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SANDWICH THEOREMS FOR ANALYTIC FUNCTIONS INVOLVING WRIGHT'S GENERALIZED HYPERGEOMETRIC FUNCTION

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Abstract. In this paper, we obtain some applications of first order differential subordination and superordination results involving Wright's generalized hypergeometric function defined by certain p-valent analytic functions.

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1.Introduction

Let $A(p)$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C}, |z| < 1\}$.
Let $H(U)$ be the class of analytic functions in U and let $H[a, p]$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} ... \ (a \in \mathbb{C}).$$

For simplicity, let $H[a] = H[a, 1]$. Also, let $A(1) = A$ be the subclass of $H(U)$ consisting of functions of the form:

$$f(z) = z + a_2 z^2 + ... \ . \tag{2}$$

If f, $g \in H(U)$, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, which (by definition) is analytic in U with $w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$, $z \in U$. Furthermore, if the function $g(z)$ is univalent in U, then we have the following equivalence, (cf., e.g.,[6], [19]; see also [20]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \ and \ f(U) \subset g(U).$$

Let $p, h \in H(U)$ and let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \to \mathbb{C}$. If p and $\varphi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second order superordination

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{3}$$

then p is a solution of the differential superordination (3). Note that if f is subordinate to g, then g is superordinate to f. An analytic function q is called a subordinant if $q(z) \prec p(z)$ for all p satisfying (3). A univalent subordinant $\widetilde{q}$ that satisfies $q \prec \widetilde{q}$ for all subordinants of (3) is called the best subordinant. Recently Miller and Mocanu [21] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{4}$$

Using the results of Miller and Mocanu [21], Bulboača [4] considered certain classes of first order differential superordinations as well as superordination-preserving integral operators [5] many researshers ([1] and [29]) have obtained sufficient conditions on normalized analytic functions $f(z)$ by means of differential subordinations and superordinations.

For analytic functions $f(z) = \sum\limits_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum\limits_{n=0}^{\infty} b_n z^n$, the Hadamard product (*orconvolution*) of $f(z)$ and $g(z)$, is defined by

$$(f * g)(z) = \sum_{n=0}^{\infty} a_n b_n z^n. \tag{5}$$

Let $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive and real parameters such that

$$1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geq 0.$$

The Wright generalized hypergeometric function [31] (see also [12])

$${}_q\Psi_s\left[(\alpha_1, A_1), ..., (\alpha_q, A_q); (\beta_1, B_1), ..., (\beta_s, B_s); z\right] = {}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right]$$

is defined by

$${}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}; (\beta_i, B_i)_{1,s}; z\right] = \sum_{n=0}^{\infty} \frac{\prod\limits_{i=1}^{q} \Gamma(\alpha_i + nA_i)}{\prod\limits_{i=1}^{s} \Gamma(\beta_i + nB_i)} . \frac{z^n}{n!} \quad (z \in U).$$

If $A_i = 1 (i = 1, ..., q)$ and $B_i = 1\,(i = 1, ..., s)$, we have the relationship:

$$\Omega_q\Psi_s\left[(\alpha_i, 1)_{1,q}\,; (\beta_i, 1)_{1,s}\,; z\right] = {}_qF_s\left(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z\right),$$

where ${}_qF_s\left(\alpha_1, ..., \alpha_q; \beta_1, ..., \beta_s; z\right)$ is the generalized hypergeometric function (see for details [9], [10], [11], [16]) and

$$\Omega = \frac{\prod\limits_{i=1}^{s}\Gamma\left(\beta_i\right)}{\prod\limits_{i=1}^{q}\Gamma\left(\alpha_i\right)}. \tag{6}$$

The Wright generalized hypergeometric functions were invoked in the geometric function theory (see [8], [9], [10], [11], [25], [26] and [27]).

By using the generalized hypergeometric function Dziok and Srivastava [10] introduced a linear operator. In [8] Dziok and Raina and in [2] Aouf and Dziok extended the linear operator by using Wright generalized hypergeometric function.

First we define a function ${}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\,; z\right]$ by

$${}_q\Phi_s^p\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\,; z\right] = \Omega z^p\,{}_q\Psi_s\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\,; z\right]$$

and consider the following linear operator

$$\theta_{p,q,s}\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\right] : A(p) \to A(p), \tag{7}$$

defined by the convolution

$$\theta_{p,q,s}\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\right] f\left(z\right) ={}_q \Phi_s^p\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\,; z\right] * f\left(z\right).$$

We observe that, for a function $f\left(z\right)$ of the form (1), we have

$$\theta_{p,q,s}\left[(\alpha_i, A_i)_{1,q}\,; (\beta_i, B_i)_{1,s}\right] f\left(z\right) = z^p + \sum_{n=p+1}^{\infty}\Omega\sigma_{n,p}\left(\alpha_1\right) a_n z^n, \tag{8}$$

where Ω is given by (6) and $\sigma_{n,p}\left(\alpha_1\right)$ is defined by

$$\sigma_{n,p}\left(\alpha_1\right) = \frac{\Gamma\left(\alpha_1 + A_1\left(n-p\right)\right)...\Gamma\left(\alpha_q + A_q\left(n-p\right)\right)}{\Gamma\left(\beta_1 + B_1\left(n-p\right)\right)...\Gamma\left(\beta_s + B_s\left(n-p\right)\right)\left(n-p\right)!}. \tag{9}$$

If, for convenience, we write

$$\theta_{p,q,s}\left[\alpha_1, A_1, B_1\right] f\left(z\right) = \theta_{p,q,s}\left[\left(\alpha_1, A_1\right), ..., \left(\alpha_q, A_q\right); \left(\beta_1, B_1\right), ..., \left(\beta_s, B_s\right)\right] f\left(z\right),$$

then one can easily verify from (8) that

$$zA_1\left(\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\right)' = \alpha_1\theta_{p,q,s}\left[\alpha_1+1,A_1,B_1\right]f(z)$$

$$-(\alpha_1-pA_1)\theta_{p,q,s}\left[\alpha_1,A_1,B_1\right]f(z)\ \ (A_1>0). \tag{10}$$

We note that for $A_i = 1\,(i=1,2,...,q)$ and $B_i = 1\,(i=1,2,...,s)$, we obtain $\theta_{p,q,s}\left[\alpha_1,1,1\right]f(z) = H_{p,q,s}[\alpha_1]f(z)$, which was introduced and studied by Dziok and Srivastava [10]. Also for $f(z)\in A$, the linear operator $\theta_{1,q,s}\left[\alpha_1,A_1,B_1\right] = \theta\left[\alpha_1\right]$ was introduced by Dziok and Raina [8] and studied by Aouf and Dziok [2].

We note that, for $f(z)\in A(p)$, $A_i = 1\,(i=1,...,q)$, $B_i = 1\,(i=1,...,s)$, $q=2$ and $s=1$, we have:

(i) $\theta_{p,2,1}\left[a,1;c\right]f(z) = L_p(a,c)f(z)\ \ (a,c>0,\ p\in\mathbb{N})$ (see Saitoh [28]);

(ii) $\theta_{p,2,1}\left[\mu+p,p;p\right]f(z) = D^{\mu+p-1}f(z)\ \ (\mu>-p,\ p\in\mathbb{N})$, where $D^{\mu+p-1}f(z)$ is the $(\mu+p-1)$ −the order Ruscheweyh derivative of a function $f(z)\in A(p)$ (see Kumar and Shukla [13] and [14];

(iii) $\theta_{p,2,1}\left[1+p,1;1+p-\mu\right]f(z) = \Omega_z^{(\mu,p)}f(z)$, where the operator $\Omega_z^{(\mu,p)}$ is defined by (see Srivastava and Aouf [30])

$$\Omega_z^{(\mu,p)}f(z) = \frac{\Gamma(1+p-\mu)}{\Gamma(1+p)}z^{\mu}D_z^{\mu}f(z)\ \ (0\le\mu<1;p\in\mathbb{N}),$$

where D_z^{μ} is the fractional derivative operator (see, for details [23] and [24]);

(iv) $\theta_{p,2,1}\left[\nu+p,1;\nu+p+1\right]f(z) = J_{\nu,p}(f)(z)$, where $J_{\nu,p}(f)(z)$ is the generalized Bernardi-Libera-Livingston-integral operator $(see\,[3],\,[15]\,and\,[18])$, defined by

$$J_{\nu,p}(f)(z) = \frac{\nu+p}{z^{\nu}}\int_0^z t^{\nu-1}f(t)\,dt\ \ (\nu>-p;p\in\mathbb{N});$$

(v) $\theta_{p,2,1}\left[p+1,1;n+p\right]f(z) = I_{n,p}f(z)\ (n\in\mathbb{Z},\ n>-p,\ p\in\mathbb{N})$, where the operator $I_{n,p}$ was considered by Liu and Noor [17];

(vi) $\theta_{p,2,1}\left[\lambda+p,c;a\right]f(z) = I_p^{\lambda}(a,c)f(z)\ \left(a,c\in R\backslash\mathbb{Z}_0^-,\ \lambda>-p,\ p\in\mathbb{N}\right)$, where $I_p^{\lambda}(a,c)$ is the Cho-Kwon-Srivastava operator [7].

The main object of the present paper is to find sufficient condition for certain normalized analytic functions $f(z)$ in U such that $\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Psi)(z)\neq 0$ and $f(z)$ to satisfy

$$q_1(z) \prec \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \prec q_2(z),$$

where q_1, q_2 are given univalent functions in U and $\Phi(z) = z^p + \sum\limits_{n=p+1}^{\infty} \lambda_n z^n$, $\Psi(z) = z^p + \sum\limits_{n=p+1}^{\infty} \mu_n z^n$ are analytic functions in U with λ_n , $\mu_n \geq 0$. Also, we obtain number of known results as special cases.

2. Definitions and Preliminaries

In order to prove our results, we shall make use of the following definition and lemmas.

Definition 1 [21]. *Denote by Q, the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where*

$$E(f) = \{\xi \in \partial U : \lim_{z \to \xi} f(z) = \infty\}$$

and are such that $f'(\xi) \neq 0$ for $\xi \in \partial U \setminus E(f)$.

Lemma 1 [20]. *Let q be univalent in the unit disk U and θ and φ be analytic in a domain D containing $q(U)$ with $\varphi(w) \neq 0$ when $w \in q(U)$. Set*

$$\psi(z) = zq'(z)\varphi(q(z)) \text{ and } h(z) = \theta(q(z)) + \psi(z). \tag{11}$$

Suppose that
(i) $\psi(z)$ is starlike univalent in U,
(ii) $Re\left\{\frac{zh'(z)}{\psi(z)}\right\} > 0$ for $z \in U$.
If p is analytic with $p(0) = q(0)$, $p(U) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)), \tag{12}$$

then $p(z) \prec q(z)$ and q is the best dominant.

Lemma 2 [4]. *Let q be convex univalent in U and ϑ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*
(i) $Re\{\vartheta'(q(z))/\phi(q(z))\} > 0$ for $z \in U$ and
(ii) $\psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in U.
If $p(z) \in H[q(0), 1] \cap Q$, with $p(U) \subseteq D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in U and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)), \tag{13}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3. Subordination results for analytic functions

Unless otherwise mentioned, we shall assume in the reminder of this paper that, $\alpha_1, A_1, ..., \alpha_q, A_q$ and $\beta_1, B_1, ..., \beta_s, B_s$ $(q, s \in \mathbb{N})$ be positive real parameters, $\Phi,\ \Psi \in A(p),\ p \in \mathbb{N},\ \xi_1, \xi_2, \xi_3 \in \mathbb{C}$ and $\xi_4 \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$.

Theorem 1. *Let q be convex univalent in U with $q(0) = 1$. Further, assume that*

$$Re\left\{\frac{\xi_3}{\xi_4} + \frac{2\xi_2}{\xi_4}q(z) + \left(1 + \frac{zq''(z)}{q'(z)}\right)\right\} > 0 \quad (z \in U). \tag{14}$$

If $f \in A(p)$ satisfies

$$\digamma(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \tag{15}$$

where

$$\digamma(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)}\right)^2 + \xi_3\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)}\right) \\ +\xi_4\left(\alpha_1\frac{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)} - (\alpha_1+1)\frac{\theta_{p,q,s}[\alpha_1+2,A_1,B_1](f*\Psi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)} + 1\right) \\ \times\left(\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)}\right), \end{cases} \tag{16}$$

then

$$\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \prec q(z)$$

and q is the best dominant.

Proof. Define the function $p(z)$ by

$$p(z) = \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \quad (z \in U). \tag{17}$$

Then the function $p(z)$ is analytic in U and $p(0) = 1$, Therefore, by making use of (17), we obtain

$$\xi_1 + \xi_2 \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \right)^2 + \xi_3 \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \right)$$

$$+\xi_4 \left(\alpha_1 \frac{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)} - (\alpha_1 + 1) \frac{\theta_{p,q,s}[\alpha_1 + 2, A_1, B_1](f * \Psi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} + 1 \right)$$

$$\times \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \right)$$

$$= \xi_1 + \xi_2 p^2(z) + \xi_3 p(z) + \xi_4 z p'(z). \tag{18}$$

By using (15) in (18), we have

$$\xi_1 + \xi_2 p^2(z) + \xi_3 p(z) + \xi_4 z p'(z) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4 z q'(z).$$

By setting

$$\theta(w) = \xi_1 + \xi_2 w^2 + \xi_3 w \qquad \textit{and} \qquad \varphi(w) = \xi_4,$$

it can be easily observed that $\theta(w)$ is analytic in $\mathbb{C}$, $\varphi(w)$ is analytic in $\mathbb{C}^*$ and that $\varphi(w) \neq 0,\ w \in \mathbb{C}^*$, then by using Lemma 1 we complete the proof of Theorem 1.

Putting $\Phi(z) = \Psi(z) = \frac{z^p}{1-z}$ (or, $\mu_n = \lambda_n = 1,\ n \geq p+1,\ p \in \mathbb{N}$) in Theorem 1, we obtain the following corollary:

Corollary 1. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If $f \in A(p)$ and satisfies*

$$Z(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ z q'(z), \tag{19}$$

where

$$Z(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2 \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)} \right)^2 + \xi_3 \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)} \right) \\ +\xi_4 \left(\alpha_1 \frac{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)} - (\alpha_1 + 1) \frac{\theta_{p,q,s}[\alpha_1+2, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)} + 1 \right) \\ \times \left(\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1] f(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1] f(z)} \right), \end{cases} \tag{20}$$

then

$$\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1]f\,(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1]f\,(z)} \prec q(z)$$

and q is the best dominant.

Remark 1. *Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$ and $p = 1$, in Corollary 1,we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.6].*

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$ and $s = \alpha_1 = \alpha_2 = \beta_1 = 1$ in Theorem 1, hence we obtain the next result:
Corollary 2. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If $f \in A(p)$ and satisfies*

$$B(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\; zq'(z), \tag{21}$$

where

$$B(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)+(1-p)(f*\Psi)(z)}\right)^2 + \xi_3\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)+(1-p)(f*\Psi)(z)}\right) \\ +\xi_4\left(\frac{z(f*\Phi)'(z)}{(f*\Phi)(z)} - \frac{z^2(f*\Psi)''(z)+z(3-p)(f*\Psi)'(z)+(1-p)(3-p)(f*\Psi)'(z)}{z(f*\Psi)'(z)+(1-p)(f*\Psi)(z)} + (2-p)\right) \\ \times\left(\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)+(1-p)(f*\Psi)(z)}\right), \end{cases} \tag{22}$$

then

$$\frac{(f*\Phi)(z)}{z(f*\Psi)'(z)+(1-p)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant.

Remark 2. *(i) Putting $p = 1$ in Corollary 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.7];*
(ii) Putting $p = 1$ and $\Phi(z) = \Psi(z)$ in Corollary 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.8];

(iii) Putting $p = 1$ *and* $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ *(or,* $p = 1$, $\mu_n = \lambda_n = 1$ *and* $n \geq 2$*) in Corollary 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.9].*

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq A < B \leq 1)$ in Theorem 1, we obtain the next result:
Corollary 3. *Let* q *be convex univalent in* U *with* $q(0) = 1$ and

$$Re\left\{\frac{\xi_3}{\xi_4} + \frac{2\xi_2}{\xi_4}\left(\frac{1+Az}{1+Bz}\right) + \frac{1-Bz}{1+Bz}\right\} > 0 \quad (z \in U).$$

If $f \in A(p)$, $-1 \leq A < B \leq 1$ *and*

$$F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2\left(\frac{1+Az}{1+Bz}\right)^2 (z) + \xi_3\left(\frac{1+Az}{1+Bz}\right) + \xi_4 \frac{z(A-B)}{(1+Bz)^2} \quad (23)$$

where $F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ is defined by (16), then

$$\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \prec \frac{1+Az}{1+Bz} \quad (24)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant.

Remark 3. *Putting* $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$ *and* $p = 1$, *in Corollary 3, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.11].*

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = a$, $\alpha_2 = 1$ and $\beta_1 = c$ $(a,\ c > 0,\ p \in \mathbb{N})$ in Theorem 1, we obtain the following corollary:
Corollary 4. *Let* q *be convex univalent in* U *with* $q(0) = 1$ *and (14) holds true. If* $f \in A(p)$ satisfies

$$K(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \quad (25)$$

where

$$K(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2 \left(\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)} \right)^2 + \xi_3 \left(\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)} \right) \\ \quad +\xi_4 \left(a \frac{L_p(a+1,c)(f*\Phi)(z)}{L_p(a,c)(f*\Phi)(z)} - (a+1) \frac{L_p(a+2,c)(f*\Psi)(z)}{L_p(a+1,c)(f*\Psi)(z)} + 1 \right) \\ \times \left(\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)} \right), \end{cases} \tag{26}$$

then

$$\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (25).

Remark 4. *Putting $p = 1$ in Corollary 4, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.5].*

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \mu+p$ $(\mu > -p,\ p \in \mathbb{N})$, $\alpha_2 = \beta_1 = p$ in Theorem 1, we obtain the following corollary:
Corollary 5. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If* $f \in A(p)$ satisfies

$$M(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \tag{27}$$

where

$$M(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2 \left(\frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)} \right)^2 + \xi_3 \left(\frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)} \right) \\ +\xi_4 \left((\mu+p) \frac{D^{\mu+p}(f*\Phi)(z)}{D^{\mu+p-1}(f*\Phi)(z)} - (\mu+p+1) \frac{D^{\mu+p+1}(f*\Psi)(z)}{D^{\mu+p}(f*\Psi)(z)} + 1 \right) \\ \times \left(\frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)} \right), \end{cases} \tag{28}$$

then

$$\frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (27).

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = 1 + p$, $\alpha_2 = 1$ and $\beta_1 = 1 + p - \mu$ $(0 \leq \mu < 1,\ p \in \mathbb{N})$, in Theorem 1, we obtain the following corollary:

Corollary 6. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If $f \in A(p)$* satisfies

$$N(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \tag{29}$$

where

$$N(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2 \left(\frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)} \right)^2 + \xi_3 \left(\frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)} \right) \\ +\xi_4 \left((1+p) \frac{\Omega_z^{(\mu+1,p)}(f*\Phi)(z)}{\Omega_z^{(\mu,p)}(f*\Phi)(z)} - (p+2) \frac{\Omega_z^{(\mu+2,p)}(f*\Psi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)} + 1 \right) \\ \times \left(\frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)} \right), \end{cases} \tag{30}$$

then

$$\frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (29).

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \nu + p$ $(\nu > -p,\ p \in \mathbb{N})$, $\alpha_2 = 1$ and $\beta_1 = \nu + p + 1$ in Theorem 1, we obtain the following corollary:

Corollary 7. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If $f \in A(p)$ satisfies*

$$L(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \tag{31}$$

where

$$L(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$= \begin{cases} \xi_1+\xi_2\left(\frac{J_{\nu,p}(f*\Phi)(z)}{J_{\nu+1,p}(f*\Psi)(z)}\right)^2+\xi_3\left(\frac{J_{\nu,p}(f*\Phi)(z)}{J_{\nu+1,p}(f*\Psi)(z)}\right) \\ +\xi_4\left((\nu+p)\frac{J_{\nu+1,p}(f*\Phi)(z)}{J_{\nu,p}(f*\Phi)(z)}-(\nu+p+1)\frac{J_{\nu+2,p}(f*\Psi)(z)}{J_{\nu+1,p}(f*\Psi)(z)}+1\right) \\ \times\left(\frac{J_{\nu,p}(f*\Phi)(z)}{J_{\nu+1,p}(f*\Psi)(z)}\right), \end{cases} \tag{32}$$

then

$$\frac{J_{\nu,p}(f*\Phi)(z)}{J_{\nu+1,p}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (31).

Putting $A_i=1$ $(i=1,2,...,q)$, $B_j=1$ $(j=1,2,...,s)$, $q=2$, $s=1$, $\alpha_1=p+1$, $\alpha_2=1$ and $\beta_1=n+p$ $(n\in\mathbb{Z},\ n>-p,\ p\in\mathbb{N})$ in Theorem 1, we obtain the following corollary:

Corollary 8. *Let q be convex univalent in U with $q(0)=1$ and (14) holds true. If $f\in A(p)$ satisfies*

$$R(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)\prec \xi_1+\xi_2 q^2(z)+\xi_3 q(z)+\xi_4\ zq'(z), \tag{33}$$

where

$$R(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$= \begin{cases} \xi_1+\xi_2\left(\frac{I_{n,p}(f*\Phi)(z)}{I_{n+1,p}(f*\Psi)(z)}\right)^2+\xi_3\left(\frac{I_{n,p}(f*\Phi)(z)}{I_{n+1,p}(f*\Psi)(z)}\right) \\ +\xi_4\left((p+1)\frac{I_{n+1,p}(f*\Phi)(z)}{I_{n,p}(f*\Phi)(z)}-(p+2)\frac{I_{n+2,p}(f*\Psi)(z)}{I_{n+1,p}(f*\Psi)(z)}+1\right) \\ \times\left(\frac{I_{n,p}(f*\Phi)(z)}{I_{n+1,p}(f*\Psi)(z)}\right), \end{cases} \tag{34}$$

then

$$\frac{I_{n,p}(f*\Phi)(z)}{I_{n+1,p}(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (33).

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \lambda + p$, $\alpha_2 = c$ and $\beta_1 = a$ $\left(a, c \in R\backslash\mathbb{Z}_0^-,\ \lambda > -p,\ p \in \mathbb{N}\right)$ in Theorem 1, we obtain the following corollary:

Corollary 9. *Let q be convex univalent in U with $q(0) = 1$ and (14) holds true. If $f \in A(p)$ satisfies*

$$G(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4) \prec \xi_1 + \xi_2 q^2(z) + \xi_3 q(z) + \xi_4\ zq'(z), \tag{35}$$

where

$$G(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$= \begin{cases} \xi_1 + \xi_2 \left(\frac{I_p^{\lambda}(a,c)(f*\Phi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)}\right)^2 + \xi_3 \left(\frac{I_p^{\lambda}(a,c)(f*\Phi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)}\right) \\ +\xi_4 \left((\lambda+p)\frac{I_p^{\lambda+1}(a,c)(f*\Phi)(z)}{I_p^{\lambda}(a,c)(f*\Phi)(z)} - (\lambda+p+1)\frac{I_p^{\lambda+2}(a,c)(f*\Psi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)} + 1\right) \\ \times \left(\frac{I_p^{\lambda}(a,c)(f*\Phi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)}\right), \end{cases} \tag{36}$$

then

$$\frac{I_p^{\lambda}(a,c)(f*\Phi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)} \prec q(z)$$

and q is the best dominant of (35).

4. Superordination results for analytic functions

Now, by applying Lemma 2, we prove the following theorem.

Theorem 2. *Let q be convex univalent in U with $q(0) = 1$, $f \in A(p)$ and $\frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$. Further, assume that*

$$Re\left\{\frac{\xi_3}{\xi_4} + \frac{2\xi_2}{\xi_4}q(z)\right\} > 0 \tag{37}$$

and $\digamma(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *be univalent in* U *and satisfy*

$$\xi_1+\xi_2 q^2(z)+\xi_3 q(z)+\xi_4\ zq'(z)\prec \digamma(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4), \tag{38}$$

where $\digamma(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *is given by (16), then*

$$q(z)\prec \frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)}$$

and q *is the best subordinant of (38).*

Proof. Define the function $p(z)$ by (17), with simple computation from (17), we get

$$\digamma(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)=\xi_1+\xi_2 p^2(z)+\xi_3 p(z)+\xi_4\ zp'(z),$$

then

$$\xi_1+\xi_2 q^2(z)+\xi_3 q(z)+\xi_4\ zq'(z)\prec \xi_1+\xi_2 p^2(z)+\xi_3 p(z)+\xi_4\ zp'(z). \tag{39}$$

By setting

$$\upsilon(w)=\xi_1+\xi_2 w^2+\xi_3 w \qquad \textit{and} \qquad \varphi(w)=\xi_4,$$

it is easy observed that $\upsilon(w)$ is analytic in $\mathbb{C}, \varphi(w)$ is analytic in $\mathbb{C}^*$ and $\varphi(w)\neq 0,\ w\in\mathbb{C}^*$. Since q is convex and univalent function, it follows that

$$Re\left\{\frac{\theta'(q(z))}{\varphi(q(z))}\right\}=\operatorname{Re}\left\{\frac{\xi_3}{\xi_4}+\frac{2\xi_2}{\xi_4}q(z)\right\}>0\ (z\in U)$$

and then, by using Lemma 2 we complete the proof of Theorem 2.

Putting $A_i=1\ (i=1,2,...,q),\ B_j=1\ (j=1,2,...,s),\ q=2,\ s=1,\ \alpha_1=a,\ \alpha_2=1$ and $\beta_1=c\ (a,\ c>0)$ in Theorem 2, we obtain the following corollary:

Corollary 10. *Let* q *be convex univalent in* U *with* $q(0)=1$, *(37) holds true* $,f\in A(p)$ *and* $\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)}\in H[q(0),1]\cap Q$. *Further, assume that* $K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *be univalent in* U *which satisfying*

$$\xi_1+\xi_2 q^2(z)+\xi_3 q(z)+\xi_4\ zq'(z)\prec K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4), \tag{40}$$

where $K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *is given by (26), then*

$$q(z) \prec \frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)}$$

and q is the best subordinant of (40).

Remark 5. *Putting $p = 1$ in Corollary 10, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.13].*

Remark 6. *(i) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$* and *$s = p = \alpha_1 = \alpha_2 = \beta_1 = 1$, in Theorem 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.14];*
(ii) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = p = \alpha_1 = \alpha_2 = \beta_1 = 1$ and $\Phi(z) = \Psi(z)$ in Theorem 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.15];
(iii) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = p = \alpha_1 = \alpha_2 = \beta_1 = 1$ and $\Phi(z) = \Psi(z) = \frac{z}{1-z}$ in Theorem 2, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.16].

Taking $q(z) = \frac{1+Az}{1+Bz}$ $(-1 \le A < B \le 1)$ in Theorem 2, we obtain the next result:
Corollary 11. *Let q be convex univalent in U with $q(0) = 1$, $f \in A(p)$,* $\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$ and

$$Re\left\{\frac{\xi_3}{\xi_4} + \frac{2\xi_2}{\xi_4}\frac{1+Az}{1+Bz}\right\} > 0 \ \ (-1 \le A < B \le 1).$$

Further, assume that $\digamma(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ be univalent in U and satisfies

$$\xi_1 + \xi_2\left(\frac{1+Az}{1+Bz}\right)^2(z) + \xi_3\left(\frac{1+Az}{1+Bz}\right) + \xi_4\,\frac{z\,(A-B)}{(1+Bz)^2} \prec \digamma(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4), \tag{41}$$

where $\digamma(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ is given by (16), then

$$\frac{1+Az}{1+Bz} \prec \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1, A_1, B_1](f*\Psi)(z)}$$

and $\frac{1+Az}{1+Bz}$ is the best subordinant (41).

Remark 7. *Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$ and $p = 1$, in Corollary 11, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 2.17].*

Remark 8. *(i) Putting* $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \mu + p$ $(\mu > -p,\ p \in \mathbb{N})$, $\alpha_2 = \beta_1 = p$ *in Theorem 2, we obtain the superordination result for the operator* $D^{\mu+p-1}$;
(ii) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = 1 + p$, $\alpha_2 = 1$ *and* $\beta_1 = 1 + p - \mu$ $(0 \le \mu < 1,\ p \in \mathbb{N})$, *in Theorem 2, we obtain the superordination result for the operator* $\Omega_z^{(\mu,p)}$;
(iii) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \nu + p$ $(\nu > -p,\ p \in \mathbb{N})$, $\alpha_2 = 1$ *and* $\beta_1 = \nu + p + 1$ *in Theorem 2, we obtain the superordination result for the operator* $J_{\nu,p}$;
(iv) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = p + 1$, $\alpha_2 = 1$ *and* $\beta_1 = n + p$ $(n \in \mathbb{Z},\ n > -p,\ p \in \mathbb{N})$ *in Theorem 2, we obtain the superordination result for the operator* $I_{n,p}$;
(v) Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \lambda + p$, $\alpha_2 = c$ *and* $\beta_1 = a$ $\left(a, c \in R \backslash \mathbb{Z}_0^-,\ \lambda > -p,\ p \in \mathbb{N}\right)$, *in Theorem 2, we obtain the superordination result for the operator* $I_p^{\lambda}(a, c)$.

4. Sandwich results

Combining Theorem 1 with Theorem 2, we get the following sandwich theorem.
Theorem 3. *Let* q_1, q_2 *be convex univalent in* U*. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f \in A(p)$, $\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$ *and* $F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is univalent in* U, which satisfying

$$\xi_1 + \xi_2 q_1^2(z) + \xi_3 q_1(z) + \xi_4\ z q_1'(z) \prec F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$\prec \xi_1 + \xi_2 q_2^2(z) + \xi_3 q_2(z) + \xi_4\ z q_2'(z), \tag{42}$$

where $F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is given by (16), then*

$$q_1(z) \prec \frac{\theta_{p,q,s}[\alpha_1, A_1, B_1](f * \Phi)(z)}{\theta_{p,q,s}[\alpha_1 + 1, A_1, B_1](f * \Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and dominant of (42).*

Taking $q_1(z) = \frac{1+A_1 z}{1+B_1 z}$ $(-1 \le A_1 < B_1 \le 1)$ and
$q_2(z) = \frac{1+A_2 z}{1+B_2 z}$ $(-1 \le A_2 < B_2 \le 1)$ in Theorem 3, we obtain the following corollary:
Corollary 12. *Let* q_1, q_2 *be convex univalent in* U*. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f \in A(p)$, $\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)(z)} \in H[q(0), 1] \cap Q$ *and* $F(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is univalent in* U, which satisfying

$$\xi_1+\xi_2\left(\frac{1+A_1z}{1+B_1z}\right)^2(z)+\xi_3\left(\frac{1+A_1z}{1+B_1z}\right)+\xi_4\ \frac{(A_1-B_1)\,z}{(1+B_1z)^2}\prec F(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$\prec\xi_1+\xi_2\left(\frac{1+A_2z}{1+B_2z}\right)^2+\xi_3\left(\frac{1+A_2z}{1+B_2z}\right)+\xi_4\ \frac{(A_2-B_2)\,z}{(1+B_2z)^2}, \tag{43}$$

where $F(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *is defined in* (16), *then*

$$\frac{1+A_1z}{1+B_1z}\prec\frac{\theta_{p,q,s}[\alpha_1,A_1,B_1](f*\Phi)\,(z)}{\theta_{p,q,s}[\alpha_1+1,A_1,B_1](f*\Psi)\,(z)}\prec\frac{1+A_2z}{1+B_2z}$$

and $\frac{1+A_1z}{1+B_1z}$, $\frac{1+A_2z}{1+B_2z}$ *are, respectively, the best subordinant and dominant of (43).*

Remark 9. *Putting* $A_i=1$ $(i=1,2,...,q)$, $B_j=1$ $(j=1,2,...,s)$, $q=2$ *and* $s=p=\alpha_1=\alpha_2=\beta_1=1$ *in Corollary 12, we obtain the result obtained by Murugusundaramoorthy and Magesh [22, Corollary 3.2].*

Putting $A_i=1$ $(i=1,2,...,q)$, $B_j=1$ $(j=1,2,...,s)$, $q=2$, $s=1$, $\alpha_1=a$, $\alpha_2=1$ and $\beta_1=c$ $(a,c>0)$ in Theorem 3, we obtain the following corollary:

Corollary 13. *Let* q_1,q_2 *be convex univalent in U. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f\in A(p)$, $\frac{L_p(a,c)(f*\Phi)(z)}{L_p(a+1,c)(f*\Psi)(z)}\in H[q(0),1]\cap Q$ *and* $K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *is univalent in* U , where $K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ *is defined in* (26), *then*

$$\xi_1+\xi_2q_1^2(z)+\xi_3q_1(z)+\xi_4\ zq_1'(z)\prec K(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$\prec\xi_1+\xi_2q_2^2(z)+\xi_3q_2(z)+\xi_4\ zq_2'(z), \tag{44}$$

implies

$$q_1(z)\prec\frac{L_p(a,c)(f*\Phi)\,(z)}{L_p(a+1,c)(f*\Psi)\,(z)}\prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and dominant of (44).*

Putting $A_i=1$ $(i=1,2,...,q)$, $B_j=1$ $(j=1,2,...,s)$, $q=2$, $s=1$, $\alpha_1=\mu+p$ $(\mu>-p,\ p\in\mathbb{N})$ and $\alpha_2=\beta_1=p$ in Theorem 3, we obtain the following corollary:

Corollary 14. *Let* q_1,q_2 *be convex univalent in U. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f\in A(p)$, $\frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)}\in H[q(0),1]\cap Q$ *and*

$M(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is univalent in U, where $M(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is defined in (28), *then*

$$\xi_1+\xi_2 q_1^2(z)+\xi_3 q_1(z)+\xi_4\ zq_1'(z)\prec M(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$\prec \xi_1+\xi_2 q_2^2(z)+\xi_3 q_2(z)+\xi_4\ zq_2'(z), \tag{45}$$

implies

$$q_1(z)\prec \frac{D^{\mu+p-1}(f*\Phi)(z)}{D^{\mu+p}(f*\Psi)(z)}\prec q_2(z)$$

and $q_1,\ q_2$ are, respectively, the best subordinant and dominant of (45).

Putting $A_i=1\ (i=1,2,...,q),\ B_j=1\ (j=1,2,...,s),\ q=2,\ s=1,\ \alpha_1=1+p,\ \alpha_2=1$ and $\beta_1=1+p-\mu\ \ (0\le\mu<1,\ p\in\mathbb{N})$ in Theorem 3, we obtain the following corollary:

Corollary 15. *Let q_1,q_2 be convex univalent in U. Suppose q_1 and q_2 satisfying (37) and (14), respectively. If $f\in A(p)$, $\frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)}\in H[q(0),1]\cap Q$ and $N(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is univalent in U, where $N(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is defined in* (30), *then*

$$\xi_1+\xi_2 q_1^2(z)+\xi_3 q_1(z)+\xi_4\ zq_1'(z)\prec N(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$$

$$\prec \xi_1+\xi_2 q_2^2(z)+\xi_3 q_2(z)+\xi_4\ zq_2'(z), \tag{46}$$

implies

$$q_1(z)\prec \frac{\Omega_z^{(\mu,p)}(f*\Phi)(z)}{\Omega_z^{(\mu+1,p)}(f*\Psi)(z)}\prec q_2(z)$$

and $q_1,\ q_2$ are, respectively, the best subordinant and dominant of (46).

Putting $A_i=1\ (i=1,2,...,q),\ B_j=1\ (j=1,2,...,s),\ q=2,\ s=1,\ \alpha_1=\nu+p\ (\nu>-p,\ p\in\mathbb{N}),\ \alpha_2=1$ and $\beta_1=\nu+p+1$ in Theorem 3, we obtain the following corollary:

Corollary 16. *Let q_1,q_2 be convex univalent in U. Suppose q_1 and q_2 satisfying (37) and (14), respectively. If $f\in A(p)$, $\frac{J_{\nu,p}(f*\Phi)(z)}{J_{\nu+1,p}(f*\Psi)(z)}\in H[q(0),1]\cap Q$ and $L(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is univalent in U, where $L(f,\Phi,\Psi,\xi_1,\xi_2,\xi_3,\xi_4)$ is defined in* (32), *then*

$$\xi_1 + \xi_2 q_1^2(z) + \xi_3 q_1(z) + \xi_4 \ zq_1'(z) \prec L(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$\prec \xi_1 + \xi_2 q_2^2(z) + \xi_3 q_2(z) + \xi_4 \ zq_2'(z), \tag{47}$$

implies

$$q_1(z) \prec \frac{J_{\nu,p}(f * \Phi)(z)}{J_{\nu+1,p}(f * \Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and dominant of (47).*

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = p+1$, $\alpha_2 = 1$ and $\beta_1 = n+p$ $(n > -p,\ p \in \mathbb{N})$ in Theorem 3, we obtain the following corollary:

Corollary 17. *Let* q_1, q_2 *be convex univalent in* U*. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f \in A(p)$, $\frac{I_{n,p}(f*\Phi)(z)}{I_{n+1,p}(f*\Psi)(z)} \in H[q(0), 1] \cap Q$ *and* $R(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is univalent in* U*, where* $R(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is defined in* (34), *then*

$$\xi_1 + \xi_2 q_1^2(z) + \xi_3 q_1(z) + \xi_4 \ zq_1'(z) \prec R(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$\prec \xi_1 + \xi_2 q_2^2(z) + \xi_3 q_2(z) + \xi_4 \ zq_2'(z), \tag{48}$$

implies

$$q_1(z) \prec \frac{I_{n,p}(f * \Phi)(z)}{I_{n+1,p}(f * \Psi)(z)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and dominant of (48).*

Putting $A_i = 1$ $(i = 1, 2, ..., q)$, $B_j = 1$ $(j = 1, 2, ..., s)$, $q = 2$, $s = 1$, $\alpha_1 = \lambda + p$ $(\lambda > -p,\ p \in \mathbb{N})$, $\alpha_2 = c$ and $\beta_1 = a$ $\left(a, c \in R\backslash\mathbb{Z}_0^-,\ \lambda > -p\right)$ in Theorem 3, we obtain the following corollary:

Corollary 18. *Let* q_1, q_2 *be convex univalent in* U*. Suppose* q_1 *and* q_2 *satisfying (37) and (14), respectively. If* $f \in A(p)$, $\frac{I_p^{\lambda}(a,c)(f*\Phi)(z)}{I_p^{\lambda+1}(a,c)(f*\Psi)(z)} \in H[q(0), 1] \cap Q$ *and* $G(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is univalent in* U*, where* $G(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$ *is defined in* (36), *then*

$$\xi_1 + \xi_2 q_1^2(z) + \xi_3 q_1(z) + \xi_4 \ zq_1'(z) \prec G(f, \Phi, \Psi, \xi_1, \xi_2, \xi_3, \xi_4)$$

$$\prec \xi_1 + \xi_2 q_2^2(z) + \xi_3 q_2(z) + \xi_4 \; z q_2'(z), \tag{49}$$

implies

$$q_1(z) \prec \frac{I_p^{\lambda}\left(a,c\right)\left(f * \Phi\right)\left(z\right)}{I_p^{\lambda+1}\left(a,c\right)\left(f * \Psi\right)\left(z\right)} \prec q_2(z)$$

and q_1, q_2 *are, respectively, the best subordinant and dominant of (49).*

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