Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 99-108

INVERSION OF REGULAR AND SINGULAR PERTURBED MATRICES

Abbas Ali Shokri and Ali Shokri

Abstract. In this paper, for matrices $A, B \in \mathbb{R}^{n\times n}$ such that A is singular, we first assume that $R(A, B; x) = (A + xB)^{-1}$ exist in a deleted neighborhood of $x = 0$ and discuss in the behavior of the $R(A, B; x)$. Finally by considering $x = 1$, we compute the inverse of R also.

2000 *Mathematics Subject Classification*: 41A58, 74G10.

Key words: Laurent series, singularly perturbed matrix, nearly singular matrix.

1. Introduction

In this paper, we study the inversion of regularly and singularly perturbed matrix $A(x) = A + xB$. In the singularly case, we are mainly interested in the case A is singular but $A(x)$ has an inverse in some punctured disc around $x = 0$. It is know that $R(A, B; x)$ can be expanded as a Laurent series at the origin. The main purpose of this paper is provided efficient computational procedures for the coefficients of this series. Let $A, B \in \mathbb{R}^{n\times n}$, we are interested here in the invertibility of the matrix $A + xB$, for real number $x > 0$. Since $\det(A + xB)$ is not identically zero(the regularity condition), if A is nonsingular then

$$R(A, B; x) = (A + xB)^{-1}, \tag{1}$$

exists in a deleted neighborhood of the origin. Then we shall be show the resolvent is holomorphic with a pole at the origin of degree at most n. Thee effect of perturbations (for small values of x) can be either small or large. The mathematical reasons for this difference underlie the classification of problems into either regular or singular perturbation problems. More precisely, we have the following definition

Definition 1.1. *Let $A(x) = A + xB$ be $n \times n$ matrix. In all cases it will be assumed that $\mathrm{rank}[A(x)] = n$ for $x > 0$ and sufficiently small. There are now two cases:*

(i) Regular Perturbation: A^{-1} *exists whenever* $(A + xB)^{-1}$ *exists for* $x > 0$ *and sufficiently small.*
(ii) Singular Perturbation: $rank[A] < n$. *That is when the matrix* A *does not have an inverse butt he perturbed matrix* $A + xB$ *dos for sufficiently small* x *but distinct from zero.*

We know that if A is nonsingular, then $R(A, B; x)$ has Taylor expansion because we may write:

$$R(A, B; x) = (A + xB)^{-1} = ((I + xBA^{-1})A)^{-1} = A^{-1}(I + xBA^{-1})^{-1}.$$

Now if we define $K = xBA^{-1}$ then we have:

$$\begin{aligned} R(A, B; x) &= (A + xB)^{-1} = A^{-1}(I + xBA^{-1})^{-1} \\ &= A^{-1}\sum_{i=0}^{\infty}(I + K)^{-1} = A^{-1}\sum_{i=0}^{\infty}(-K)^{i} \\ &= A^{-1}(I - xBA^{-1} + x^2(BA^{-1})^2 - x^3(BA^{-1})^3 + \cdots), \end{aligned}$$

in a neighborhood of $x = 0$ ($\|K\| < 1$ or $|x| < \frac{1}{\|BA^{-1}\|}$). The interesting case then is when A is singular ($\det A = 0$), so that the expansion of $R(A, B; x)$ has pole, and in this case when x is near 0, $(A + xB)^{-1}$ is "nearly singular" when x is near zero. But if A is nonsingular, we may also write:

$$(A + xB)^{-1} = x^{-1}A^{-1}(x^{-1}I + BA^{-1})^{-1} \ ,$$

so in this case the expansion of $(A + xB)^{-1}$ in a deleted neighborhood of $x = 0$ is related to the expansion of the resolvent operator $(\xi I - K)^{-1}$ of $K = -BA^{-1}$ about $\xi = \infty$. The inversion of nearly singular perturbed matrices was probably first studied in the paper by Keldysh [10]. In particular, he showed that the principal part of the Laurent series expansion for the inverse of $R(A, B; x)$ can be given in terms of generalized Jordan chains. The generalized Jordan chains were initially developed in the context of matrix and operator polynomials. Some authors [6,7,8] studied the linear perturbation $A(x) = A + xB$ and showed that one can express $R(A, B; x)$ as a Laurent series as long as $A(x)$ is invertible in some punctured neighborhood of the origin. Langenhop in [5] showed that the coefficients of the regular part of the Laurent series for the inverse of a linear perturbation from a geometric sequence. Shokri [1] and Howlett [9] provided a computational procedure for the Laurent series coefficients based on a sequence of row and column matrices on the coefficients of the original power series. Howlett used the rank test of Sain and Massey to determine s, the order of the pole. He also showed that the coefficients of the Laurent series satisfy a finite linear recurrence relation in the case of a polynomial perturbation.

2. Inversion Of Singularly Perturbed Matrix

Assume that $A, B \in R^{n\times n}$, and $rank(A) < n$. Then for all sufficiently small x we define $A(x) = A + xB$. In this section we will discuss in invertibility of $A(x)$. We have,

$$R(A,B;x) = \frac{adj(A+xB)}{\det(A+xB)}.$$

But we know $\det(A+xB)$ is a polynomial of x of at most of degree n. (To derivation this claim we can use induction rule). Then it has at most n zero also, and we know that the $adj(A+xB)$ is a $n\times n$ matrix of polynomials of degree at most $n-1$ (because they are determinate of matrixes be $(n-1)\times(n-1)$).
Then $R(A,B;x)$ is a $n\times n$ matrix by components of rational function $R_{ij}(x)$, such that, $R_{ij}(x) = \frac{p_{ij}(x)}{q(x)}$ where $p_{ij}(x) = [adj(A+xB)]_{ij}$, $q(x) = \det(A+xB)$. Then we have

$$R(A,B;x) = \begin{bmatrix} R_{11}(x) & \dots & R_{1n}(x) \\ \vdots & & \vdots \\ R_{n1}(x) & \dots & R_{nn}(x) \end{bmatrix}, \tag{2}$$

Lemma 2.1. *suppose $c(x) = \frac{a(x)}{b(x)}$ is a rational function with the degrees of the polynomial $a(x)$ and $b(x)$ being p and q , respectively . Then the function $c(x)$ can be expanded as a laurent series in some punctured neighborhood of zero with the order of pole s that is at most q. Moreover, if $c^{(-s)} = \ldots = c^{(p)} = 0$ then $c(x) = 0$*

Proof. see [1].

We will know that the division of two analytic function with poles with finite orders is also an analytic function with a pole of finite order. Then $c(x)$ can be expanded as a Laurent series near $x = 0$ that converges in a punctured disc of the origin with nonzero radius. Then for $1 \leq i$ and $j \leq n$ we can write

$$R_{ij}(x) = \sum_{t=-k}^{\infty} x^t (p_{ij})_t, \tag{3}$$

where $k \leq n$ and the quantities p_{ij} are independent of x. Now if we replace $R_{ij}(x)$ by (3) in (2) then we can write

$$R(A,B;x) = \begin{bmatrix} \sum_{t=-k}^{\infty} x^t (p_{11})_t & \dots & \sum_{t=-k}^{\infty} x^t (p_{1n})_t \\ \vdots & & \vdots \\ \sum_{t=-k}^{\infty} x^t (p_{n1})_t & \dots & \sum_{t=-k}^{\infty} x^t (p_{nn})_t \end{bmatrix}. \tag{4}$$

So we have

$$R(A,B;x) = \sum_{t=-k}^{\infty} x^t \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}_t . \tag{5}$$

Then

$$R(A,B;x) = \sum_{t=-k}^{\infty} x^t Q_t, \tag{6}$$

where

$$Q_t = \begin{bmatrix} p_{11} & \cdots & p_{1n} \\ \vdots & & \vdots \\ p_{n1} & \cdots & p_{nn} \end{bmatrix}_t .$$

And $k \leq n$ and the matrices Q_t are independent of x. If $A(x)$ becomes singular at $x = 0$, then above series will have pole of order k at $x = 0$ and will contain a nontrivial singular part, defined by

$$A^S(x) = \frac{1}{x^k} Q_{(-k)} + \cdots + \frac{1}{x} Q_{(-1)}. \tag{7}$$

Similarly, a regular part of (2.5) is defined by

$$A^R(x) = Q_{(0)} + xQ_{(1)} + \cdots \tag{8}$$

In recent years Shokri [1], Langenhop [5], Lamond [2], Huang [4], in some papers have investigated the resolvent (1), giving necessary and sufficient conditions for the Laurent expansion (6) to exist. Moreover, he has shown that the matrices Q_t in the Laurent expansion are uniquely determined by Q_{-1} and Q_0 as is summarized in the following theorem.

Theorem 2.2. *If $R(A,B;x)$ exists in a deleted neighborhood of the origin, then it has a Laurent expansion of the form* (6) *where*

$$Q_i = (-Q_0 B)^i Q_0, \qquad 0 \leq i, \tag{9}$$

$$Q_{-i} = (-Q_{-1} A)^{i-1} Q_{-1}, \qquad 1 \leq i \leq k, \tag{10}$$

and

$$0 = (-Q_{-1} A)^k Q_{-1} \tag{11}$$

Moreover, if we separate $R(A,B;x)$ into its regular and singular parts by writing

$$R(A,B;x) = R(A,B;x)^R + R(A,B;x)^S \ ,$$

where

$$R(A,B;x)^R = \sum_{i=0}^{\infty} Q_i x^i \quad and \quad R(A,B;x)^S = \sum_{i=-1}^{-k} Q_i x^i \ , \tag{12}$$

and set $P = BQ_{-1}$ *and* $\tilde{P} = Q_{-1}B$ *then* P *and* $\tilde{P}$ *are idempotent matrices satisfying:*

$$R(A,B;x)^R = R(A,B;x)(I-P) \quad and \quad R(A,B;x)^S = R(A,B;x)P \tag{13}$$

and

$$R(A,B;x)^R = (I-\tilde{P})R(A,B;x) \quad and \quad R(A,B;x)^S = \tilde{P}R(A,B;x) \tag{14}$$

Proof. We begin this proof whit the identity

$$(A+xB)R(A,B;x) = I. \tag{15}$$

Then the regular part of (15) is

$$(A+xB)R(A,B;x)^R + BQ_{-1} = I,$$

which on premultiplication by $(A+xB)^{-1}$ yields the left part of (13)

$$R(A,B;x)^R = R(A,B;x)(I-BQ_{-1}) = R(A,B;x)(I-P)$$

It then follow that

$$R(A,B;x)^S = R(A,B;x) - R(A,B;x)^R = R(A,B;x)P \tag{16}$$

which is the right part of (13). The coefficient of x^{-1} in (16) is

$$Q_{-1} = Q_{-1}P \tag{17}$$

Premultiplication of (17) by B gives $P^2 = P$, which establishes the idempotency of P. The corresponding result for $\tilde{P}$ are obtained in a similar manner from the identity $R(A,B;x)(A+xB) = I$ For later use note that the coefficient of x^{-1} in (15) is

$$AQ_{-1} + BQ_{-2} = 0 \tag{18}$$

The remaining derivations employ the resolvent equation

$$R(A,B;x_2) - R(A,B;x_1) = (x_1 - x_2)R(A,B;x_2)BR(A,B;x_1) \tag{19}$$

which may by derived by observing that both sides of (19) are expressions for

$$R(A,B;x_2)[(A+x_1B)-(A+x_2B)]R(A,B;x_1)$$

By projection of (19) we obtain separate resolvent equations for the regular and singular parts:

$$R(A,B;x_2)^R - R(A,B;x_1)^R = (x_1-x_2)R(A,B;x_2)^R BR(A,B;x_1)^R \tag{20}$$

and

$$R(A,B;x_2)^s - R(A,B;x_1)^s = (x_1-x_2)R(A,B;x_2)^s BR(A,B;x_1)^s \tag{21}$$

To derive (2.8), set $x_2 = 0$ in (19) to get

$$(I + x_1Q_0B)R(A,B;x_1)^R = Q_0.$$

Then

$$R(A,B;x)^R = (I+xQ_0B)^{-1}Q_0 = \sum_{i=0}^{\infty}(-xQ_0B)^iQ_0$$

from which (9) follows immediately. To derive (10) and (11) note first that the coefficient of x_1^{-1} in (21) is

$$-Q_{-1} = R(A,B;x_2)^s BQ_{-2} - x_2R(A,B;x_2)^s BQ_{-1} \tag{22}$$

On the right side of (22) replace BQ_{-2} by the value obtained from (18) and replace $R(A,B;x_2)^s BQ_{-1}$ by

$$R(A,B;x_2)^s BQ_{-1} = R(A,B;x_2)^s P = R(A,B;x_2)^s$$

to get $R(A,B;x_2)^s(x_2I + AQ_{-1}) = Q_{-1}$ Hence for all sufficiently large x

$$R(A,B;x)^s = x^{-1}Q_{-1}(I+AQ_{-1})^{-1} = \sum_{i=0}^{\infty} x^{-i-1}Q_{-1}(-AQ_{-1})^i \tag{23}$$

Equating coefficients of powers of x in the sides of (23) yields (11) and (12).

3. The Inversion Of Regularly Perturbed Matrix

For a given $n\times n$ non-singular matrix A, its inverse matrix A^{-1} is first evaluated. If the original matrix A is perturbed by an $n \times n$ diverting matrix B, the inverse of this perturbed matrix $(A + B)$ may be found from [1-3]

$$(A+B)^{-1} = A^{-1} - A^{-1}(A^{-1}+B^{-1})^{-1}A^{-1}. \tag{24}$$

Obviously, (24) is not a feasible formula for computing $(A+B)^{-1}$, even though A^{-1} is already known beforehand. Besides, it requires B and $(A^{-1}+B^{-1})$ to be non-singular. For convenience, let

$$A^{-1} = D; \tag{25}$$

then

$$\begin{aligned} (A+B)^{-1} &= D - D(I+BD)^{-1}BD \\ &= D - DB(I+DB)^{-1}D. \end{aligned} \tag{26}$$

The non-singularity requirement for B in (24) is thus removed.

Let the matrices B and D be partitioned as

$$[B] = \begin{bmatrix} \overline{\underline{B}} & 0 \\ 0 & 0 \end{bmatrix}, \qquad [D] = \begin{bmatrix} \overline{\underline{D}} & D_2 \\ D_1 & D_3 \end{bmatrix} \tag{27}$$

and

$$\overline{D} = \begin{bmatrix} \overline{\underline{D}} \\ D_1 \end{bmatrix}, \qquad \underline{D} = \begin{bmatrix} \overline{\underline{D}} & D_2 \end{bmatrix} \tag{28}$$

where the element positions of $\overline{\underline{D}}$ are the transport element positions of $\overline{\underline{B}}$. That is, the partitioned matrices $\overline{\underline{D}}$, $\overline{\underline{D}}$, $\overline{D}$ and $\underline{D}$ are of order of $m_1 \times m_2$, $m_2 \times m_1$, $n \times m_2$, and $m_1 \times n$, respectively. Then (26) may be rewritten as

$$(A+B)^{-1} = D + H \tag{29}$$

where

$$\begin{aligned} H &= -\overline{D}(I + \overline{\underline{B}}\ \overline{\underline{D}})^{-1}\overline{\underline{B}}\ \underline{D} \\ &= \overline{D}\ \overline{\underline{B}}(I + \overline{\underline{D}}\ \overline{\underline{B}})^{-1}\underline{D}. \end{aligned} \tag{30}$$

It is noted that $(I + \overline{\underline{B}}\ \overline{\underline{D}})^{-1}$ and $(I + \overline{\underline{D}}\ \overline{\underline{B}})^{-1}$ are respectively the inverses of square matrices of order m_1 and m_2, where $m_1,\ m_2 \leq n$. In particular, if $m_1 = 1$ or $m_2 = 1$, one of these inversion factors becomes a scalar quantity (a single element matrix), that can be easily evaluated directly without going through the usual matrix inversion procedure. It is noted also that $\overline{\underline{B}}$ does not necessary have to be a solid matrix located at the upper left corner of $[B]$. If non-zero elements are scattering within $[B]$, then $\overline{\underline{B}}$ is formed through the selected rows and columns that must cover these elements entirely. It follows that $\overline{\underline{D}}$ as well as $\overline{D}$ and $\underline{D}$ are determined from $[D]$ accordingly.

4. Example

For a given matrix A, and its inverse A^{-1} which has been computed:

$$[A] = \begin{bmatrix} 1.5 & -0.5 & -1.5 & 2.5 & -3.0 \\ -3.0 & 1.5 & 2.0 & -3.0 & 4.0 \\ -1.0 & 0.5 & 1.0 & -1.0 & 1.0 \\ 2.0 & -0.5 & -1.0 & 2.0 & -2.0 \\ -1.0 & 0.5 & 0.0 & -0.5 & 0.5 \end{bmatrix}$$

$$[D] = [A]^{-1} = \begin{bmatrix} 4.0 & 4.0 & -3.0 & -1.0 & -6.0 \\ 4.0 & 4.0 & -2.0 & 0.0 & -4.0 \\ -2.0 & -2.0 & 3.0 & 1.0 & 2.0 \\ -6.0 & -5.0 & 4.0 & 3.0 & 8.0 \\ -2.0 & -1.0 & 0.0 & 1.0 & 2.0 \end{bmatrix}$$

we want to find the inverse matrix $[A+B]^{-1}$, where A is perturbed by various matrices B.

(1) For

$$[B] = \begin{bmatrix} 0. & 0. & 0. & 0. & 0. \\ 0. & -1.5 & 0. & -1.0 & +2.0 \\ 0. & 0. & 0. & 0. & 0. \\ 0. & +1.5 & 0. & 0. & +1.0 \\ 0. & 0. & 0. & 0. & 0. \end{bmatrix}$$

Then we have

$$\underline{\overline{B}} = \begin{bmatrix} -1.5 & -1.0 & +2.0 \\ +1.5 & 0.0 & +1.0 \end{bmatrix}, \qquad \underline{\overline{D}} = \begin{bmatrix} 4.0 & 0.0 \\ -5.0 & 3.0 \\ -1.0 & 1.0 \end{bmatrix}$$

$$\overline{D} = \begin{bmatrix} 4.0 & -1.0 \\ 4.0 & 0.0 \\ -2.0 & 1.0 \\ -5.0 & 3.0 \\ -1.0 & 1.0 \end{bmatrix}, \qquad \underline{D} = \begin{bmatrix} 4.0 & 4.0 & -2.0 & 0.0 & -4.0 \\ -6.0 & -5.0 & 4.0 & 3.0 & 8.0 \\ -2.0 & -1.0 & 0.0 & 1.0 & 2.0 \end{bmatrix}$$

and

$$[I + \underline{\overline{B}}\ \underline{D}]^{-1} = \begin{bmatrix} -2.0 & -1.0 \\ 5.0 & 2.0 \end{bmatrix}^{-1} = \begin{bmatrix} 2.0 & 1.0 \\ -5.0 & -2.0 \end{bmatrix}$$

$$[I + \underline{\overline{D}}\ \overline{B}]^{-1} = \begin{bmatrix} -5.0 & -4.0 & 8.0 \\ 12.0 & 6.0 & -7.0 \\ 3.0 & 1.0 & 0.0 \end{bmatrix}^{-1} = \begin{bmatrix} 7.0 & 8.0 & -20.0 \\ -21.0 & -24.0 & 61.0 \\ -6.0 & -7.0 & 18.0 \end{bmatrix}$$

Then

$$H = \begin{bmatrix} 28.0 & 9.0 & 31.0 & 7.0 & -2.0 \\ 16.0 & 4.0 & 20.0 & 4.0 & 0.0 \\ -20.0 & -7.0 & -21.0 & -5.0 & 2.0 \\ -56.0 & -20.0 & -58.0 & -14.0 & 6.0 \\ -16.0 & -6.0 & -16.0 & -4.0 & 2.0 \end{bmatrix}$$

Thus

$$[A + B]^{-1} = [D + H] = \begin{bmatrix} 32.0 & 13.0 & 28.0 & 6.0 & -8.0 \\ 20.0 & 8.0 & 18.0 & 4.0 & -4.0 \\ -22.0 & -9.0 & -18.0 & -4.0 & 4.0 \\ -62.0 & -25.0 & -54.0 & -11.0 & 14.0 \\ -18.0 & -7.0 & -16.0 & -3.0 & 4.0 \end{bmatrix}$$

ACKNOWLEDGEMENTS. Taken from the "A study of reversibility of singular perturbed matrix and presentation of a modern method in the application of asymptotic simplex" project which has been performed by the budget provided by the vice research of Islamic Azad University - Ahar Branch.

References

[1] A. Shokri, *A.M. Method for finding inversion of matrices that are singular at the origin*, 36th Annual Iranian Mathematics Conference, pp80, Sept. 10-13, 2005.

[2] B. F. Lamond, *An efficient basis update for asymptotic linear programming*, Linear Algebra and Appl., 184 (1993), 83-102.

[3] B. F. Lamond, *A generalized inverse method for asymptotic linear programming*, Math. program. 43(1987), 71-86.

[4] Y. Huang, *A canonical form for pencils of matrices with applications to asymptotic linear programs*, Linear Algebra programming and Appl., 234(1996), pp.97-123.

[5] C.E. Langenhop, *On the invertibility of a nearly singular matrix*, Linear Algebra. Appl. 7: 361-365 (1973).

[6] I. Gohberg, M. A. Kaashoek and F.VanSchagen, *On the local theory of regular analytic matrix functions*, Linear Algebra and its Applications, v. 182, pp. 9-25, 1993.

[7] I. Gohberg, P. Lancaster and L. Rodman, *Matrix Polynomials*, Computer Science and Applied Mathematics Series, Academic Press, 1982.

[8] I. C. Gohberg and E. I. Sigal, *An operator generalization of the logarithmic residue theorem and the theorem of Rouché*, Math. USSR Sbornik, 13 (1971), 603-625.

[9] P. G. Howlett, *Input retieval in finite dimensional linear systems*, J. Austral. Math. Soc.(seriesB),23(1982), 357-382.

[10] M. V. Keldysh, *On the characteristic values and characteristic functions of certain classes of non-self adjoint equations*, Dokl. Akad. NaukUSSR, 77 (1951), 11-14.

Abbas Ali Shokri
Department of Mathematics
Ahar Branch, Islamic Azad University
Ahar, Iran
email:*a-shokri@iau-ahar.ac.ir*

Ali Shokri
Department of Mathematics
University of Maragheh
Maragheh, Iran
email: *shokri@maragheh.ac.ir*