COHOMOLOGY GROUPS OF A GROUPOID

Dorin Wainberg[1]

Abstract. In this paper we construct a cohomology theory for Brandt groupoids which extends the usual cohomology theory for groups. We construct a cohomology theory for cochains of a Brandt groupoid Γ with values in an abelian group A. This construction was inspired by that of M. Hall [2] in the case of groups.

1. Introduction

In this section preliminary definitions regarding the category of groupoids and some important examples of groupoids are given.

Definition 1.1 ([1]) *A groupoid Γ over Γ_0 is a pair (Γ, Γ_0) endowed with:*

a) two surjections α and β (called the source, respectively target maps), $\alpha, \beta : \Gamma \longrightarrow \Gamma_0$;

b) a product map $m : \Gamma_2 \longrightarrow \Gamma, (x, y) \longrightarrow m(x, y)$ where $\Gamma_2 = \{(x, y) \in \Gamma \times \Gamma \mid \beta(x) = \alpha(y)\}$ is a subset of $\Gamma \times \Gamma$ called the set of composable pairs;

c) an injection $\varepsilon : \Gamma_0 \longrightarrow \Gamma$ (identity),

d) an inverse map $i : \Gamma \longrightarrow \Gamma$,

such that the fallowing conditions are satisfied:

i) for $(x, y); (y, z) \in \Gamma_2$ we have $(m(x, y), z); (x, m(y, z)) \in \Gamma_2$ and $m(m(x, y), z) = m(x, m(y, z))$ (associative law);

ii) for each $x \in \Gamma$ we have $(\varepsilon(\alpha(x)), x); (x, \varepsilon(\beta(x))) \in \Gamma_2$ and $m(\varepsilon(\alpha(x)), x) = m(x, \varepsilon(\beta(x)))$ (identities);

iii) for each $x \in \Gamma$ we have $(i(x), x); (x, i(x)) \in \Gamma_2$ and $m(x, i(x)) = \varepsilon(\alpha(x))$; $m(i(x), x) = \varepsilon(\beta(x))$ (inverses).

Example 1.1 Let Γ_0 be an abstract set and $\Gamma = \Gamma_0 \times \Gamma_0$. It is easy to prove that Γ is a groupoid over Γ_0 with the following strutcure:

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$$\begin{array}{l} \alpha(x,y)=x; \beta(x,y)=y \\ m((x,y),(y,z))=(x,z) \\ \varepsilon(x,x)=x; \\ i(x,y)=(y,x). \end{array}$$

where $x, y, z \in \Gamma_0$. The grupiod Γ is called the coarse groupoid.

Definition 1.2 *A groupoid Γ over Γ_0 is called principal groupoid if the map:*

$$\alpha \times \beta : x \in \Gamma \longrightarrow (\alpha(x), \beta(x)) \in \Gamma_0 \times \Gamma_0$$

is one-to-one.

Example 1.2 Let B be a nonempty set and G a multiplicative group with the unit element e. Then $B \times B \times G$ is a groupoid over B (called the trivial groupoid) with respect to the following structure:

$$\begin{array}{l} \alpha(x,y,g)=x;\ \beta(x,y,g)=y \\ m((x,y,g),(y,z,g'))=(x,z,gg') \\ \varepsilon(u)=(u,u,e); \\ i(x,y,g)=(y,x,g^{-1}). \end{array}$$

In the particular case $\Gamma = \{e\}$ it can be canonically identified with the coarse groupoid.

The following proposition gives some properties of the groupoids:

Proposition 1.1 *Let Γ be a groupoid over Γ_0. Then we have:*
g1) $\alpha \circ \varepsilon = \beta \circ \varepsilon = id$;
g2) $\alpha(m(x,y)) = \alpha(x)$ *and* $\beta(m(x,y)) = \beta(y)$;
g3) *If* $m(x,y_1) = m(x,y_2)$ *or* $m(y_1,z) = m(y_2,z)$ *then* $y_1 = y_2$;
g4) *For each* $x \in \Gamma_0$ *we have* $m(\varepsilon(x), \varepsilon(x)) = \varepsilon(x)$;
g5) $i \circ i = id_\Gamma$;
g6) $\alpha \circ i = \beta$ *and* $\beta \circ i = \alpha$.

Definition 1.3 *A groupoid Γ over Γ_0, with $\Gamma_0 \subseteq \Gamma$ is called Brandt groupoid.*

It is easy to observe that in the case of the Brandt groupoids we have:

i) $\varepsilon(\Gamma_0) = \Gamma_0$.
ii) $\varepsilon(u) = u$, $(\forall)$ $u \in \Gamma_0$.
Moreover, for each $u \in \Gamma_0$ the set $\alpha^{-1}(u) \cap \beta^{-1}(u)$ is a group under the restriction of the multiplication in Γ, called the isotropy group at u.

Doing these changes for the definition 1, in [3] we can find the next definition for the Brandt groupoids:

DEFINITION 1.4 *A Brandt groupoid is a nonempty set* Γ *endowed with:*
a) *two maps* d *and* r *(called source, respectively target),* $d, r : \Gamma \longrightarrow \Gamma$*;*
b) *a product map*

$$m : \Gamma_{(2)} \longrightarrow \Gamma,$$
$$(x, y) \longrightarrow m(x, y) \stackrel{def}{=} xy$$

where $\Gamma_{(2)} = \{(x, y) \in \Gamma \times \Gamma \mid d(x) = r(y)\}$ *is a subset of* $\Gamma \times \Gamma$ *called the set of composable pairs;*
c) *an inverse map*

$$i : \Gamma \longrightarrow \Gamma,$$
$$x \longrightarrow i(x) \stackrel{def}{=} x^{-1}$$

such that the fallowing conditions are satisfied:
i) $(x, y); (y, z) \in \Gamma_{(2)} \Longrightarrow (xy, z); (x, yz) \in \Gamma_{(2)}$ *and* $(xy)z = x(yz)$*;*
ii) $x \in \Gamma \Longrightarrow (r(x), x); (x, d(x)) \in \Gamma_{(2)}$ *and* $r(x)x = xd(x) = x$*;*
iii) $x \in \Gamma \Longrightarrow (x^{-1}, x); (x, x^{-1}) \in \Gamma_{(2)}$ *and* $x^{-1}x = d(x); xx^{-1} = r(x)$.
If Γ *is a groupoid then* $\Gamma_0 := d(\Gamma) = r(\Gamma)$ *is the unit set of* Γ *and we say that* Γ *is a* Γ_0*-groupoid.*

EXAMPLE 1.3 The grouopoid $\Gamma^{(n)} (n \geq 2)$.
Let Γ be a Γ_0-groupoid and by $\Gamma^{(n)}$ we denote the set of n-tuples $(x_0, \ldots, x_{n-1})$ of Γ such that $(x_{i-1}, x_i) \in \Gamma_{(2)}$ for $i = 1, 2, \ldots, n-1$. We give to $\Gamma^{(n)}$ the following groupoid structure:

- $d^{(n)}, r^{(n)} : \Gamma^{(n)} \longrightarrow \Gamma^{(n)}$;
- $d^{(n)}(x_0, \ldots, x_{n-1}) \stackrel{def}{=} (x_0 x_1, x_1 x_2, \ldots, x_{n-2} x_{n-1}, d(x_{n-2} x_{n-1}))$;
- $r^{(n)}(x_0, \ldots, x_{n-1}) \stackrel{def}{=} (x_0, x_1, \ldots, x_{n-2}, r(x_{n-1}))$;

- $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ are composable if $y_0 = x_0x_1, y_1 = x_1x_2, ..., y_{n-2} = x_n - 2x_{n-1}$ and

$$(x_0, \ldots, x_{n-1})(x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, y_{n-1}) \overset{def}{=} (x_0, \ldots, x_{n-2}, x_{n-1}y_{n-1})$$

- the inverse of $(x_0, \ldots , x_{n-1})$ is defined by:

$$(x_0, \ldots, x_{n-1})^{-1} \overset{def}{=} (x_0x_1, x_1x_2, \ldots, x_{n-2}x_{n-1}, x)$$

Definition 1.5 *Let* Γ *and* Γ' *be groupoids.*

i) *a map* $f : \Gamma \longrightarrow \Gamma'$ *is a morphism if for any* $(x, y) \in \Gamma_{(2)}$ *we have* $(f(x), f(y)) \in \Gamma'_{(2)}$ *and* $f(x, y) = f(x)f(y)$.

ii) *two morphisms* $f, g : \Gamma \longrightarrow \Gamma'$ *are similar (and we write* $f \sim g$*) if there exists a map* $\theta : \Gamma_0 \longrightarrow \Gamma'$ *such that* $\theta(r(x)) \cdot f(x) = g(x) \cdot \theta(d(x))$ *for any* $x \in \Gamma$.

iii) *the groupoids* Γ *and* Γ' *are similar (*$\Gamma \sim \Gamma'$*) if there exists two morphisms* $f : \Gamma \longrightarrow \Gamma'$ *and* $g : \Gamma' \longrightarrow \Gamma$ *such that* $g \circ f$ *and* $f \circ g$ *are similar to identity isomorphisms.*

Example 1.4 Let $\Gamma^{(0)} = \Gamma_0$; $\Gamma^{(1)} = \Gamma$; $\Gamma^{(n)}$ be the groupoid given in example 3 and $\phi : \Gamma \longrightarrow \Gamma'$ be a morphism of groupoids. The map $\phi^{(n)} : \Gamma^{(n)} \longrightarrow \Gamma^{(n)}$ defined by:

$$\phi^{(n)}(x_0, \ldots , x_{n-1}) \overset{def}{=} (\phi(x_0), \ldots , \phi(x_{n-1}))$$

is a morphism of groupoids for any $n \geq 0$.

Example 1.5 The trivial groupoid $\Gamma = B \times B \times G$ and the group G are similar.

2. Cohomology

We assume that:

a) Γ is a Γ_0-groupoid;

b) $(A, +)$ is an abelian group;

c) Γ operates on the left on A , i.e. $\Gamma \times A \longrightarrow A$, $(x,a) \longrightarrow x.a$ subject of the following conditions:

i)$x.(y.a) = (xy).a$ for all $(x,y) \in \Gamma_{(2)}$ and $a \in A$;

ii) $u.a = a$ for all $u \in \Gamma_0$ and $a \in A$;

iii) $x.(a+b) = x.a + x.b$ for all $x \in \Gamma$ and $a, b \in A$.

In this hypothesis we say that A is a Γ-module.

DEFINITION 2.1 *Given a Γ-module A, a function $f : \Gamma^{(n)} \longrightarrow A$, $(x_0, \dots, x_{n-1}) \longrightarrow f(x_0, \dots, x_{n-1})$, where $\Gamma^{(1)} = \Gamma$ and $\Gamma^{(n)}$ for $n \geq 2$ is the groupoid given in example 3, is called a n-cochain of the groupoid Γ with values in A.*

We denote by $C^{(n)}(\Gamma, A) = 0$ the additive group of n-cochains of Γ. By definition $C^{(n)}(\Gamma, A) = 0$ if $n < 0$ and $C^{(0)}(\Gamma, A) = A$. Define the coboundary operator:

$$\delta^n : C^n(\Gamma, A) \longrightarrow C^{n+1}(\Gamma, A)$$

by:

$\delta^0 f(x) = x.f(d(x)) - f(r(x))$

for all $x \in \Gamma$, $f \in C^0(\Gamma, A)$ if $n = 0$;

$$\delta^n f(x_0, \dots, x_n) = x_0.f(x_1, \dots, x_n) +$$
$$\sum_{i=1}^{n}(-1)^i f(x_0, \dots, x_{i-2}, x_{i-1}\, x_i, x_{i+1}, ..., x_n) + (-1)^{n+1} f(x_0, \dots, x_{n-1})$$

if $n \geq 1$.

The map $f \longrightarrow \delta^n f$ is a homomorphism with respect to addition and we have that $(C^n(\Gamma, A), \delta^n)$ is a cochain complex. The cohomology n-groups $H^n(\Gamma, A)$ of Γ-module A are defined by $H^n(\Gamma, A) = Z^n(\Gamma, A)/B^n(\Gamma, A)$, where $Z^n(\Gamma, A) = \ker \delta^n$ and $B^n(\Gamma, A) = Im\delta^{n-1}$.

If $\phi : \Gamma \longrightarrow \Gamma'$ is a morphism of groupoids then the morphism of groupoids $\phi^{(n)} : \Gamma^{(n)} \longrightarrow \Gamma'^{(n)}$ given by example 1.4 induce a homomorphism of groups

$$\bar{\phi}'^n : C^n(\Gamma', A) \longrightarrow C^n(\Gamma, A)$$

defined by:

$$\bar{\phi}'^n(f) \overset{def}{=} f \circ \phi^{(n)} \text{ for each } f \in C^n(\Gamma', A)$$

and satisfying the following:

$$\delta^n \circ \bar{\phi}'^n = \bar{\phi}'^{n+1}\delta^n \text{ for all } n \geq 0.$$

From here it follows that $\bar{\phi}'^n$ induce a homomorphism of cohomology groups $(\phi^n)^* : H^n(\Gamma', A) \longrightarrow H^n(\Gamma, A)$ given by:

$$(\phi^n)^*([f]) = [\bar{\phi}'^n(f)] \text{ forevery } [f] \in H^n(\Gamma', A).$$

REMARK 2.1 *We have:*

$H^0(\Gamma, A) = \{x \in A \mid x.a = a \text{ for all } x \in \Gamma\}$ *is the set of elements of* A *such that* Γ *operates simply on* A*. In particular, if* Γ *operates trivially on* A*, i.e.* $x.a = a$ *for all* $x \in \Gamma$*, then* $H^0(\Gamma, A) = A$*.*

$Z^1(\Gamma, A) = \{f : \Gamma \longrightarrow A \mid f(x_0x_1) = f(x_0) + x_0f(x_1),\ \forall\ (x_0, x_1) \in \Gamma_{(2)}\}$ *is the group of crossed morphisms of groupoids.*

$B^1(\Gamma, A) = \{f : \Gamma \longrightarrow A \mid f(x) = x.a - a,\ \forall\ x \in \Gamma\}$ *is the group of principal morphisms of groupoids.*

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Author:

Dorin Wainberg
"1 Decembrie 1918" University of Alba Iulia
N. Iorga, no.11-13, Alba Iulia, 510009, Romania
e-mail:dwainberg@uab.ro