Acta Universitatis Apulensis No. 27/2011
ISSN: 1582-5329 pp. 51-56

OPERATOR ON HILBERT SPACE AND ITS APPLICATION TO CERTAIN UNIVALENT FUNCTIONS WITH A FIXED POINT

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Abstract. By making use of the operators on Hilbert space, the authors introduce a new class of univalent functions with a fixed point. Coefficient estimate, distortion bounds and extreme points are obtained. Also the effect of a operator on functions in this class is investigated.

2000 *Mathematics Subject Classification*: 30C45, 30C50.

1. Introduction and Motivation

Let w be a fixed point and S_w denote the class of functions $f(z)$ of the form

$$f(z) = \frac{A}{z-w} + \sum_{n=1}^{+\infty} \alpha_n (z-w)^n, \tag{1}$$

where A is the Residue of $f(z)$ in $z = w$, $0 < A \leq 1$.
Let N_w denoted the subclass of S_w consisting of functions in the form

$$f(z) = \frac{A}{z-w} - \sum_{n=1}^{+\infty} \alpha_n (z-w)^n. \tag{2}$$

For the function $f(z) \in N_w$, we consider the operator I^k as follow:

$$I^0 f(z) = f(z),$$

$$I^1 f(z) = (z-w) f'(z) + \frac{2A}{z-w} \quad ,$$

and for $k \geq 2$,

$$\begin{aligned} I^k f(z) &= (z-w)(I^{k-1} f(z))' + \frac{2A}{z-w} \\ &= \frac{A}{z-w} - \sum_{n=1}^{+\infty} n^k \alpha_n (z-w)^n. \end{aligned} \tag{3}$$

For more information about the operator I^k see [3,4].

Definition 1. *The function $f \in N_w$ is said to be a member of the class $N_w^k(\beta,\gamma,\theta)$ if it satisfies*

$$\left|\frac{(z-w)^3[I^kf(z)]''+(z-w)[I^kf(z)]'-A}{2(z-w)[I^kf(z)]-\beta(1+\theta)A}\right|<\gamma, \tag{4}$$

where β,γ,θ belong to $[0,1)$.

Let H be a Hilbert space on the $\mathbb{C}$ and T be a linear operator on H. Also $f(T)$ be the operator on H defined by Riesz-Dunford integral [1]

$$2\pi i f(T)=\int_c f(z)(zI-T)^{-1}dz, \tag{5}$$

where c is a positively oriented simple closed rectifiable contour lying in $\Delta=\{z:|z|<1\}$ and containing the spectrum of T in its interior domain and I is the identity operator on H. see [2].

Definition 2. *A function $f(z)$ given by (2) is in the class $N_w^k(\beta,\gamma,\theta,T)$ if for all operator T with $\|T\|<1$ and $T\neq 0$ it satisfy the inequality*

$$\|T^3[I^kf(T)]''+T^2[I^kf(T)]'-A\|<\gamma\|-2TI^kf(T)-\beta(1+\theta)A\|, \tag{6}$$

where β,γ,θ are in $[0,1)$.

The operators on Hilbert space were considered recently by Ghanim and Darus [5], Joshi [6], and Xiaopei [7].

2. Main results

In this section we obtain coefficient bounds and distortion property for a function $f\in N_w^k(\beta,\gamma,\theta,T)$.

Theorem 2.1. *A function $f(z)$ given by (4) is in the class $N_w^k(\beta,\gamma,\theta,T)$ for all $T\neq 0$ if and only if*

$$\sum_{n=1}^{+\infty}\frac{(n^2+2\gamma)}{A\gamma(2-\beta(1+\theta))}a_n\leq 1. \tag{7}$$

The result is sharp for the function $F(z)$ given by

$$F(z)=\frac{A}{z-w}-\frac{A\gamma(2-\beta(1+\theta))}{(n^2+2\gamma)}(z-w)^n \quad ,n\geq 1 \tag{8}$$

Proof. Suppose that (7) holds, we have

$$\|T^3f''(T)+T^2f'(T)-A\|-\gamma\|2Tf(T)-\beta(1+\theta)A\|=$$

$$\| - \sum_{n=1}^{+\infty} n^2 n^k a_n T^{n+1}\| - \gamma \|A(2-\beta(1+\theta)) - \sum_{n=1}^{+\infty} 2n^k a_n T^{n+1}\|$$

$$\leq \sum_{n=1}^{+\infty}(n^2+2\gamma)a_n - \gamma A(2-\beta(1+\theta)) \leq 0.$$

Hence f is in the class $N_w^k(\beta,\gamma,\theta,T)$.
Conversely, suppose that

$$\|T^3 f''(T) + T^2 f'(T) - A\| < \gamma \|2Tf(T) - \beta(1+\theta)A\|,$$

so

$$\| - \sum_{n=1}^{+\infty} n^2 n^k a_n T^{n+1}\| <$$

$$\gamma \|A(2-\beta(1+\theta)) - \sum_{n=1}^{+\infty} 2n^k a_n T^{n+1}\|.$$

Setting $T = qI \quad (0 < q < 1)$ in the above inequality, we get

$$\frac{\sum_{n=1}^{+\infty} n^2 n^k a_n q^{n+1}}{A(2-\beta(1+\theta)) - \sum_{n=1}^{+\infty} 2n^k a_n q^{n+1}} < \gamma. \tag{9}$$

Upon clearing denominator in (9) and letting $q \to 1$, we obtain

$$\sum_{n=1}^{+\infty} n^2 n^k a_n < A\gamma(2-\beta(1+\theta)) - \sum_{n=1}^{+\infty} 2\gamma n^k a_n,$$

or

$$\sum_{n=1}^{+\infty}(n^2+2\gamma)a_n \leq A\gamma(2-\beta(1+\theta)),$$

which completes the proof.

Corollary: *If $f(z)$ given by (4) be in the class $N_w^k(\beta,\gamma,\theta,T)$ Then*

$$a_n \leq \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma} \quad , n \in \mathbb{N}. \tag{10}$$

Theorem 2.2. *If $f(z)$ of the form (4) be in the class $N_w^k(\beta,\gamma,\theta,T)$, $\|T\| < 1$ and $\|T\| \neq 0$. Then*

$$\|\frac{A}{T}\| - \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}\|T\|^n \leq \|f(T)\| \leq \|\frac{A}{T}\| + \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}\|T\|^n. \tag{11}$$

The result is sharp for the function $F(z)$ given by (8).

Proof. According to the Theorem 2.1, we get

$$\sum_{n=1}^{+\infty} a_n \leq \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}.$$

So we have

$$\|f(T)\| \geq \|\frac{A}{T}\| - \|T\|^n \sum_{n=1}^{+\infty} a_n$$

$$\geq \|\frac{A}{T}\| - \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}\|T\|^n$$

and

$$\|f(T)\| \leq \|\frac{A}{T}\| + \|T\|^n \sum_{n=1}^{+\infty} a_n$$

$$\leq \|\frac{A}{T}\| + \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}\|T\|^2.$$

Hence the proof is complete.

3. Extreme points and operators

In this section we discuss about extreme points of $N_w^k(\beta,\gamma,\theta,T)$ and effect of operator on functions in this class.

Theorem 3.1. *Let $f_0(z) = \frac{A}{z-w}$ and*

$$f_n(z) = \frac{A}{z-w} - \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}(z-w)^n \quad ,n \geq 1.$$

Then $f(z) \in N_w^k(\beta,\gamma,\theta,T)$ if and only if it can be expressed by

$$f(z) = \sum_{n=0}^{+\infty} t_n f_n(z),$$

where $t_n \geq 0$ and $\sum_{n=0}^{+\infty} t_n = 1$.

Proof. Let

$$f(z) = \sum_{n=0}^{+\infty} t_n f_n(z)$$

$$= \frac{A}{z-w} + \sum_{n=1}^{+\infty} t_n \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}(z-w)^n.$$

Since

$$\sum_{n=1}^{+\infty} \frac{n^2+2\gamma}{A\gamma(2-\beta(1+\theta))} \times \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma} t_n = \sum_{n=1}^{+\infty} t_n = 1 - t_0 \le 1,$$

so by Theorem 2.1 we get $f(z) \in N_w^k(\beta, \gamma, \theta, T)$.
Conversely, suppose that $f(z) \in N_w^k(\beta, \gamma, \theta, T)$. Then by (10) we have

$$a_n \le \frac{A\gamma(2-\beta(1+\theta))}{n^2+2\gamma}.$$

Setting

$$t_n = \frac{n^2+2\gamma}{A\gamma(2-\beta(1+\theta))} a_n,$$

and

$$t_0 = 1 - \sum_{n=1}^{+\infty} t_n,$$

we conclude the required result.
Theorem 3.2. *If $f(z) \in N_w^k(\beta, \gamma, \theta, T)$, then the function $F_c(z)$ defined by*

$$F_c(z) = c\int_0^1 [\nu^c f(z\nu + w(1-\nu))]d\nu, \quad c \ge 1,$$

is also in the same class.

Proof. Since $f(z) \in N_w^k(\beta, \gamma, \theta, T)$ is of the form (4), so

$$F_c(z) = c\int_0^1 \left\{ \nu^c \left[\frac{A}{\nu(z-w)} - \sum_{n=1}^{+\infty} a_n[\nu(z-w)]^n \right] \right\} d\nu$$

$$= \frac{A}{z-w} - \sum_{n=1}^{+\infty} \frac{c}{c+n+1} a_n (z-w)^n.$$

Since $\frac{c}{c+n+1} < 1$, by using Theorem 2.1 we conclude that

$$F_c(z) \in N_w^k(\beta, \gamma, \theta, T).$$

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