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ON SOME P-CONVEX SEQUENCES

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Abstract. The aim of this paper is to give some properties of the p-convex sequences.

Let K be the set of all real sequences, K_+ the set of all real positive sequence and $p \in \mathbf{R} \setminus \{0\}$. We define a linear operator $\Delta_p^m : K \to K$, $m \in \mathbf{N}^*$

$$\Delta_p^1 = \Delta_p a_n := a_{n+1} - p a_n, \tag{1}$$

$$\Delta_p^{m+1} a_n = \Delta_p(\Delta_p^m a_n)\ , \quad \text{for every} \ \ n \in \mathbf{N}.$$

Definition 1. *A sequence $(a_n)_{n \in \mathbf{N}}$ from K is said to be p-convex of order $m \in \mathbf{N}^*$ if and only if*

$$\Delta_p^m a_n \geq 0\ , \quad \textit{for all } n \in \mathbf{N}. \tag{2}$$

We denote by K_m^p the set of all real sequences p-convex.

Proposition 1. *For $n \in \mathbf{N}$, $m \in \mathbf{N}^*$ and $p \in \mathbf{R} \setminus \{0\}$ the equalities*

$$\Delta_p^{m+1} a_n = \Delta_p^m a_{n+1} - p \Delta_p^m a_n \tag{3}$$

$$\Delta_p^m a_n = \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} p^{m-k} a_{n+k} \tag{4}$$

holds.

Remark 1. *For $p = 1$, we obtain $\Delta_1^m a_n = \Delta^m a_n$, where $\Delta^m a_n$ represents the m-th order difference of the sequence $(a_n)_{n \in \mathbf{N}}$.*

Example 1. For $p \neq 1$, a sequences $(a_n)_{n \in \mathbf{N}}$, where $a_n = p^n$, is p-convex the m-th order, but is not a 1-convex the m-th order for every $m \in \mathbf{N}^*$.

Example 2. Let $(a_n)_{n \in \mathbf{N}}$ from K, $a_n = n^r p^n$, $r \in \{0, 1, ..., m-1\}$ where $m\mathbf{Z}$, $m \geq 2$, $\Delta_p^m a_n = 0$.

We have $A = \{\pm p^n, \pm n p^n, \pm n^2 p^n, ..., \pm n^{m-1} p^n\} \subset K_m^p$.

Proposition 2. *We have*

$$\Delta_p^{m+r} a_n = \Delta_p^m(\Delta_p^r a_n) = \Delta_p^r(\Delta_p^m a_n), \tag{5}$$

for every $m, r \in \mathbf{N}^*$, $n \in \mathbf{N}$, $p \in \mathbf{R}$.

Theorem 1. *For* $n, m \in \mathbf{N}$, *the equality*

$$a_{n+m} = \sum_{k=0}^{m} \binom{m}{k} p^{m-k} \Delta_p^k a_n, \tag{6}$$

is verified.

Proof:

$$\begin{aligned} a_{n+m} &= \sum_{k=0}^{m} \binom{m}{k} p^{m-k} \left[\sum_{i=0}^{k} (-1)^{k-i} \binom{k}{i} p^{k-i} a_{n+i} \right] = \\ &= \sum_{k=0}^{m} b(m,k) \sum_{i=0}^{k} c(k,i) a_{n+i} \end{aligned}$$

where

$$b(m,k) = \binom{m}{k} p^{m-k}, \qquad c(k,i) = (-1)^{k-i} \binom{k}{i} p^{k-i}$$

$$\begin{aligned} a_{n+m} &= \sum_{k=0}^{m} a_{n+k} \sum_{i=k}^{m} b(m,i) c(i,k) = \\ &= \sum_{k=0}^{m} a_{n+k} \sum_{i=0}^{m-k} b(m,i+k) c(i+k,k) = \\ &= \sum_{k=0}^{m} a_{n+k} p^{m-k} \sum_{i=0}^{m-k} (-1)^i \binom{m}{k} \binom{m-k}{i} = a_{n+m}. \end{aligned}$$

Theorem 2. *We have*

$$\Delta_p^m a_n b_n = \sum_{k=0}^{m} \binom{m}{k} \Delta_1^k a_n \Delta_p^{m-k} b_{n+k}, \tag{7}$$

for all $n \in \mathbf{N}$ *and* $m \in \mathbf{N}$.

Proof: We proceed by mathematical introduction.

Remark 2. *For $p = 1$, we obtain the results of T. Popoviciu (see [3]).*

We consider a linear operator $T : K_m^p \to K$ defined by

$$T(a;n)=\sum_{k=0}^{n} \rho_k(n) a_{s+k}, \text{ where } s \in \mathbf{N} \text{ is arbitrary, } \rho_k(n) \in \mathbf{R},\ n \in \mathbf{N},\ k = \overline{0,n}. \tag{8}$$

Next, we will give some necesary and sufficienty condition for a matrix $\rho = \|\rho_k(n)\|_{\substack{n\in\mathbf{N} \\ k=\overline{0,n}}}$ to verify

$$T(K_m^p) \subseteq K_+. \tag{9}$$

Lemma 1. *Let $m, s \in \mathbf{N}$, $m \geq 2$, $\rho = \|\rho_k(n)\|_{\substack{n\in\mathbf{N} \\ k=\overline{0,n}}}$. If*

$$\sum_{k=0}^{n} p^k \rho_k(n) = 0 \ , \quad \sum_{k=0}^{n} p^k \rho_k(n) k^i = 0 \ , \quad i = 1, 2, ..., m-1 \tag{10}$$

then

$$T(a;n) = \sum_{k=0}^{n-m} q_k(n) \Delta_p^m a_{s+k} \tag{11}$$

where $q_k(n) = \dfrac{(-1)^m}{(m-1)!} \displaystyle\sum_{i=0}^{k} \rho_i(n) p^{i-k-m} (k-i+1)_{m-1}$, $(x)_l = x(x+1)...(x+l-1)$.

Proof: We proceed by mathematical induction.

Theorem 3. *Let $T : K \to K$, $T(a;n) = \sum_{k=0}^{n} \rho_k(n) a_{s+k}$, $s \in \mathbf{N}$. $T(K_m^p) \subseteq K_+$ if and only if*

i) $\displaystyle\sum_{k=0}^{n} p^k \rho_k(n) = 0$, $\displaystyle\sum_{k=0}^{n} p^k \rho_k(n) k^i = 0$, *for* $i = 1, 2, ..., m-1$

ii) $\displaystyle\sum_{i=0}^{k} \rho_i(n)(i-(k+1))(i-(k+2))...(i-(k+m-1)) p^{i-k-m} \leq 0$,
$k = \overline{0, n-m}$.

Proof: Necessity: We consider $T(K_m^p) \subseteq K_+$. In example 2 it is shown that

$$A = \{\pm p^n, \pm np^n, ..., \pm n^{m-1}p^n\} \subseteq K_m^p.$$

For $(a_n)_{n\in\mathbf{N}}$ from A we obtain condition i) and ii).

Sufficienty: From lemma **??** we have

$$T(a;n) = \sum_{k=0}^{n-m} q_k(n)\Delta_p^m a_{s+k},$$

where $q_k = -\frac{1}{(m-1)!}\sum_{i=0}^{k}\rho_i(n)(i-(k+1))...(i-(k+m-1))p^{i-k-m}$.

From condition $q_k \geq 0$, $k = 0,1,...,n-m$ and $(a_n)_{n\in\mathbf{N}} \in K_m^p$ it follows $T(K_m^p) \subseteq K_+$.

For $p = 1$, we obtain the results of T. Popoviciu (see [4]).

Next, let $(A_n(a))_{n\in\mathbf{N}}$ be the sequence of the means, that is $A_n(a) = \frac{1}{n+1}\sum_{k=0}^{n} a_k$, $n \in \mathbf{N}$.

THEOREM 4. *Let $p < 1$. If $A_n(a) \in K_{m-1}^p$, $m \geq 1$, then operator A_n : $K_m^p \to K$, $A_n(a) = \frac{1}{n+1}\sum_{k=0}^{n} a_k$ verify*

$$A_n(K_m^p) \subseteq K_m^p. \tag{12}$$

Proof: $a_k = (k+1)A_k - kA_{k-1}$, $k = 0,1,....$ Then

$$\Delta_p^m a_k = \Delta_p^m(k+1)A_k - \Delta_p^m kA_{k-1}.$$

From (**??**) we have

$$\begin{aligned}
\Delta_p^m a_k &= \sum_{i=0}^{m}\binom{m}{i}\Delta_1^i(k+1)(\Delta_p^{m-i}A_{k+i}) - \sum_{i=0}^{m}\binom{m}{i}\Delta_1^i k(\Delta_p^{m-i}A_{k+i-1}) = \\
&= (k+1)\Delta_p^m A_k + m\Delta_p^{m-1}A_{k+1} - k\Delta_p^m A_{k-1} - m\Delta_p^{m-1}A_k \\
\Delta_p^m A_k &= \Delta_p^{m-1}(\Delta_p^1 A_k) = \Delta_p^{m-1}(A_{k+1} - pA_k) = \Delta_p^{m-1}A_{k+1} - p\Delta_p^{m-1}A_k \\
\Delta_p^{m-1}A_{k+1} &= \Delta_p^m A_k + p\Delta_p^{m-1}A_k.
\end{aligned}$$

After that

$$\Delta_p^m a_k = (m+k+1)\Delta_p^m A_k - k\Delta_p^m A_{k-1} + m(p-1)\Delta_p^{m-1} A_k. \tag{13}$$

$$\begin{aligned} \frac{(m+k)!}{k!}\Delta_p^m a_k &= \frac{(m+k+1)!}{k!}\Delta_p^m A_k - \frac{(m+k)!}{(k-1)!}\Delta_p^m A_{k-1} + \\ &+ m(p-1)\frac{(m+k)!}{k!}\Delta_p^{m-1} A_k. \\ \sum_{k=1}^{n}\frac{(m+k)!}{k!}\Delta_p^m a_k &= \frac{(m+n+1)!}{n!}\Delta_p^m A_n - (m+1)!\Delta_p^m A_0 + \\ &+ m(p-1)\sum_{k=1}^{n}\frac{(m+k)!}{k!}\Delta_p^{m-1} A_k\ , \quad n \geq 1 \end{aligned}$$

$$\begin{aligned} \Delta_p^m &= \frac{n!}{(m+n+1)!}\left[\sum_{k=1}^{n}\frac{(m+k)!}{k!}\Delta_p^m a_k + \right. \\ &\left. + (m+1)!\Delta_p^m a_0 + m(1-p)\sum_{k=1}^{n}\frac{(m+k)!}{k!}\Delta_p^{m-1} A_k\right]. \end{aligned} \tag{14}$$

In (**??**) let $k=0$, we obtain

$$\Delta_p^m a_0 = (m+1)\Delta_p^m A_0 + m(p-1)\Delta_p^{m-1} A_0$$

$$\Delta_p^m A_0 = \frac{1}{m+1}\left[\Delta_p^m a_0 + m(1-p)\Delta_p^{m-1} A_0\right]. \tag{15}$$

From (**??**) and (**??**) we obtain (**??**).

Remark 3. *If we consider $p=1$ in (**??**), we obtain the result of A. Lupaş (see [2]).*

We consider following question: What are the conditions for a sequences $(a_n)_{n\in\mathbf{N}}$ from K to verify

$$A \leq \Delta_p^m a_n \leq B\ , \quad \text{for all } \ n\in\mathbf{N}\ , \ \ m\in\mathbf{N}^*, \ \text{ arbitrary} \tag{16}$$

where A and B are constants that are not dependent by m and n.

Theorem 5. *The sequence $(a_n)_{n\in\mathbf{N}}$ satisfied the condition (**??**) if and only if exists the real sequence $(b_n)_{n\in\mathbf{N}}$ which verify*

$$A \le b_n \le B \ , \quad \textit{for every} \ \ n \ge m \tag{17}$$

and

$$a_n = \sum_{k=0}^{n} b_k \frac{(n-k+1)_{m-1}}{(m-1)!} p^{n-k} \ , \quad \textit{where} \ \ (x)_l = x(x+1)...(x+l-1). \tag{18}$$

Proof: Sufficienty:

$$\begin{aligned}
\Delta_p^m a_n \ &= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} p^{m-k} a_{n+k} = \\
&= \sum_{k=0}^{m} (-1)^{m-k} \binom{m}{k} p^{m-k} \sum_{i=0}^{n+k} b_i \frac{(n+k-i+1)_{m-1}}{(m-1)!} p^{n+k-i} = \\
&= \sum_{k=0}^{m} c_{m,k} \sum_{i=0}^{n+k} d_{n,k}(m,i) b_i
\end{aligned}$$

where

$$\begin{aligned}
c_{m,k} &= (-1)^{m-k} \binom{m}{k} p^{m-k}, \\
d_{n,k}(m,i) &= \frac{(n+k-i+1)_{m-1}}{(m-1)!} p^{n+k-i}
\end{aligned}$$

$$\begin{aligned}
\Delta_p^m a_n \ &= \sum_{j=0}^{n} b_j \sum_{r=0}^{m} c_{m,r} d_{n,r}(m,j) + \sum_{j=1}^{m} b_{n+j} \sum_{r=j}^{m} c_{m,r} d_{n,r}(m,n+j) = \\
&= \sum_{j=0}^{n} b_j S_{m,r}(n,j) + \sum_{j=1}^{m} b_{n+j} S'_{m,r}(n,j)
\end{aligned}$$

with

$$\begin{aligned}
S_{m,r}(n,j) &= \sum_{r=0}^{m} c_{m,r} d_{n,r}(m,j) \\
S'_{m,r}(n,j) &= \sum_{r=j}^{p} c_{m,r} d_{n,r}(m,n+j)
\end{aligned}$$

$$\begin{aligned} S_{m,r}(n,j) &= \sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}p^{m-r}\frac{(n+r-j+1)_{m-1}}{(m-1)!}p^{n+r-j} = \\ &= \frac{p^{m+n-j}}{(m-1)!}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}\frac{\Gamma(n+r+m-j)}{\Gamma(n+r-j+1)}\cdot\frac{\Gamma(n-j+m)}{\Gamma(n-j+m)} = \\ &= \frac{p^{m+n-j}}{(m-1)!}\sum_{r=0}^{m}(-1)^{m-r}\binom{m}{r}(n-j+m)_r(n-j+r+1) \\ &(n-j+r+2)...(n-j+m-1) = \\ &= \frac{p^{m+n-j}}{(m-1)!}\sum_{r=0}^{m}\binom{m}{r}(j-n-r-1)(j-n-r-2)... \\ &\quad ...(j-n-m+1)(n-j+m)_r = \\ &= \frac{-p^{m+n+j}}{(m-1)!}\sum_{r=0}^{m}\binom{m}{r}(n-j+m)_r\frac{(j-n-m)_{m-r}}{j-n-m} = \\ &= -\frac{p^{m+n-j}}{(m-1)!}\cdot\frac{(0)_m}{j-n-m} = 0. \end{aligned}$$

In the previously calculations we have use the Chy-Vandermonde formula

$$(x+y)_m = \sum_{k=0}^{m}\binom{m}{k})(x)_k(y)_{m-k}$$

$$\begin{aligned} S'_{m,r}(n,j) &= \sum_{r=j}^{m}(-1)^{m-r}\binom{m}{r}p^{m-r}\frac{(r-j+1)_{m-1}}{(m-1)!}p^{r-j} = \\ &= \frac{p^{m-j}}{(m-1)!}\sum_{r=0}^{m-j}(-1)^{m-r-j}\binom{m}{r+j}p^{m-r-j}(r+1)_{m-1} \end{aligned}$$

$$\begin{aligned}
S'_{m,r} &= \frac{p^{m-j}}{(m-1)!}\sum_{r=0}^{m-j}(-1)^{m-j-r}\binom{m-j}{r}\frac{\binom{m}{r+j}}{\binom{m-j}{r}}(r+1)_{m-1} = \\
&= \frac{p^{m-j}}{(m-1)!}\sum_{r=0}^{m-j}(-1)^{m-j-r}\binom{m-j}{r}\frac{m!}{(m-j)!}\cdot\frac{r!}{(r+j)!}(r+1)_{m-1} = \\
&= \frac{p^{m-j}}{(m-1)!}\cdot\frac{m!}{(m-j)!}\sum_{r=0}^{m-j}(-1)^{m-j-r}\binom{m-j}{r}\frac{\Gamma(m)}{\Gamma(r+j+1)}\cdot\frac{\Gamma(m+r)}{\Gamma(m)} = \\
&= \frac{mp^{m-j}}{(m-j)!}\sum_{r=0}^{m-j}(-1)^{m-j-r}\binom{m-j}{r}(r+j+1)(r+j+2)... \\
&\quad ...(m-1)(m)_r = \\
&= \frac{p^{m-j}}{(m-j)!}\sum_{r=0}^{m-j}\binom{m-}{r}(-m)_{m-j-r}(m)_r = \\
&= \frac{p^{m-j}}{(m-j)!}(0)_{m-j}.
\end{aligned}$$

For $j \leq m-1$ we obtain

$$S'_{m,r}(n,j) = 0.$$

For $j = m$, we have

$$S'_{m,r} = c_{m,n}d_{n,m}(m,n+m) = \frac{(1)_{m-1}}{(m-1)!} = 1.$$

Results $\Delta_m^p a_n = b_{n+m}$.

Necessity: For all real sequence we may consider the sequence (b_n) which define by

$$\begin{aligned}
&b_0 = a_0, \\
&b_n = a_n - \sum_{k=0}^{n-1} b_k \frac{(n-k+1)_{m-1}}{(m-1)!}p^{n-k}\ , \qquad n = 1, 2, ...
\end{aligned}$$

Because $\Delta_p^m a_n = b_{n+m}$ we obtain (**??**).

Example 3. Let $(a_n)_{n\in\mathbf{N}}$, $(b_n)_{n\in\mathbf{N}}$ be the sequences defined by

$$b_n = p^n\ , \quad p \in (0,1),$$
$$a_n = \sum_{k=0}^{n} b_k \frac{(n-k+1)_{m-1}}{(m-1)!} p^{n-k} = \frac{p^n}{n!}(m+1)_n.$$

From theorem **??** we obtain

$$0 \le \Delta_p^m a_n \le 1\ , \quad \text{for all} \quad n \in \mathbf{N}\ , \quad m \in \mathbf{N}^* \quad \text{and} \quad p \in (0,1).$$

We observe that then sequence $(a_n)_{n\in\mathbf{N}}$ defined by $a_n = \dfrac{p^n}{n!}(m+1)_n$ is p-convex with the order m.

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