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LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTIONS INVOLVING $p-$ GAMMA FUNCTIONS

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Abstract. In this paper we prove that the function $f_{\alpha,\beta,p}(x) = \dfrac{\Gamma_p(x+\beta)}{p^x x^{x+\beta-\alpha}}(x+p)^{x+p+\beta-\alpha}$ is logarithmically completely monotonic on $(0,\infty)$ if $0 \leq \beta \leq \frac{1}{2}$, and $\alpha \leq \beta - e^{-4}(1-\beta)^2 exp\Big(\frac{2}{1-\beta}\Big)$.

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1. Introduction

A function f is said to be *completely monotonic* on an open interval I, if f has derivatives of all orders on I and satisfies

$$(-1)^n f^{(n)}(x) \geq 0, (x \in I, n = 0, 1, 2, \ldots). \tag{1.1}$$

If the inequality (1.1) is strict, then f is said to be *strictly completely monotonic* on I. Bernstein' Theorem (see [10]) states that f is completely monotonic on $(0,\infty)$ if and only if $f(x) = \int_0^\infty e^{-xt} d\mu(t)$, where μ is a nonnegative measure on $[0,\infty)$ such that for all $x > 0$ the integral converges.

A positive function f is said to be *logarithmically completely monotonic* (see [6]) on an open interval I, if f satisfies

$$(-1)^n [\ln f(x)]^{(n)} \geq 0, (x \in I, n = 1, 2, \ldots). \tag{1.2}$$

If the inequality (1.2) is strict, then f is said to be *strictly logarithmically completely monotonic*. Let C and L denote the set of completely monotonic functions and the set of logarithmically completely monotonic functions, respectively. The relationship between completely monotonic functions and logarithmically completely monotonic functions can be presented (see [6]) by $L \subset C$.

The *Euler gamma* function $\Gamma(x)$ is defined for $x > 0$ by $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$. The *digamma* (or *psi*) function is defined for positive real numbers x as the logarithmic derivative of Euler's gamma function, that is $\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. The following integral and series representations are valid (see [1]):

$$\psi(x) = -\gamma + \int_0^\infty \frac{e^{-t} - e^{-xt}}{1 - e^{-t}}dt = -\gamma - \frac{1}{x} + \sum_{n\geq 1}\frac{x}{n(n+x)}, \tag{1.3}$$

where $\gamma = 0.57721\cdots$ denotes Euler's constant. Euler gave another equivalent definition for the $\Gamma(x)$ (see [7]),

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\cdots(x+p)} = \frac{p^x}{x(1+\frac{x}{1})\cdots(1+\frac{x}{p})}, \quad x > 0, \tag{1.4}$$

where p is a positive integer, and

$$\Gamma(x) = \lim_{p\to\infty}\Gamma_p(x). \tag{1.5}$$

The p-analogue of the psi function is defined as the logarithmic derivative of the Γ_p function (see [7],[9]), that is

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}. \tag{1.6}$$

The function ψ_p defined in (1.6) satisfies the following properties (see [8]). It has the following series representation

$$\psi_p(x) = \ln p - \sum_{k=0}^{p}\frac{1}{x+k} = \ln p - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{1.7}$$

It is increasing on $(0,\infty)$ and it is strictly completely monotonic on $(0,\infty)$. Its derivatives are given by (see [7], [8]):

$$\psi_p^{(n)}(x) = \sum_{k=0}^{p}\frac{(-1)^{n-1}\cdot n!}{(x+k)^{n+1}} = (-1)^{n+1}\int_0^\infty \frac{t^n e^{-xt}}{1-e^{-t}}(1-e^{-pt})dt. \tag{1.8}$$

2. Main Results

Let $\alpha \in R$ and $\beta \geq 0$ be real numbers, define

$$f_{\alpha,\beta,p}(x) = \frac{\Gamma_p(x+\beta)}{p^x x^{x+\beta-\alpha}}(x+p)^{x+p+\beta-\alpha}. \tag{3.9}$$

Theorem 3.1 *The function $f_{\alpha,\beta,p}(x)$ defined by (3.9) is logarithmically completely monotonic on $(0,\infty)$ if $0 \leq \beta \leq \frac{1}{2}$ and $\alpha \leq \beta - e^{-4}(1-\beta)^2 exp\left(\frac{2}{1-\beta}\right)$.*

Proof. It is clear that

$$\ln f_{\alpha,\beta,p}(x) = \ln \Gamma_p(x+\beta) + (x+p+\beta-\alpha)\ln(x+p) - (x+\beta-\alpha)\ln x - x\ln p$$

$$\begin{aligned}[\ln f_{\alpha,\beta,p}(x)]' &= \psi_p(x+\beta) + \frac{x+p+\beta-\alpha}{x+p} + \ln(x+p) - \ln px - \frac{x+\beta-\alpha}{x} \\ &= \psi_p(x+\beta) + \frac{\beta-\alpha}{x+p} - \frac{\beta-\alpha}{x} + \ln(x+p) - \ln x - \ln p\end{aligned}$$

Using the representation of:

$$\frac{1}{x^n} = \frac{1}{\Gamma(n)}\int_0^\infty t^{n-1}e^{-xt}dt, \qquad n = 1, 2, \dots \tag{3.10}$$

$$\begin{aligned}(-1)^n[\ln f_{\alpha,\beta,p}(x)]^{(n)} &= (-1)^n\psi_p^{(n-1)}(x+\beta) - \frac{(n-2)!}{x^{n-1}} + \frac{(n-2)!}{(x+p)^{n-1}} \\ &+ \frac{(\beta-\alpha)(n-1)!}{x^n} - \frac{(\beta-\alpha)(n-1)!}{(x+p)^n} \\ &= (-1)^n\psi_p^{(n-1)}(x+\beta) - (n-2)!\Big[\frac{1}{x^{n-1}} - \frac{1}{(x+p)^{n-1}}\Big] \\ &+ (\beta-\alpha)(n-1)!\Big[\frac{1}{x^n} - \frac{1}{(x+p)^n}\Big] \\ &= \int_0^\infty \frac{t^{n-1}e^{-(x+\beta)t}}{1-e^{-t}}(1-e^{-pt})dt - \int_0^\infty (t^{n-2}e^{-xt}(1-e^{-pt})dt \\ &+ (\beta-\alpha)\int_0^\infty (t^{n-1}e^{-xt}(1-e^{-pt})dt \\ &= \int_0^\infty g_{\alpha,\beta}(t)\frac{t^{n-2}e^{-xt}}{e^t-1}(1-e^{-pt})dt, \end{aligned} \tag{3.11}$$

where

$$g_{\alpha,\beta}(t) = (\beta-\alpha)te^{(1-\beta)t} + e^{(1-\beta)t} + (\beta-\alpha-1)e^t + \alpha - \beta. \tag{3.12}$$

Since $g_{\alpha,\beta}(t) > 0$ for $0 \le \beta \le \frac{1}{2}$ and $\alpha \le \beta - e^{-4}(1-\beta)^2 exp\Big(\frac{2}{1-\beta}\Big)$ (see [5]), from (3.11) we see that when $n \ge 2$

$$(-1)^n[\ln f_{\alpha,\beta,p}(x)]^{(n)} > 0$$

in $(0,\infty)$ for $0 \le \beta \le \frac{1}{2}$. Since $[\ln f_{\alpha,\beta,p}(x)]'$ is increasing, we have

$$[\ln f_{\alpha,\beta,p}(x)]' < \lim_{x\to\infty}\left[\psi_p(x+\beta)+\frac{\beta-\alpha}{x+p}-\frac{\beta-\alpha}{x}+\ln\left(1+\frac{p}{x}\right)-\ln p\right]=0. \quad (3.13)$$

Hence, for $0 \le \beta \le \frac{1}{2}$ and $\alpha \le \beta - e^{-4}(1-\beta)^2 exp\left(\frac{2}{1-\beta}\right)$ and $n \in \mathbb{N}$, $(-1)^n[\ln f_{\alpha,\beta,p}(x)]^{(n)} > 0$ in $(0,\infty)$. The proof is complete. □

Remark 3.2 Let p tend to infinty, then we obtain Thorem 1,(1) of (see [5]).

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