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RICCI FLOW ON SOME TYPE OF DIFFERENTIABLE MANIFOLDS

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Abstract. In the present paper we study first the behavior of Ricci flow on Riemannian manifold satisfying certain condition on the Ricci tensor. Then we study the uniform boundedness of $R(x,t)$ and $|\nabla f(x,t)|$ and using maximum principle we obtain uniform boundedness of $f(x,t)$, where $f(x,t) = -log\phi(x,t)$ and the metric $g(x,t) = \phi(x,t)g_E, g_E$ being the standard Euclidean metric on $\Re^n$. Then we study the behavior of scalar curvature, Riemannian curvature tensor and Weyl tensor on η-Einstein manifolds under Ricci flow. Next we study the volume form of different type of manifolds under Ricci flow. We have also obtained the value of k on $N(k)$-contact η-Einstein manifold ($k \neq 0$) using critical points under gradient Ricci soliton. Finally we study the eigenvalues of symmetric endomorphism Q on a special type of trans-Sasakian manifold and on LP-Sasakian manifold satisfying certain condition under gradient Ricci soliton.

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1. Introduction

Ricci flow on a smooth, compact and without boundary Riemannian manifold M, equipped with a Riemannian metric g, means the process by which the metric g is allowed to evolve under the parabolic PDE [21]

$$\frac{\partial g}{\partial t} = -2Ric(g) \tag{1}$$

where $Ric(g)$ is Ricci curvature tensor which depends upon g.
The behaviour of the flow depends on the topology of the underlying manifolds.

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It is introduced by R.S.Hamilton [8] in the year 1982 and proved its existence. Later much simpler proof has been given by DeTurck [21]. This concept was developed to answer Thurston's geometric conjecture which says that each closed three-manifold admits a geometric decomposition.
Hamilton himself and many other researchers like Cao [4], Yau [25], B.Chow, P.Lu, L.Ni [5], G.Perelman[16], [17], J.W.Morgan and G.Tian [14] developed the theory of Ricci flow.
In this paper we study Ricci flow on some type of differentiable manifolds. Ricci tensor plays an important role in differential geometry.
In 2006 De and Matsuyama studied quasi conformally flat manifolds [7] satisfying

$$Ric(g) = R\eta \otimes \eta. \tag{2}$$

Later in [6], authors studied pseudo-projectively flat manifolds satisfying the condition (2). So firstly we study Ricci flow where the Ricci tensor satisfies (2), where R is the scalar curvature, η is a non-zero 1-form and we study the behavior of Ricci flow on a closed Riemannian manifold satisfying the equation (2).
In 2009 J.Isenberg and M.Javaheri studied convergence of Ricci flow on $\Re^2$ to flat space in [10] and obtained some interesting results. In 2010 Li Ma and L.Cheng in [11] studied some conditions to control curvature tensors of Ricci flow. Motivated by their papers we have studied unifom boundedness of scalar curvature and we have also studied behavior of scalar curvature, Weyl curvature tensor and Riemannian curvature tensor on η-Einstein manifold under some conditions. Next we recall that in any dimension $n \geq 3$, the Riemannian curvature tensor admits an orthogonal decomposition [11]

$$Rm = -\frac{R}{2(n-1)(n-2)} g \bigodot g + \frac{1}{n-2} Ric \bigodot g + W \tag{3}$$

where W is the Weyl tensor and $\bigodot$ denotes the Kulkarni-Nomizu product [1].
A Ricci soliton is a generalisation of an Einstein metric. In a Riemannian manifold (M, g), g is called a Ricci soliton [9] if

$$\pounds_X g + 2\ Ric + 2\lambda g = 0 \tag{4}$$

where $\pounds$ is the Lie derivative, X is a complete vector field on M and λ is a constant. If the vector field X is the gradient of a potential function $-f$, then g is called a gradient Ricci soliton and equation (4) assumes the form

$$\nabla^2 f = Ric + \lambda g \tag{5}$$

An odd-dimensional differentiable manifold M^{2n+1} is said to be an almost contact manifold [22] if it admits a (1,1) tensor field ϕ, a vector field ξ and a 1-form η, satisfying

$$\eta(\xi) = 1 \quad and \quad \phi^2 = -I + \eta \otimes \xi \tag{6}$$

One can deduce from (6) that

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \tag{7}$$

If an almost contact manifold M^{2n+1} admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{8}$$

for vector fields X, Y, then M^{2n+1} is said to have an almost contact metric structure and g is called compatible metric.
From (8) we have

$$g(\phi X, Y) = -g(X, \phi Y), \quad g(X, \xi) = \eta(X) \tag{9}$$

An almost contact manifold M is said to be $\eta-$Einstein if its non-zero Ricci tensor Ric is of the form

$$Ric(g) = ag + b\eta \otimes \eta \tag{10}$$

for vector fields X, Y on M, and a, b are smooth functions on M.
An almost contact metric structure becomes a contact metric structure if

$$g(\phi X, Y) = d\eta(X, Y) \tag{11}$$

for all $X, Y \epsilon\, TM$. In a contact metric manifold M, the (1,1)-tensor field h defined by $2h = \pounds_\xi \phi$, is symmetric and satisfies

$$h\xi = 0, \qquad h\phi + \phi h = 0 \tag{12}$$

$$\nabla\xi = -\phi - \phi h \tag{13}$$

where ∇ is the Levi-Civita connection.
Let M be a (2n+1)-dimensional almost contact manifold with almost contact structure (ϕ, ξ, η). We define a linear map J on the product manifold $M \times \Re$ by

$$J(X, f\tfrac{d}{dt}) = (\phi X - f\xi, \eta(X)\tfrac{d}{dt})$$

then $J^2 = -I$. Thus J induces an almost complex structure on $M \times \Re$. The almost complex structure J is said to be integrable if its Nijenhuis tensor N_J vanishes, that is

$$N_J(X,Y) = J^2[X,Y] + [JX,JY] - J[JX,Y] - J[X,JY] = 0$$

If the almost complex structure J on $M \times \Re$ is integrable, we say the almost complex structur (ϕ,ξ,η) is normal. A contact manifold with a normal contact metric structure is said to be a Sasakian manifold.
In [3], Blair, Koufogiorgos and Papantoniou introduced a class of contact metric manifold M, which satisfies

$$Rm(X,Y)\xi = (kI + \mu h)(\eta(Y)X - \eta(X)Y), \qquad X,Y \epsilon TM \tag{14}$$

where k,μ are real constants. A contact metric manifold belonging to this class is called a (k,μ)-manifold. When $\mu = 0$, the manifold is called a $N(k)$-contact metric manifold and the Ricci operator Q satisfies

$$Q\xi = 2nk\xi \tag{15}$$

where $dim\ M = 2n+1$.
If a $N(k)-$contact metric manifold is $\eta-$Einstein, then we call it a $N(k)$ $\eta-$Einstein manifold.
We denote the Ricci curvature by

$$Ric(X,Y) = trRm(X,.,Y,.).$$

An almost contact manifold M is called trans-Sasakian manifold of type (α,β) ([12], [15], [18]) if it admits a (1,1) tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Riemannian metric g satisfying (6), (7), (8) and (9) such that

$$(\nabla_X\phi)Y = \alpha(g(X,Y)\xi - \eta(Y)X) + \beta(g(\phi X,Y)\xi - \eta(Y)\phi X) \tag{16}$$

for some smooth functions α and β on M. From (16) it follows that

$$\nabla_X\xi = -\alpha\phi X + \beta(X - \eta(X)\xi). \tag{17}$$

From [18] we have

$$\begin{aligned} Rm(\xi,X)\xi &= (\alpha^2 - \beta^2 - \xi\beta)(\eta(X)\xi - X), & (18)\\ Ric(X,\xi) &= (2n(\alpha^2-\beta^2) - \xi\beta)\eta(X) - (2n-1)X\beta - (\phi X)\alpha & (19)\end{aligned}$$

When $\phi(grad\ \alpha) = (2n-1)grad\ \beta$, then (19) reduces to

$$\begin{aligned} Ric(X,\xi) &= 2n(\alpha^2-\beta^2)\eta(X), & (20)\\ Q\xi &= 2n(\alpha^2-\beta^2)\xi & (21)\end{aligned}$$

where Ric denotes the Ricci curvature tensor.
A differentiable manifold M of dimension n is called LP-Sasakian [13], if it admits an (1,1) tensor field ϕ, a contravariant vector field ξ, a covariant vector field η and a Lorentzian metric g such that

$$\begin{aligned} \eta(\xi) &= -1, && (22)\\ \phi^2 &= I + \eta \otimes \xi, && (23)\\ g(\phi X, \phi Y) &= g(X,Y) + \eta(X)\eta(Y), && (24)\\ g(X,\xi) &= \eta(X),\ \nabla_X \xi = \phi X, && (25)\\ (\nabla_X \phi)Y &= [g(X,Y) + \eta(X)\eta(Y)]\xi + [X + \eta(X)\xi]\eta(Y), && (26) \end{aligned}$$

where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g.
It can be easily seen that in an LP-Sasakian manifold the following relations hold:

$$\phi\xi = 0, \qquad \eta(\phi X) = 0, \qquad rank\ \phi = n-1.$$

Let M^{2n+1} be an almost contact metric manifold with (ϕ, ξ, η, g) structure. The vector field ξ is called the killing vector field with respect to g if

$$(\pounds_\xi g)(X,Y) = 0$$

Let M be a $(2n+1)$-dimensional almost contact metric manifold. If the vector field ξ is a killing vector field, then M is said to be a K-contact Riemannian manifold. Here we recall the following significant results.

Theorem 1.1 *Every three-dimensional K-contact manifold is Sasakian.*

Theorem 1.2 *If a K-contact manifold M Ricci-symmetric, then the manifold is Einstein.*

Theorem 1.3 *If a Sasakian manifold of dimension $n(=2m+1)$ is a $(m \geq 1)$ Einstein manifold, then its scalar curvature is $R = n(n-1)$.*

The conharmonic curvature tensor H of type (1,3) on a Riemannian manifold (M, g) of dimension n is defined by [2]

$$\begin{aligned} H(X,Y)Z &= Rm(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \qquad (27) \end{aligned}$$

for all vector fields X, Y, Z on M and $g(QX, Y) = Ric(X, Y)$. If H vanishes identically on M, then we say that the manifold is conharmonically flat.

A Riemannian manifold M of dimension n is conformally flat if and only if the Weyl conformal curvature tensor C defined by [7]

$$\begin{aligned} C(X,Y)Z &= Rm(X,Y)Z - \frac{1}{n-2}[Ric(Y,Z)X - Ric(X,Z)Y \\ &+ g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{R}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{28}$$

where R is the scalar curvature of the manifold.

In this paper we have also discussed about the volume form of K-contact manifold which is Ricci symmetric, volume form of a LP-Sasakian manifold satisfying $Rm(X,Y).C = 0$. In last three sections we have studied the behavior of gradient Ricci soliton for a $N(k)$-contact η-Einstein manifold on a special type of conharmonically flat trans-Sasakian manifold and LP-Sasakian manifold.
Finally we state the weak maximum principle for scalars [5].

Theorem 1.4 *Suppose $g(t)$ is a family of metrics on a closed manifold M^n and $u \epsilon M^n \times [0,T) \to \Re$ satisfies*

$$\frac{\partial u}{\partial t} \leq \triangle_{g(t)} u + g(X(t), \nabla u) + F(u) \tag{29}$$

where $X(t)$ is a time-dependent vector field and F is a Lipschitz function. If $u \leq p$ at $t = 0$ for some $p \epsilon \Re$, then $u(x,t) \leq \varphi(t)$ for all $x \epsilon\ M^n$ and $t \epsilon [0,T], 0 < T < \infty$, where $\varphi(t)$ is the solution to the ODE

$$\frac{d\varphi(t)}{dt} = F(\varphi(t)) \qquad with \qquad \varphi(0) = p \tag{30}$$

All these results will be required in next sections.

2. The behavior of Ricci flow satisfying (2).

Theorem 2.1 *Suppose $g(t)$, $t\epsilon[0,T]$ is a Ricci flow, satisfying (2), on a closed Riemannian manifold M. If $R \geq \alpha\epsilon\Re$ at time $t=0$, then for all times $t\epsilon[0,T]$,*

$$g(t) \leq n\eta \otimes \eta logc(\frac{2\alpha t}{n} - 1), \quad t\epsilon[0,T] \tag{31}$$

where c is a constant. When $t = \frac{(c+1)n}{2c\alpha}$, then $g(t)$ will collapse.

Proof. Let $g(t)$ be a Ricci flow on a closed Riemannian manifold M^n where $t\epsilon$ $[0,T]$. From [21] we have, if $R \geq \alpha\epsilon\Re$ at time $t=0$, then for all times $t\epsilon[0,T]$,

$$R \geq \frac{\alpha}{1-(\frac{2\alpha}{n})t}.$$

We consider the Ricci flow satisfying (2). Then we have from (1), if $R \geq \alpha\epsilon\Re$ at $t=0$, then

$$\frac{\partial g}{\partial t} = -2R\eta \otimes \eta$$

$$\leq \frac{2\alpha}{(\frac{2\alpha}{n})t-1}\eta \otimes \eta$$

Hence

$$g(t) \leq n\eta \otimes \eta logc(\frac{2\alpha t}{n} - 1), \quad t\epsilon[0,T]$$

where c is a constant.

Corollary 2.1 *Suppose $g(t)$, $t\epsilon(0,T]$ is a Ricci flow, satisfying (2), on a closed Riemannian manifold M. Then for all $t\epsilon(0,T]$,*

$$g_{ij} \leq n\eta \otimes \eta logct, \quad t\epsilon(0,T] \tag{32}$$

where c is a constant. When $t = \frac{1}{c}$, then $g(t)$ will collapse.

Proof. From [21] we have, for a Ricci flow $g(t), t\epsilon(0,T]$ on a closed manifold M, the scalar curvature $R \geq -\frac{n}{2t}$. So if the Ricci flow satisfies (2), then from (1)

$$\frac{\partial g}{\partial t} = -2R\eta \otimes \eta$$

$$\leq \frac{n}{t}\eta \otimes \eta$$

Hence

$$g_{ij} \leq n\eta \otimes \eta logct, \quad t\epsilon(0,T]$$

where c is a constant.

3.The uniform boundedness of scalar curvature

Theorem 3.1 *Let $g(t) = \phi(x,t)g_E$ be the Ricci flow starting at $g(0) = g_0$ and $f(x,t) = -log\phi(x,t)$. Then $f(x,t)$ is uniformly bounded for all $(x,t)\epsilon\ \Re^n \times [0,\infty)$.*

Proof. Consider the metric of the form

$$g(x,t) = \phi(x,t)g_E$$

where g_E is the standard Euclidean metric on $\Re^n$. We assume here that $g_0 = \phi_0 g_E$ has bounded scalar curvature $|R_0| < k_0$ and that $\phi_0(x) = \phi(x,0)$ is bounded. Then it follows from standard elliptic gradient estimates that $|\nabla u_0|$ is bounded on $\Re^n$. Let $g(t) = \phi(x,t)g_E$ be the Ricci flow starting at $g(0) = g_0$. The long-term existence of the flow follows from [24]. Replacing the quantity $u(x,t)$ for the moment by

$$f(x,t) = -log\phi(x,t)$$

we have the following initial-value problem

$$\frac{\partial}{\partial t}f = \triangle_{g(t)}f = R_{g(t)}, \qquad f(x,0) = f_0(x).$$

Applying theorem 2.4 from [24] to this flow we obtain a uniform bound on $R(x,t)$ as well as a uniform bound on $|\nabla f(x,t)|$. Hence by theorem 1.4, $f(x,t)$ is uniformly bounded for all $(x,t)\epsilon\ \Re^n \times [0,\infty)$.

4.The behavior of scalar curvature, Weyl tensor and Riemannian curvature tensor on η-Einstein manifold under Ricci flow.

In [23] Wang has shown that if $Ric(g)$ is uniformly bounded from below on $[0,T)$, where $T < \infty$ with the bound of R

$$||R||_\alpha = (\int_0^T \int_M |R|^\alpha d\mu dt)^{\frac{1}{\alpha}}, \qquad \alpha \geq \frac{n+2}{2}$$

then $||Rm||$ is uniformly bounded. With the help of this result Li Ma and Liang Cheng in [11] have shown that the uniform bounds about $||R||_{\frac{n+2}{2}}$ and $||W||_{\frac{n+2}{2}}$ are enough to control $||Rm||$. So in this section we apply these results on η-Einstein manifold.

Theorem 4.1 *Let* $(M^n, g(t)), t\epsilon[0,T)$, *where* $T < \infty$, *be a solution to the Ricci flow (1) on a closed* η*-Einstein manifold, then* $\sup_{M\times[0,T)} |Rm| < \infty$.

Proof. Here $|R| = (an+b)$. So, $\int_0^T \int_M |R|^{\frac{n+2}{2}} d\mu dt = (an+b)^{\frac{n+2}{2}} VT$. Hence

$$||R||_{\frac{n+2}{2}} = [(an+b)^{\frac{n+2}{2}} VT]^{\frac{2}{n+2}} \tag{33}$$

Again $|W| = |Rm| + \frac{4n}{n-2}(|a|+|b|) + \frac{2Rn}{(n-1)(n-2)} = A_n$,say.

So, $\int_0^T \int_M |W|^{\frac{n+2}{2}} d\mu dt = (A_n)^{\frac{n+2}{2}} VT$

and hence

$$||W||_{\frac{n+2}{2}} = (\int_0^T \int_M |W|^{\frac{n+2}{2}} d\mu dt)^{\frac{2}{n+2}} = [(A_n)^{\frac{n+2}{2}} VT]^{\frac{2}{n+2}} \tag{34}$$

Since $||R||_{\frac{n+2}{2}} < \infty$ and $||W||_{\frac{n+2}{2}} < \infty$, so from [11] (see theorem 1.1) we have the required result.

Corollary 4.1 *Let* $(M^n, g(t)), t\epsilon[0,T)$, *where* T *may be infinite, be a solution to the Ricci flow (1) on a complete* η*-Einstein manifold with bounded sectional curvature at* $t = 0$, *then* $\sup_{M\times[0,T)} |Rm| < \infty$.

Proof. Since $\sup_{M\times[0,T)} |R| < \infty$ and $\sup_{M\times[0,T)} |W| < \infty$, then from [11] (see theorem 1.2) we get the required result.

5. The volume form of different type of manifolds under Ricci flow

Theorem 5.1 *Let* M *be a three-dimensional K-contact manifold which is Ricci-symmetric, then the volume form is given by*

$$V = V_0 + ce^{-6t} \tag{35}$$

.

Proof. From [21] we have

$$\frac{\partial}{\partial t}dV = \frac{1}{2}(tr\ h)dV$$

where $V(t) = Vol((M, g(t)))$ and $h = \frac{\partial g}{\partial t}$.

So for a Ricci flow, $h = -2Ric(g)$.

Hence $tr\ h = -2R$. So we have under Ricci flow

$$\frac{\partial}{\partial t}dV = -RdV$$

Hence

$$\frac{dV}{dt} = -\int RdV. \tag{36}$$

Now let M be a three-dimensional K-contact manifold which is Ricci-symmetric. Then using theorem 1.1, theorem 1.2 and theorem 1.3 we have the value of scalar curvature is 6. Then using (36) the volume form is given by

$$V = V_0 + ce^{-6t}$$

where V_0 is the initial volume and c is the constant of integration.

Theorem 5.2 *Let M be a LP-Sasakian manifold satisfying $Rm(X,Y).C = 0$, then the volume form is given by*

$$V = V_0 + ce^{-n(n-1)t}. \tag{37}$$

Proof. If we consider a LP-Sasakian manifold satisfying $Rm(X,Y).C = 0$ where $Rm(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y, then from [19] the manifold is conformally flat. Again the scalar curvature of a conformally flat LP-Sasakian manifold is $R = n(n-1)$ [19]. So using (36) the volume form is given by

$$V = V_0 + ce^{-n(n-1)t}.$$

where V_0 is the initial volume and c is the constant of integration.

6. The behavior of gradient Ricci soliton for an N(k)-contact η-Einstein manifold ($k \neq 0$) at critical points

Theorem 6.1 *Let* $(M^n, g(t))$ *be a N(k)-contact* η*-Einstein manifold with* $k \neq 0$ *and* g *a gradient Ricci soliton. Then the value of* k *is given by,* $k = -\frac{\lambda}{2n}$

Proof. Let M be a $(2n+1)$-dimensional $N(k)$ η-Einstein manifold and g a gradient Ricci soliton. Then the equation (5) can be written as

$$\nabla_X Df = QX + \lambda X \tag{38}$$

for all vector fields X on M, where D denotes the gradient operator of g.
From (38) it follows that

$$Rm(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \qquad X, Y \epsilon TM. \tag{39}$$

Since g is a metric connection, so it follows that

$$\xi\eta(Df) = (2nk + \lambda) \tag{40}$$

where (15) has been used. So

$$g(Rm(\xi, Y)Df, \xi) = g(k(Df - (2nk + \lambda)), Y), \qquad Y \epsilon TM \tag{41}$$

where (14) and (40) are used.
Also in a $N(k)$ $\eta-$Einstein manifold

$$g(Rm(\xi, Y)Df, \xi) = 0, \qquad Y \epsilon TM. \tag{42}$$

From (41) and (42) we get

$$k(Df - (2nk + \lambda)) = 0$$

that is, either $k = 0$ or

$$Df = 2nk + \lambda. \tag{43}$$

Here we suppose $k \neq 0$. Now at critical point p, $(Df)(p) = 0$.
So, using (43) we have $k = -\frac{\lambda}{2n}$.

7.Gradient Ricci soliton on a special type of conharmonically flat trans-Sasakian manifold

Theorem 7.1 *Let M be a conharmonically flat trans-Sasakian manifold of type (α, β) satisfying*

$$\phi(grad\ \alpha) = (2n-1) grad\ \beta \tag{44}$$

and

$$\alpha^2 - \beta^2 \neq \xi\beta \tag{45}$$

together with $Rm(X,Y).Ric = 0$ and g a gradient Ricci soliton. Then

$$\lambda = -2n(\alpha^2 - \beta^2) \text{ or } 4n(\alpha^2 - \beta^2)$$

where λ is given by (4).

Proof. Let (M, g) be a trans-Sasakian manifold of type (α, β) satisfying (44) and (45) and g a gradient Ricci soliton. Then the equation (5) can be written as

$$\nabla_X Df = QX + \lambda X \tag{46}$$

for all vector fields X on M, where D denotes the gradient operator of g.
From (46) it follows that

$$Rm(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \qquad X, Y \epsilon TM. \tag{47}$$

Since g is a metric connection, so it follows that

$$\xi\eta(Df) = (2n(\alpha^2 - \beta^2) + \lambda) \tag{48}$$

where (21) has been used. So with the help of (18) and (48) we have

$$g(Rm(\xi, Y)Df, \xi) = g((\alpha^2 - \beta^2 - \xi\beta)(Df - (2n(\alpha^2 - \beta^2) + \lambda)), Y), \qquad Y \epsilon TM. \tag{49}$$

Also in a trans-Sasakian manifold

$$g(Rm(\xi, Y)Df, \xi) = 0, \qquad Y \epsilon TM. \tag{50}$$

From (49) and (50) we get

$$(\alpha^2 - \beta^2 - \xi\beta)(Df - (2n(\alpha^2 - \beta^2) + \lambda)) = 0$$

that is, either $\alpha^2 - \beta^2 - \xi\beta = 0$ or

$$Df = 2n(\alpha^2 - \beta^2) + \lambda. \tag{51}$$

By the hypothesis (45) we have (51) holds true. Hence

$$g(\nabla_X Df, Y) = 0, \qquad X, Y \epsilon TM. \tag{52}$$

Let μ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then

$$QX = \mu X. \tag{53}$$

Using (53) in (46) and taking $Y = \xi$ we have from (52), $\lambda = -\mu$, since η is a non-zero 1-form.
Now from [20] we have if, moreover, the manifold is conharmonically flat together with $Rm(X,Y).Ric = 0$, then there are two values of μ, namely, $2n(\alpha^2 - \beta^2)$ or $-4n(\alpha^2 - \beta^2)$.

8.Gradient Ricci soliton on conharmonically flat LP-Sasakian manifold

Theorem 8.1 *Let M^n $(n \geq 3)$ be a conharmonically flat LP-Sasakian manifold satisfying $Rm(X,Y).Ric = 0$ and g a gradient Ricci soliton. Then*

$$\lambda = -(n-1) \qquad \text{or} \qquad 2(n-1)$$

where λ is given by (4).

Proof. For a conharmonically flat LP-Sasakian manifold [2]

$$QX = -X - n\eta(X)\xi. \tag{54}$$

Then taking $X = \xi$ and using (22) we have

$$Q\xi = (n-1)\xi. \tag{55}$$

Let g be a gradient Ricci soliton. Then the equation (5) can be written as

$$\nabla_X Df = QX + \lambda X \tag{56}$$

for all vector fields X on M, where D denotes the gradient operator of g.
From (56) it follows that

$$Rm(X,Y)Df = (\nabla_X Q)Y - (\nabla_Y Q)X, \qquad X, Y \epsilon TM. \tag{57}$$

Since g is a metric connection, so it follows that

$$\xi\eta(Df) = ((n-1) + \lambda) \tag{58}$$

where (55) has been used. So

$$g(Rm(\xi, Y)Df, \xi) = g(-(Df + ((n-1) + \lambda)), Y), \qquad Y \epsilon TM \tag{59}$$

where (57) and (58) are used.
Also in a LP-Sasakian manifold

$$g(Rm(\xi, Y)Df, \xi) = 0, \qquad Y \epsilon TM. \tag{60}$$

From (59) and (60) we get

$$Df + ((n-1) + \lambda) = 0$$

that is

$$Df = -((n-1) + \lambda). \tag{61}$$

Hence

$$g(\nabla_X Df, Y) = 0, \qquad X, Y \epsilon TM. \tag{62}$$

Let μ be the eigenvalue of the endomorphism Q corresponding to an eigenvector X. Then

$$QX = \mu X. \tag{63}$$

Using (63) in (56) and taking $Y = \xi$ we have from (62), $\lambda = -\mu$, since η is a non-zero 1-form.
Now from [2] we have if, moreover, the manifold satisfies $Rm(X, Y).Ric = 0$, then there are two values of μ, namely, $-2(n-1)$ and $(n-1)$.

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