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MEROMORPHIC FUNCTIONS WITH A FIXED POINT INVOLVING DZIOK-RAINA OPERATOR

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Abstract. In the present investigation, the authors define a new class of meromorphic functions defined in the punctured unit disk $\Delta^* := \{z \in C : 0 < |z| < 1\}$ by making use of the generalized Dziok-Srivastava operator $\mathcal{L}^{\tau,\alpha_1}_{\eta,l,m}$. Coefficient inequalities, growth and distortion inequalities, as well as closure results are obtained. We also establish some results concerning the partial sums of meromorphic functions and neighbourhood results for functions in new class.

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1. Introduction

Let ξ be a fixed point in the unit disc $\Delta := \{z \in C : |z| < 1\}$. Denote by $\mathcal{H}(\Delta)$ the class of functions which are regular and

$$\mathcal{A}(\xi) = \{f \in H(\Delta) : f(\xi) = f'(\xi) - 1 = 0\}$$

also denote by $\mathcal{S}_\xi = \{f \in \mathcal{A}(\xi) : f \text{ is Univalent in } \Delta\}$, the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z) = (z - \xi) + \sum_{n=2}^{\infty} a_n (z - \xi)^n, \tag{1}$$

that are analytic in the open unit disc Δ. Note that $\mathcal{S}_0 = \mathcal{S}$ be a subclass of $\mathcal{A}$ consisting of univalent functions in Δ. By $\mathcal{S}^*_\xi(\beta)$ and $\mathcal{K}_\xi(\beta)$, respectively, we mean the classes of analytic functions that satisfy the analytic conditions

$$\mathrm{Re}\left\{\frac{(z-\xi)f'(z)}{f(z)}\right\} > \beta, \mathrm{Re}\left\{1 + \frac{(z-\xi)f''(z)}{f'(z)}\right\} > \beta \qquad \text{and} \qquad z \in \Delta$$

for $0 \leq \beta < 1$ introduced and studied by Kanas and Ronning [15].The class $\mathcal{S}^*_\xi(0)$ is defined by geometric property that the image of any circular arc centered at ξ is starlike with respect to $f(\xi)$ and the corresponding class $\mathcal{K}^*_\xi(0)$ is defined by the property that the image of any circular arc centered at ξ is convex. We observe that the definitions are somewhat similar to the ones introduced by Goodman in [13] and [14] for uniformly starlike and convex functions, except that in this case the point ξ is fixed. In particular, $\mathcal{K} = \mathcal{K}_0(0)$ and $\mathcal{S}^*_0 = \mathcal{S}^*(0)$ respectively, are the well-known standard class of convex and starlike functions(see [26]).

Let Σ denote the class of normalized meromorphic functions f of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n, \tag{2}$$

defined on the punctured unit disk

$$\Delta^* := \{z \in C : 0 < |z| < 1\}.$$

Denote by Σ_ξ be the subclass of $\mathcal{A}(\xi)$ consist of the functions of the form

$$f(z) = \frac{1}{(z-\xi)} + \sum_{n=1}^{\infty} a_n (z-\xi)^n, a_n \geq 0; z \neq \xi \tag{3}$$

A function $f \in \Sigma^*_\xi$ is *meromorphic starlike of order* α $(0 \leq \alpha < 1)$ if

$$-\Re\left(\frac{(z-\xi)f'(z)}{f(z)}\right) > \alpha \quad (z-\xi \in \Delta := \Delta^* \cup \{0\}).$$

The class of all such functions is denoted by $\Sigma^*_\xi(\alpha)$. A function $f \in \Sigma_\xi$ is *meromorphic convex of order* α $(0 \leq \alpha < 1)$ if

$$-\Re\left(1 + \frac{(z-\xi)f''(z)}{f'(z)}\right) > \alpha \quad (z-\xi \in \Delta := \Delta^* \cup \{0\}).$$

For functions $f(z)$ given by (3) and $g(z) = \frac{1}{(z-\xi)} + \sum_{n=1}^{\infty} b_n(z-\xi)^n$ we define the Hadamard product or convolution of f and g by

$$(f * g)(z) := \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_n b_n (z-\xi)^n.$$

For complex parameters $\alpha_1, \ldots, \alpha_l$ and $\beta_1, \ldots, \beta_m$ $(\beta_j \neq 0, -1, \ldots; j = 1, 2, \ldots, m)$ the *generalized hypergeometric function* ${}_lF_m(z)$ is defined by

$$\begin{aligned} {}_lF_m(z) \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) \quad &:= \quad \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n} \frac{z^n}{n!} \\ (l \leq m+1;\ l, m \in N_0 \quad &:= \quad N \cup \{0\}; z \in U) \end{aligned} \tag{4}$$

where N denotes the set of all positive integers and $(a)_n$ is the Pochhammer symbol defined by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & n=0 \\ a(a+1)(a+2)\ldots(a+n-1), & n \in N; a \in C \end{cases} \tag{5}$$

For positive real parameters $\alpha_1, A_1 \ldots, \alpha_l, A_l$ and $\beta_1, B_1 \ldots, \beta_m, B_m$ $(l, m \in N = 1, 2, 3, ...$ such that

$$1 + \sum_{n=1}^{m} B_n - \sum_{n=1}^{l} A_n \geq 0. \quad z \in U. \tag{6}$$

The Wright generalized hypergeometric function[31]

$${}_l\Psi_m[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m); z] =_l \Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z]$$

is defined by

$${}_l\Psi_m[(\alpha_t, A_t)_{1,l}(\beta_t, B_t)_{1,m}; z] = \sum_{n=0}^{\infty}\{\prod_{t=0}^{l}\Gamma(\alpha_t + nA_t)\}\{\prod_{t=0}^{m}\Gamma(\beta_t + nB_t\}^{-1}\frac{z^n}{n!}, z \in U.$$

If $A_t = 1(t = 1, 2, ..., l)$ and $B_t = 1(t = 1, 2, ..., m)$we have the relationship:

$$\Omega_l\Psi_m[(\alpha_t, 1)_{1,l}(\beta_t, 1)_{1,m}; z] \equiv {}_lF_m(\alpha_1, \ldots \alpha_l; \beta_1, \ldots, \beta_m; z) = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n \ldots (\alpha_l)_n}{(\beta_1)_n \ldots (\beta_m)_n}\,\frac{z^n}{n!} \tag{7}$$

$(l \leq m+1;\ l, m \in N_0 = N \cup \{0\}; z \in U)$ is the generalized hypergeometric function(see for details[8]) where N denotes the set of all positive integers and $(\alpha)_n$ is the Pochhammer symbol and

$$\Omega = \left(\prod_{t=0}^{l}\Gamma(\alpha_t)\right)^{-1}\left(\prod_{t=0}^{m}\Gamma(\beta_t)\right). \tag{8}$$

By using the generalized hypergeometric function Dziok and Srivastava [8] introduced the linear operator. further [7] Dziok and Raina extended the linear operator by using Wright generalized hypergeometric function .

Now we define a new linear operator for meromorphic functions with a fixed pointξ given by (3).

Let $\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}] : \Sigma_\xi \to \Sigma_\xi$ be a linear operator defined by

$$\mathcal{W}[(\alpha_t, A_t)_{1,p}; (\beta_t, B_t)_{1,q}]f(z) \quad := \quad (z-\xi)^{-1}\ {}_l\phi_m[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}; (z-\xi)] * f(z)$$

We observe that , for $f(z)$ of the form(3),we have

$$\mathcal{W}[(\alpha_t, A_t)_{1,l}; (\beta_t, B_t)_{1,m}]f(z) = (z-\xi)^{-1} + \sum_{n=1}^{\infty} \omega_n(\alpha_1)\ a_n(z-\xi)^n \tag{9}$$

where $\omega_n(\alpha_1)$is defined by

$$\omega_n(\alpha_1) = \frac{\Omega\Gamma(\alpha_1 + nA_1)\ldots\Gamma(\alpha_l + nA_l)}{\Gamma(\beta_1 + nB_1)\ldots\Gamma(\beta_m + nB_m)}\frac{1}{n!}\ . \tag{10}$$

and Ω is given by (8).

For convenience, we write

$$\mathcal{W}^l_m f(z) = \mathcal{W}[(\alpha_1, A_1), \ldots, (\alpha_l, A_l); (\beta_1, B_1), \ldots, (\beta_m, B_m)]f(z) \tag{11}$$

introduced by Dziok and Raina[7].

In view of the relationship (10) the linear operator(11) and by setting $A_t = 1(t = 1, ..., l)$ and $B_t = 1(t = 1, ..., m)$, the linear operator $\mathcal{H}^l_m$ is called Dziok-Srivastava operator (see [8]), includes (as its special cases) various other linear operators introduced and studied by Carlson and Shaffer, Ruscheweyh and Srivastava-Owa studied in the literature (see Aouf [1], and Uralegaddi and Somanatha [28]; Duren [[6] , Srivastava and Owa [[27], also [1, 16, 17, 19]). Further, the class $\Sigma^*_0(\alpha)$ and various other subclasses of Σ have been studied rather extensively by Clunie [5], Nehari and Netanyahu [20], , Pommerenke ([22], [23]), Royster [24], and others (cf., e.g., Bajpai [2] , Mogra et al. [18], Uralegaddi and Ganigi [29], Cho et al. [3].

Now by making use of the generalized Raina-Dziok operator $\mathcal{W}^l_m$, we define a new subclass of functions in Σ_ξ as follows.

For $0 \le \gamma < 1$ and $0 \le \lambda \le 1/2$, we let $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda, \gamma)$ denote a subclass of Σ_ξ consisting functions of the form (3) satisfying the condition that

$$\begin{aligned} - &\ \Re\left(\frac{(z-\xi)(\mathcal{W}^l_m f(z))' + \lambda(z-\xi)^2(\mathcal{W}^l_m f(z))''}{(1-\lambda)\mathcal{W}^l_m f(z) + \lambda(z-\xi)(\mathcal{W}^l_m f(z))'} - \gamma\right) \\ > &\ \beta\left|\frac{(z-\xi)(\mathcal{W}^l_m f(z))' + \lambda(z-\xi)^2(\mathcal{W}^l_m f(z))''}{(1-\lambda)\mathcal{W}^l_m f(z) + \lambda(z-\xi)(\mathcal{W}^l_m f(z))'} + 1\right| \end{aligned} \tag{12}$$

where $\mathcal{W}^l_m$ is given by (11).Further shortly we can state this condition by

$$-\Re\left(\frac{zG'(z)}{G(z)} - \gamma\right) > \beta\left|\frac{zG'(z)}{G(z)} + 1\right|, \tag{13}$$

where

$$G(z) = (1-\lambda)F(z) + \lambda(z-\xi)F'(z) = \frac{1-2\lambda}{z-\xi} + \sum_{n=1}^{\infty}(n\lambda - \lambda + 1)\omega_n(\alpha_1)a_n(z-\xi)^n, \quad a_n \ge 0. \tag{14}$$

and $F(z) = \mathcal{W}_m^l f(z)$. In this paper, we obtain the coefficient inequalities, growth and distortion inequalities, as well as closure results for the class $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$. Properties of a certain integral operator and its inverse defined on the new class $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$ are also discussed.

2. Coefficients Inequalities

Our first theorem gives a necessary and sufficient condition for a function f to be in the class $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$.

Lemma 1 Suppose that $\beta \in [0,1)$, $r \in (0,1]$ and the function H is of the form

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n, \quad 0 < |z| < r, \tag{15}$$

with $b_n \geq 0$. Then the condition

$$-\Re \frac{zH'(z)}{H(z)} > \beta \quad \text{for} \quad |z| < r \tag{16}$$

is equivalent to the condition

$$\sum_{n=1}^{\infty} (n+\beta) b_n r^{n+1} \leq 1 - \beta. \tag{17}$$

Theorem 1 Let $f(z) \in \Sigma_\xi$ be given by (3). Then $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$ if and only if

$$\sum_{n=1}^{\infty} [n(1+\gamma) + (\gamma+\beta)](n\lambda + \lambda - 1)\ \omega_n(\alpha_1) a_n \leq (1-2\lambda)(1-\gamma). \tag{18}$$

Proof. If $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$, then by (12) we have ,

$$\begin{aligned} - \quad & \Re\left(\frac{(z-\xi)(\mathcal{W}_m^l f(z))' + \lambda(z-\xi)^2(\mathcal{W}_m^l f(z))''}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda(z-\xi)(\mathcal{W}_m^l f(z))'} - \gamma\right) \\ > \quad & \beta \left| \frac{(z-\xi)(\mathcal{W}_m^l f(z))' + \lambda(z-\xi)^2(\mathcal{W}_m^l f(z))''}{(1-\lambda)\mathcal{W}_m^l f(z) + \lambda(z-\xi)(\mathcal{W}_m^l f(z))'} + 1 \right| \end{aligned} \tag{19}$$

That is,

$$- \quad \Re\left(\frac{[(z-\xi)G'(z)](1+\beta e^{i\theta}) + \beta e^{i\theta}[G(z)]}{G(z)} - \gamma\right) > 0,$$

where $G(z)$ is given by (14). By letting $z \to 1^-$, we have

$$\left\{ \frac{(1-2\lambda)(1-\gamma) - \sum_{n=1}^{\infty}[n(1+\gamma)+(\gamma+\beta)](n\lambda-\lambda+1)\omega_n(\alpha_1)a_n}{(1-2\lambda) - \sum_{n=1}^{\infty} n(n\lambda-\lambda+1)\omega_n(\alpha_1)a_n} \right\} > 0.$$

This shows that (18) holds.Conversely assume that (18) holds. Since

$$\Re(w) > \gamma \quad \text{ifandonlyif} \quad |w-1| < |w+1-2\gamma|,$$

it is sufficient to show that

$$\left| \frac{w-1}{w+(1-2\gamma)} \right| < 1 \quad \text{and} \quad |w+1-2\gamma| \neq 0 \quad \text{for} \quad |z-\xi| < r \leq 1, \quad (z-\xi) \in \Delta.$$

Using (18), we see that

$$\begin{aligned} &= \left| \frac{-\sum_{n=1}^{\infty}(n\lambda+\lambda-1)(n+1)\omega_n(\alpha_1)a_n(z-\xi)^{n+1}}{-2(1-\alpha)+\sum_{n=1}^{\infty}(n\lambda+\lambda-1)(n-1+2\alpha)\omega_n(\alpha_1)a_n(z-\xi)^{n+1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty}(n\lambda+\lambda-1)(n+1)\omega_n a_n}{2(1-\alpha)-\sum_{n=1}^{\infty}(n\lambda+\lambda-1)(n-1+2\alpha)\omega_n a_n} \leq 1. \end{aligned}$$

Thus we have $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$.

For the sake of brevity throughout this paper we let

$$d_n(\lambda,\gamma) := [n(1+\gamma)+(\gamma+\beta)](n\lambda-\lambda+1) \tag{20}$$

$$d_1(\lambda,\gamma) = (1+2\gamma+\beta)$$

unless otherwise stated. Our next result gives the coefficient estimates for functions in $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$.

Theorem 2. If $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$, then

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}, \qquad n = 1,2,3,\ldots.$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z-\xi} + \frac{1-\gamma}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}(z-\xi)^n, \qquad n = 1,2,3,\ldots.$$

Proof.

If $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$, then we have, for each n,

$$d_n(\lambda,\gamma)a_n \leq \sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_n \leq (1-\gamma)(1-2\lambda).$$

Therefore we have

$$a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}.$$

Since

$$F_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}(z-\xi)^n$$

satisfies the conditions of Theorem 1, $F_n(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ and the equality is attained for this function.

Theorem 3. Suppose that there exists a positive number ν

$$v = \inf_{n\in N}\{d_n(\lambda,\alpha)\omega_n(\alpha;l;m)\}. \tag{21}$$

If $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, then

$$\left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right| \leq |f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r \quad (|z-\xi| = r).$$

If $\nu = d_1(\lambda,\gamma)\omega_1(\alpha_1) = (1+2\gamma+\beta)\omega_1(\alpha_1)$, then the result is sharp for

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+2\gamma+\beta)\omega_1(\alpha_1)}(z-\xi). \tag{22}$$

Proof.

Since $f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_n(z-\xi)^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r\sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\gamma)(1-2\lambda)}{\nu}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{(1-\gamma)(1-2\lambda)}{\nu}r.$$

Similarly

$$|f(z)| \geq \left|\frac{1}{r} - \frac{(1-\gamma)(1-2\lambda)}{\nu}r\right|.$$

The result is sharp for function (22) with $\nu = d_1(\lambda,\gamma)\omega_1(\alpha_1) = (1+2\gamma+\beta)\omega_1(\alpha_1)$.

Similarly we have the following:
Theorem 4. If $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, then

$$\left|\frac{1}{r^2} - \frac{(1-\gamma)(1-2\lambda)}{\nu}\right| \le |f'(z)| \le \frac{1}{r^2} + \frac{(1-\gamma)(1-2\lambda)}{\nu} \quad (|z-\xi| = r).$$

The result is sharp for the function (22)with $\nu = d_1(\lambda,\gamma)\omega_1(\alpha_1) = (1+2\gamma+\beta)\omega_1(\alpha_1)$.

3. Closure Theorems

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_{n,k}(z-\xi)^n, \quad k = 1,2,...,m. \tag{23}$$

We shall prove the following closure theorems for the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$.

Theorem 5. Let the function $F_k(z)$ defined by (23) be in the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ for every $k = 1,2,...,m$. Then the function $f(z)$ defined by $f(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} a_{n,k}(z-\xi)^n$, $(a_{n,k} \ge 0)$ belongs to the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, where $a_{n,k} = \frac{1}{m}\sum_{k=1}^{m} a_{n,k}$ $(n = 1,2,..)$.

Proof.

Since $F_k(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, it follows from Theorem 1 that

$$\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_{n,k} \le (1-\gamma)(1-2\lambda), \forall k = 1,2,..,m. \tag{24}$$

Hence

$$\begin{aligned}
\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_n &= \sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)\left(\frac{1}{m}\sum_{k=1}^{m} a_{n,k}\right) \\
&= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_{n,k}\right) \\
&\le (1-\gamma)(1-2\lambda).
\end{aligned}$$

By Theorem 1, we have $f(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$.
Theorem 6. The class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ is closed under convex linear combination.

Proof.

Let the function $F_k(z)$ given by (23) be in the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Then it is enough to show that the function

$$H(z) = \mu F_1(z) + (1-\mu)F_2(z) \quad (0 \le \mu \le 1)$$

is also in the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Since for $0 \leq \mu \leq 1$,

$$H(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty}[\mu a_{n,1} + (1-\mu)a_{n,2}](z-\xi)^n,$$

we observe that

$$\sum_{n=1}^{\infty}[\mu a_{n,1} + (1-\mu)a_{n,2}]d_n(\lambda,\gamma)\omega_n(\alpha_1)$$

$$\begin{aligned} &= \mu\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_{n,1} + (1-\mu)\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_{n,2} \\ &\leq (1-\gamma)(1-2\lambda). \end{aligned}$$

By Theorem 1, we have $H(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$.

Theorem 7. Let $F_0(z) = \frac{1}{z-\xi}$ and $F_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}(z-\xi)^n$ for $n = 1, 2, \ldots$. Then $f(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ if and only if $f(z)$ can be expressed in the form $f(z) = \sum_{n=0}^{\infty}\eta_n F_n(z)$ where $\eta_n \geq 0$ and $\sum_{n=0}^{\infty}\eta_n = 1$.

Proof.

Let

$$\begin{aligned} f(z) &= \sum_{n=0}^{\infty}\eta_n F_n(z) \\ &= \frac{1}{z-\xi} + \sum_{n=1}^{\infty}\frac{\eta_n((1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}(z-\xi)^n. \end{aligned}$$

Then

$$\sum_{n=1}^{\infty}\eta_n\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}\frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{((1-\gamma)(1-2\lambda)} = \sum_{n=1}^{\infty}\eta_n = 1-\eta_0 \leq 1.$$

By Theorem 2, we have $f(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Conversely, let $f(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. From Theorem 2, we have $a_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_n(\alpha_1)}$; for $n = 1, 2, ..$ we may take $\eta_n = \frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)}a_n$, for $n = 1, 2, ...$ and $\eta_0 = 1 - \sum_{n=1}^{\infty}\eta_n$. Then $f(z) = \sum_{n=0}^{\infty}\eta_n F_n(z)$.

4. Radius of starlikeness

In the following theorem we obtain the radius of starlikeness for the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. We say that f given by 2 is meromorphically starlike of order ρ, $(0 \leq \rho < 1)$, in $|z| < r$.

Theorem 8. Let the function f given by 2 be in the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Then, if there exists

$$\inf_{n\geq 1}\left[\frac{(1-\rho)d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(n+\rho)(1-\gamma)(1-2\lambda)}\right]^{\frac{1}{n+1}} := r_1(\gamma,\lambda,\rho) \tag{25}$$

and it is positive, then f is meromorphically starlike of order ρ in $|z| < r \leq r_1(\gamma, \lambda, \rho)$.

Proof.

Let the function $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda, \gamma)$ be of the form 2. If $0 < r \leq r_1(\gamma, \lambda, \rho)$, then by 25

$$r^{n+1} \leq \frac{(1-\rho)d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(n+\rho)(1-\gamma)(1-2\lambda)} \tag{26}$$

for all $n \in N$. From 26 we get

$$\frac{n+\rho}{1-\rho} r^{n+1} \leq \frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)}$$

for all $n \in N$, thus

$$\sum_{n=1}^{\infty} \frac{n+\rho}{1-\rho} a_n r^{n+1} \leq \sum_{n=1}^{\infty} \frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)} a_n \leq 1. \tag{27}$$

If $f \in \Sigma_P$, then by Lemma 1, the function f is meromorphically starlike of order ρ, in $|z| < r$ if and only if

$$\sum_{n=1}^{\infty} (n+\rho) a_n r^{n+1} \leq 1-\rho. \tag{28}$$

Therefore, 27 and 28 give that f is meromorphically starlike of order ρ in $|z| < r \leq r_1(\gamma, \lambda, \rho)$.

Suppose that there exists a number $\widetilde{r}$, $\widetilde{r} > r_1(\gamma, \lambda, \rho)$ such that each $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda, \gamma)$ is meromorphically starlike of order ρ in $|z| < \widetilde{r} \leq 1$. The function

$$f(z) = \frac{1}{z} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_1(\alpha_1)} z^n$$

is in the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda, \gamma)$, thus it should satisfy 28 with $\widetilde{r}$:

$$\sum_{n=1}^{\infty} (n+\rho) a_n \widetilde{r}^{n+1} \leq 1-\rho, \tag{29}$$

while the left–hand site of 29 becomes

$$(n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_1(\alpha_1)}\widetilde{r}^{n+1} > (n+\rho)\frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\gamma)\omega_1(\alpha_1)}\frac{(1-\rho)d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(n+\rho)(1-\gamma)(1-2\lambda)} = 1-\rho$$

what contradicts with 29. Therefore the number $r_1(\gamma, \lambda, \rho)$ in Theorem 8 cannot be replaced with a grater number. This means that $r_1(\gamma, \lambda, \rho)$ is so called radius of meromorphically starlikness of order ρ for the class $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda, \gamma)$.

5. Integral Operators

In this section, we consider integral transforms of functions in the class $\mathcal{M}^{l,m}_{w,\alpha_1}(\alpha,\lambda)$.

Theorem 9. Let the function $f(z)$ given by (3) be in $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Then the integral operator

$$F(z) = c\int_0^1 u^c f(uz)du \qquad (0 < u \leq 1, 0 < c < \infty)$$

is in $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, where

$$\delta \leq \frac{(c+n+1)d_n(\lambda,\gamma) - cn(1-\gamma)(1-2\lambda)}{c((1-\gamma)(1-2\lambda))\{1-\lambda(1+n)\} + d_n(\lambda,\gamma)(c+n+1)}.$$

The result is sharp for the function $f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{(1+2\gamma+\beta)\omega_1(\alpha_1)}(z-\xi)$.

Proof.

Let $f(z) \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$. Then

$$F(z) = c\int_0^1 u^c f(uz)du = \frac{1}{z-w} + \sum_{n=1}^{\infty}\frac{c}{c+n+1}a_n(z-\xi)^n.$$

It is sufficient to show that

$$\sum_{n=1}^{\infty}\frac{c\ d_n(\lambda,\delta)\omega_n(\alpha_1)}{(c+n+1)(1-\delta)}a_n \leq 1. \tag{30}$$

Since $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, we have

$$\sum_{n=1}^{\infty}\frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)}a_n \leq 1.$$

Note that (30) is satisfied if

$$\frac{c\ d_n(\lambda,\delta)\omega_n(\alpha_1)}{(c+n+1)(1-\delta)} \leq \frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)}.$$

Solving for δ, we have

$$\delta \leq \frac{(c+n+1)d_n(\lambda,\gamma) - cn(1-\gamma)(1-2\lambda)}{c(1-\gamma)(1-2\lambda)\{1-\lambda(1+n)\} + d_n(\lambda,\gamma)(c+n+1)} = \Phi(n).$$

A simple computation will show that $\Phi(n)$ is increasing and $\Phi(n) \geq \Phi(1)$. Using this, the results follows.

Theorem 10. Let $f(z)$, given by (3), be in $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$,

$$F(z) = \frac{1}{c}[(c+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty}\frac{c+n+1}{c}a_n(z-\xi)^n, \qquad c > 0. \tag{31}$$

Then $F(z)$ is in $\Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ for $|z-\xi| \leq r(\gamma,\lambda,\beta)$ where

$$r(\gamma,\lambda,\beta) = \inf_n \left(\frac{c(1-\beta)d_n(\lambda,\gamma)}{((1-\gamma)(1-2\lambda))(c+n+1)d_n(\lambda,\beta)} \right)^{1/(n+1)}, \quad n = 1,2,3,\ldots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_n(\lambda,\alpha)\omega_n(\alpha_1)}(z-\xi)^n, \quad n = 1,2,3,\ldots.$

Proof.

Let $w = \frac{[(z-\xi)G'(z)](1+\beta e^{i\theta})+\beta e^{i\theta}[G(z)]}{G(z)}$ where $G(z)$ is given by (14).Then it is sufficient to show that $\left|\frac{w-1}{w+1-2\beta}\right| < 1$. A simple computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{(c+n+1)d_n(\lambda,\beta)\omega_n(\alpha_1)}{c\ (1-\beta)} a_n|z-\xi|^{n+1} \leq 1. \tag{32}$$

Since $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, by Theorem 1, we have

$$\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_n \leq (1-\gamma)(1-2\lambda).$$

The equation (32) is satisfied if

$$\frac{(c+n+1)d_n(\lambda,\beta)\omega_n(\alpha_1)}{c\ (1-\beta)} a_n|z-\xi|^{n+1} \leq \frac{d_n(\lambda,\alpha)\omega_n(\alpha_1)a_n}{(1-\gamma)(1-2\lambda)}.$$

Solving for $|z-\xi|$, we get the desired result.

6. Integral Operators

Silverman [26] determined sharp lower bounds on the real part of the quotients between the normalized starlike or convex functions and their sequences of partial sums. As a natural extension, one is interested to search results analogous to those of Silverman for meromorphic univalent functions. In this section, motivated essentially by the work of silverman [26] and Cho and Owa [4] we will investigate the ratio of a function of the form(3) to its sequence of partial sums

$$f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n(z-\xi)^n \tag{33}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} d_n(\lambda,\gamma)\omega_n(\alpha_1)a_n \leq (1-\gamma)(1-2\lambda).$$

More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known results that$\Re\left\{\frac{1+w(z)}{1-w(z)}\right\} > 0 \quad (z-\xi \in \Delta)$ if and only if $\omega(z) = \sum\limits_{n=1}^{\infty} c_n(z-\xi)^n$ satisfies the inequality $|\omega(z)| \leq |z-\xi|$.

Unless otherwise stated, we will assume that f is of the form (3) and its sequence of partial sums is denoted by $f_k(z) = \frac{1}{z-\xi} + \sum_{n=1}^{k} a_n(z-\xi)^n$.

Theorem 11. Let $f(z) \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$ be given by (3)satisfies condition, (18)

$$\operatorname{Re}\left\{\frac{f(z)}{f_k(z)}\right\} \geq \frac{(1+\gamma)d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)-(1-\beta)}{(1+\gamma)d_{k+1}(\lambda,\alpha)\omega_{k+1}(\alpha_1)} \qquad (z-w\in\Delta) \tag{34}$$

where

$$d_n(\lambda,\gamma) \geq \begin{cases} (1-\gamma)(1-2\lambda), & if \quad n=1,2,3,\ldots,k \\ d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1), & if \quad n=k+1,k+2,\ldots. \end{cases} \tag{35}$$

The result (34) is sharp with the function given by

$$f(z) = \frac{1}{z-\xi} + \frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}(z-\xi)^{k+1}. \tag{36}$$

Proof.

Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\left[\frac{f(z)}{f_k(z)} - \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)-1+\alpha}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}\right]$$

$$= \frac{1+\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1} + \left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}}{1+\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1}} \tag{37}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (37) we can write

$$w(z) = \frac{\left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}^{\tau}(\alpha_1;l;m)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}}{2+2\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1} + \left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}^{\tau}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{k=n+1}^{\infty} a_n(z-\xi)^{n+1}}.$$

Hence we obtain

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{k=n+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n| - \left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1;l;m)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)\sum_{n=k+1}^{\infty}|a_n| \leq 2-2\sum_{n=1}^{k}|a_n|$$

or, equivalently,

$$\sum_{n=1}^{k}|a_n| + \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1;l;m)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| \leq 1.$$

From the condition (18), it is sufficient to show that

$$\sum_{n=1}^{k}|a_n| + \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1;l;m)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| \leq \sum_{n=1}^{\infty}\frac{d_n(\lambda,\alpha)\omega_n(\alpha_1)}{(1-\gamma)(1-2\lambda)}|a_n|$$

which is equivalent to

$$\begin{aligned} &\sum_{n=1}^{k}\left(\frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)-1+\alpha}{(1-\gamma)(1-2\lambda)}\right)|a_n| \\ &+\sum_{n=k+1}^{\infty}\left(\frac{d_n(\lambda,\gamma)\omega_n(\alpha_1)-d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)|a_n| \\ \geq\ & 0 \end{aligned} \tag{38}$$

To see that the function given by (36) gives the sharp result, we observe that for $z = re^{i\pi/(k+2)}$

$$\begin{aligned} \frac{f(z)}{f_k(z)} &= 1+\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}(z-\xi)^n \to 1-\frac{(1-\gamma)(1-2\lambda)}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)} \\ &= \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)-1+\alpha}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)} \quad \text{when } r \to 1^- \end{aligned}$$

which shows the bound (34) is the best possible for each $k \in N$.

We next determine bounds for $f_k(z)/f(z)$.

Theorem 12. *If f of the form (3) satisfies the condition (18), then*

$$\mathrm{Re}\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)+(1-\gamma)(1-2\lambda)} \qquad (z-w\in\Delta), \tag{39}$$

where

$$d_k(\lambda,\gamma)\omega_k(\alpha_1) \geq \begin{cases} (1-\gamma)(1-2\lambda), & if \quad k=1,2,3,\ldots,n \\ d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1), & if \quad k=n+1,n+2,\ldots. \end{cases} \tag{40}$$

The result (39) is sharp with the function given by (36).

Proof.

We write

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)+(1-\gamma)(1-2\lambda)}{(1-\gamma)(1-2\lambda)}\left[\frac{f_k(z)}{f(z)} - \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)+(1-\gamma)(1-2\lambda)}\right]$$

$$= \frac{1+\sum\limits_{n=1}^{k} a_n(z-\xi)^{n+1} - \left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty} a_n(z-\xi)^{n+1}}{1+\sum\limits_{n=1}^{\infty} a_n(z-\xi)^{n+1}},$$

where

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)+(1-\gamma)(1-2\lambda)}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty}|a_n|}{2-2\sum\limits_{n=1}^{k}|a_n| - \left(\frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)-1+\alpha}{(1-\gamma)(1-2\lambda)}\right)\sum\limits_{n=k+1}^{\infty}|a_n|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{n=1}^{k}|a_n| + \frac{d_{k+1}(\lambda,\gamma)\omega_{k+1}(\alpha_1)}{(1-\gamma)(1-2\lambda)}\sum_{n=k+1}^{\infty}|a_n| \leq 1.$$

Make use of (18) to get (38). Finally, equality holds in (39) for the extremal function $f(z)$ given by (36).

7. Neighborhoods for the class $\Sigma_{w,\alpha_1}^{l,m}(\alpha,\lambda)$

Following the earlier works on neighborhoods of analytic functions by Goodman [12] and Ruscheweyh [25], we begin by introducing here the δ-neighborhood of a function $f \in \Sigma_\xi$ of the form(3) by means of the definition below:

$$N_\delta(f) := \left\{ g \in \Sigma_\xi \ : \ g(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} b_n(z-\xi)^n \text{and} \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{41}$$

Particulary for the identity function $e(z) = \frac{1}{z}$, we have

$$N_\delta(e) := \left\{ g \in \Sigma : g(z) = \frac{1}{z-\xi} + \sum_{n=1}^{\infty} b_n(z-\xi)^n \ \text{ and } \ \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}. \tag{42}$$

Theorem 13. If

$$\delta := \frac{(1-\gamma)(1-2\lambda)}{(1+(1-2\lambda)\alpha)\omega_1(\alpha_1)} \tag{43}$$

then $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma) \subset N_\delta(e)$.

Proof.

For $f \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$, Theorem 1 immediately yields

$$d_1(\lambda,\gamma)\omega_1(\alpha_1)\sum_{n=1}^{\infty} a_n \leq 1-\alpha,$$

so that

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\alpha)}{d_1(\lambda,\gamma)\omega_1(\alpha_1)} \tag{44}$$

On the other hand, from (18) and (44) that

$$\begin{aligned} \omega_1(\alpha_1)\sum_{n=2}^{\infty} na_n &\leq ((1-\gamma)(1-2\lambda)) - (1-2\lambda)\alpha\ \omega_1(\alpha_1)\sum_{n=1}^{\infty} a_n \\ &\leq ((1-\gamma)(1-2\lambda)) - (1-2\lambda)\alpha\ \omega_1(\alpha_1)\frac{(1-\alpha)}{d_1(\lambda,\gamma)\omega_1(\alpha_1)} \\ &\leq \frac{(1-\gamma)(1-2\lambda)}{1+(1-2\lambda)\alpha}, \end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n \leq \frac{(1-\gamma)(1-2\lambda)}{(1+(1-2\lambda)\alpha)\omega_1(\alpha_1)} := \delta \tag{45}$$

which, in view of the definition (42) proves Theorem .

Definition 1.

A function $f \in \Sigma_\xi$ is said to be in the class $\mathcal{M}^{l,m}_{w,\alpha_1}(\lambda,\alpha,\gamma)$ if there exists a function $g \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < (1-\gamma)(1-2\lambda), \qquad (z \in \Delta, 0 \leq \gamma < 1). \tag{46}$$

Theorem 14. If $g \in \Sigma^{l,m}_{\xi,\alpha_1}(\lambda,\gamma)$ and

$$\gamma = 1 - \frac{\delta\ d_1(\lambda,\gamma)\omega_1(\alpha_1)}{d_1(\lambda,\gamma)\omega_1(\alpha_1) - ((1-\gamma)(1-2\lambda))}, \tag{47}$$

then

$$N_\delta(g) \subset \mathcal{M}^{l,m}_{w,\alpha_1}(\lambda,\alpha,\gamma).$$

Proof. Let $f \in N_\delta(g)$. Then we find from (41) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta, \tag{48}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad (n \in N). \tag{49}$$

Since $g \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$, we have [cf. equation (18)]

$$\sum_{n=1}^{\infty} b_n \leq \frac{(1-\gamma)(1-2\lambda)}{d_1(\lambda,\gamma)\omega_1(\alpha_1)}, \tag{50}$$

so that

$$\begin{aligned} \left|\frac{f(z)}{g(z)} - 1\right| &< \frac{\sum_{n=1}^{\infty} |a_n - b_n|}{1 - \sum_{n=1}^{\infty} b_n} \\ &= \frac{\delta\ d_1(\lambda,\gamma)\omega_1(\alpha_1)}{d_1(\lambda,\gamma)\omega_1(\alpha_1) - ((1-\gamma)(1-2\lambda))} \\ &= (1-\gamma)(1-2\lambda), \end{aligned}$$

provided γ is given by (47). Hence, by definition, $f \in \Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$ for γ given by (47), which completes the proof.

Concluding Remarks: By specializing the parameters l, m, λ,the various results presented in this paper would provide interesting extensions and generalizations of those considered earlier by [1, 2, 3, 5, 18, 29, 28, 27] .In fact, by appropriately selecting these arbitrary sequences, the results presented in this paper would find further applications for the class of meromorphic functions which would incorporate a generalized form of the DziokSrivastava linear operator involving the Hadamard product (or convolution) of the function in (3) with the Fox-Wright generalization ${}_l\psi_m$ of the hypergeometric function ${}_lF_m$. Theorem 1 to 14 would thus eventually lead us further to new results for the class of functions (defined analogously to the class $\Sigma_{\xi,\alpha_1}^{l,m}(\lambda,\gamma)$ by associating instead the Fox-Wright generalized hypergeometric function ${}_l\psi_m$.

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