomorphic.*

Proof. Denote by $P(M_0/F)$ the set of prime divisors of $[M_0 : F]$. The minimality of E and Proposition 4.2 (ii) imply that $M_{0,w(p)'} \otimes_{F_{w(p)}} E_{v(p)}$ is a field isomorphic to $M_{v(p)'}$ over $E_{v(p)}$, where $v(p)'$ is a prolongation of $v(p)$ on M_0, for each $p \in P(M_0/F)$. This, combined with the fact that the groups $G(M/E)$ and $G(M_{v(p)'}/E_{v(p)})$ embed in $G(M_0/F)$ and $G(M/E)$, respectively, and also, with the condition on the extension $M_{0,w(p)'}/F_{w(p)}$, proves that $G(M/E)$ is isomorphic to $G(M_0/F)$.

Lemma 4.6. *Let M/E be a Galois extension with a Galois group G embeddable in the symmetric group S_n, for some $n \in \mathbb{N}$. Then there exists a polynomial $f(X) \in E[X]$ of degree n with a root field (over E) equal to M.*

Proof. Denote by s the number of G-orbits of the set $\{1, ..., n\}$, fix a system $\{g_j : j = 1, ..., s\}$ of representatives of these orbits, and for each index j, let U_j be the intermediate field of M/E corresponding by Galois theory to the stabilizer $\mathrm{Stab}_G(g_j) := G_j$. It is easily verified that $[U_j : E] = |G : G_j|$, $\sum_{j=1}^{s} |G : G_j| = n$, and $\cap_{j=1}^{s} V_j = \{1\}$, where V_j is the intersection of the subgroups of G conjugate to G_j, for each $j \in \{1, ..., s\}$. Therefore, one can take as $f(X)$ the product $\prod_{j=1}^{s} f_j(X)$, where $f_j(X)$ is the minimal polynomial over E of any primitive element of U_j over E, for any j.

Lemma 4.7. *Let E_0 be an algebraic number field, n an integer number greater than one, and P_n the set of prime numbers $\leq n$. Assume that E_0 possesses a system $\{w(p) : p \in P_n\}$ of pairwise nonequivalent absolute values, such that the completion $E_{0,w(p)}$ admits a Galois extension $\widetilde{M}_p$ with $G(\widetilde{M}_p/E_{0,w(p)})$ isomorphic to the Sylow p-subgroups of the symmetric group S_n, for any $p \in P_n$. Then there exists a Galois extension M_0 of E_0 with $G(M_0/E_0)$ isomorphic to S_n, and such that the completion $M_{0,w(p)'}$ is $E_{0,w(p)}$-isomorphic to $\widetilde{M}_p$, for each $p \in P_n$, and any prolongation $w(p)'$ of $w(p)$ on M.*

Proof. It follows from Lemma 4.6 and the assumptions of the present lemma that $\widetilde{M}_p$ is a root field over $E_{0,w(p)}$ of a separable polynomial $f_p(X) = X^n + \sum_{j=1}^{n} c_{p,j}X^{n-j} \in E_{0,w(p)}[X]$, for any $p \in P_n$. Since the absolute values $w(p) : p \in P_n$ are pairwise nonequivalent, the weak approximation theorem (cf. [17, Ch. XII, Sect. 1]) and the density of E_0 in $E_{0,w(p)}$ ensure, for each real positive number ε, the existence of a polynomial $g_\varepsilon(X) \in E_0[X]$ equal to $X^n + \sum_{j=0}^{n-1} b_{\varepsilon,j}X^{n-j}$, and such that $w(p)(b_{\varepsilon,j} - c_{p,j}) < \varepsilon$, for every $p \in P_n$. This enables one to deduce from Krasner's lemma (cf. [18, Ch. II, Proposition 3]) that if ε is sufficiently small, then the quotient rings $E_{0,w(p)}[X]/f_p(X)E_{0,w(p)}[X]$ and $E_{0,w(p)}[X]/g_\varepsilon(X)E_{0,w(p)}[X]$ are isomorphic as $E_{0,w(p)}$-algebras, which implies that $\widetilde{M}_p$ is a root field of $g_\varepsilon(X)$ over $E_{0,w(p)}$, for each $p \in P_n$. When this occurs, it becomes clear from Galois theory and the obtained result that the root field of $g_\varepsilon(X)$ over E_0 is a normal extension of E_0 with a Galois group G_ε of order divisible by $n!$ As G_ε obviously embeds in S_n, this means that $G_\varepsilon \cong S_n$, so Lemma 4.7 is proved.

We are now in a position to prove Proposition 4.3. Retaining assumptions and notations in accordance with Lemma 4.5, note that every intermediate field K of M_0/E_0 possesses a system $\{\nu(p) : p \in P(M_0/E_0)\}$ of absolute values, such that $\nu(p)$ is a prolongation of $w(p)$ and $M_{\nu(p)'}/K_{\nu(p)}$ is a normal extension with a Galois group isomorphic to the Sylow p-subgroups of $G(M_0/K)$, for each $p \in P(M_0/K)$. To show this, take a prime $p \in P(M_0/K)$, fix a Sylow p-subgroup P of $G(M_0/K)$ as well as a Sylow p-subgroup P_0 of $G(M_0/E_0)$ including P, and denote by F_0 and F the extensions of E_0 in M_0 corresponding by Galois theory to P_0 and P, respectively. The local behaviour of M_0/E_0 at $w(p)'/w(p)$ implies the existence of a prolongation $\tau_0(p)$ of $w(p)$ on F_0, such that $F_{0,\tau_0(p)}$ is a completion of E_0 with respect to $w(p)$; moreover, it becomes clear that $\tau_0(p)$ is uniquely extendable to an absolute value $\tau(p)$ of M_0 (cf. [3, Ch. II, Theorem 10.2]). Observing that $[F : F_0] = |P_0 : P|$ and p does not divide $[F : K]$, one concludes that the absolute value $\nu(p)$ of K induced by $\tau(p)$ has the required property. Since finite groups of order n are embeddable in the symmetric group S_n, for each $n \in \mathbb{N}$ (Cayley's theorem), the obtained result and the previous three lemmas indicate that Proposition 4.3 will be proved, if we show the existence of an algebraic number field E_0 satisfying the conditions of Lemma 4.7.

Fix a natural number $n > 1$ as well as an odd integer $m > n!$, suppose that P_n is defined as in Lemma 4.7, put $\tilde{n} = \prod_{p \in P_n} p$, and denote by Φ_0 the extension of $\mathbb{Q}$ in $\overline{\mathbb{Q}}$ obtained by adjoining a root of the polynomial $X^m - \tilde{n}$. Also, let $\Gamma_s = \mathbb{Q}(\delta_s + \delta_s^{-1})$, $\Phi_s = \Phi_0(\delta_s + \delta_s^{-1})$, where δ_s is a primitive 2^s-th root of unity in $\overline{\mathbb{Q}}$, for any $s \in \mathbb{N}$, and $\Phi_\infty = \cup_{s=1}^{\infty} \Phi_s$. The choice of Φ_0 indicates that the p-adic absolute value of $\mathbb{Q}$ is uniquely extendable to an absolute value $w_0(p)$ of Φ_0, for each $p \in P_n$; one obtains similarly that the 2-adic absolute value of $\mathbb{Q}$ has a unique prolongation $w_s(2)$ on Φ_s, for every $s \in \mathbb{N}$ (cf. [3, Ch. I, Theorem 6.1]), and also, a unique prolongation w_∞ on Φ_∞. Furthermore, our argument proves that $\Phi_{s,w_s(2)}/\mathbb{Q}_2 : s \in \mathbb{N}$ and $\Phi_{0,w_0(p)}/\mathbb{Q}_p : p \in P_n, p > 2$, are totally ramified extensions of degrees $2^s.m$ and m, respectively. We first show that $\Phi_{0,w_0(p)}$ admits a Galois extension with a Galois group isomorphic to the Sylow p-subgroups of S_n, provided that $p \in (P_n \setminus \{2\})$. Note that $\Phi_{0,w_0(p)}$ does not contain a primitive p-th root of unity. This follows from the fact that m is odd whereas $p-1$ equals the degree of the extension of $\mathbb{Q}_p$ obtained by adjoining a primitive p-th root of unity (cf. [12, Ch. IV, (1.3)]). Hence, by the Shafarevich theorem [28] (cf. also [27, Ch. II, Theorem 3]), the Galois group of the maximal p-extension of $\Phi_{0,w_0(p)}$ is a free pro-p-group of rank $m+1$. In view of Galois theory, this means that a finite p-group is realizable as a Galois group of a p-extension of $\Phi_{0,w_0(p)}$ if and only if it is of rank at most equal to $m+1$. The obtained result, combined with the fact that $m > n!$ and the ranks of the p-subgroups of S_n are less than $n!$, proves our assertion. Taking now into consideration that Φ_s/Φ_0 is a cyclic extension of degree 2^s, one obtains by applying [3, Ch. II, Theorem 10.2] and [17, Ch. IX, Proposition 11] that $\Phi_{s,w_s(p)}$ is a cyclic extension of $\Phi_{0,w_0(p)}$ of degree dividing 2^s, for each absolute value $w_s(p)$ of Φ_s extending $w_0(p)$. It is therefore clear from Galois theory that Proposition 4.3 will be proved, if we show that $\Phi_{s,w_s(2)}$ admits a normal extension with a Galois group isomorphic to the Sylow 2-subgroups of S_n, for every sufficiently large index s. Identifying $\Phi_{0,w_0(2)}$ with the closure of Φ_0 in Φ_{∞,w_∞}, one obtains from the uniqueness of the prolongation $w_\infty/w_0(2)$ that $\Phi_\infty \cap \Phi_{0,w_0(2)} = \Phi_0$ and the compositum $\Phi_\infty\Phi_{0,w_0(2)} := \Phi_{\infty,2}$ is a $\mathbb{Z}_2$-extension of $\Phi_{0,w_0(2)}$. This implies that $\mathrm{Br}(\Phi_{\infty,2})_2 = \{0\}$ and $\Phi_{\infty,2}(2) \neq \Phi_{\infty,2}$, which means that $G(\Phi_{\infty,2}(2)/\Phi_{\infty,2})$ is a free pro-2-group of countably infinite rank (cf. [27, Ch. II, 5.6, Theorem 4 and Lemma 3] and [29, p. 725]). Hence, finite 2-groups are realizable as Galois groups of normal extensions of $\Phi_{\infty,2}$. In particular, there exists a 2-extension $T_{\infty,2}$ of $\Phi_{\infty,2}$ with $G(T_{\infty,2}/\Phi_{\infty,2})$ isomorphic to the Sylow 2-subgroups of S_n, so it follows from [8, (1.3)] that one can find an index $\tilde{s}$ and a Galois extension $T_{\tilde{s},2}$ of $\Phi_{\tilde{s},w_{\tilde{s}}(2)}$ in $T_{\infty,2}$, such that $T_{\tilde{s},2} \otimes_{\Phi_{\tilde{s},w_{\tilde{s}}(2)}} \Phi_{\infty,2}$ is isomorphic to $T_{\infty,2}$ as an algebra over $\Phi_{\tilde{s},w_{\tilde{s}}(2)}$. In view of the general properties of tensor products (cf. [22, Sect. 9.4, Corollary a]), this implies that if s is an integer $\geq \tilde{s}$, then the $\Phi_{s,w_s(2)}$-algebra $T_{\tilde{s},2} \otimes_{\Phi_{\tilde{s},w_{\tilde{s}}(2)}} \Phi_{s,w_s(2)} := T_{s,2}$ is a field, and more precisely, a Galois extension of $\Phi_{s,w_s(2)}$ with $G(T_{s,2}/\Phi_{s,w_s(2)})$ isomorphic to the Sylow 2-subgroups of S_n. Furthermore, in this case, the field $\Phi_s := E_0$ and its absolute values $w_s(p)$, $p \in P_n$, satisfy the conditions of Lemma 4.7, which completes the proof of Proposition 4.3 (and Theorem 1.2).

It would be of interest to know whether every nonnilpotent finite group G is isomorphic to the Galois group of a finite Galois extension $M(G)/E(G)$, for a suitably chosen strictly PQL-field $E(G)$. Proposition 2.7 of [9] gives an affirmative answer to this question (with $E(G)$ algebraic over $\mathbb{Q}$ and $P(E(G))$ equal to the set of prime numbers), in case the Sylow p-subgroups of G are abelian, for every prime p dividing the order $o(G)$ of G. Let now $P(G)$ be the set of prime divisors of $o(G)$, and for each $p \in P(G)$, let N_p be the minimal normal subgroup of G of p-primary index, n_p the rank of G/N_p as a p-group, G_p a Sylow p-subgroup of G, and p^{m_p} the exponent of G_p. Applying Lemma 4.5 and modifying the proof of [9, Proposition 2.7], one obtains the same answer, if some of the following two conditions is in force, for each $p \in P(G)$: (i) the natural short exact sequence $1 \to N_p \to G \to G/N_p \to 1$ splits; (ii) the quotient group of G/N_p by its commutator subgroup is of order $p^{n_p m_p}$. In particular, this occurs, if G is a symmetric group of degree ≥ 3, the exponent of G is a square-free number, or G equals its commutator subgroup.

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