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GENERALIZED SEQUENCE SPACES ON SEMINORMED SPACES

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Abstract. In this paper we define the sequence space $\ell_M(u,p,q,s)$ on a seminormed complex linear space by using Orlicz function and we give various properties and some inclusion relations on this space. This study generalized some results of Bektaş and Altın [1].

Keywords: Orlicz functions, sequence spaces, convex function.

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1. Introduction

Let ω be the set of all sequences $x=(x_k)$ with complex terms.

Lindenstrauss and Tzafriri [3] used the idea of Orlicz function to construct the sequence space $\ell_M=\{x\in\omega:\sum_{k=1}^{\infty}M(\frac{|x_k|}{\rho})<\infty \;\; for\ some\ \rho>0\}$. The space ℓ_M with the norm $\|x\|=\inf\{\rho>0:\sum_{k=1}^{\infty}M(\frac{|x_k|}{\rho})\leq 1\}$ becomes a Banach space which is called an Orlicz sequence space. The space ℓ_M is closely related to the space ℓ_p which is an Orlicz sequence space with $M(x)=x^p$ for $1\leq p<\infty$. An Orlicz function is a function $M:[0,\infty)\to[0,\infty)$, which is continuous, non-decreasing and convex with $M(0)=0$, $M(x)>0$ for $x>0$ and $M(x)\to\infty$ as $x\to\infty$.

Remark 1.1. If M is a convex function and $M(0)=0$, then $M(\lambda x)\leq\lambda M(x)$ for all λ with $0\leq\lambda\leq 1$.

Let X be a complex linear space with zero element θ and (X, q) be a seminormed space with the seminorm q. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $(x_k) \in X$ and the usual coordinatewise operations:

$$\alpha x = (\alpha x_k) \text{ and } x + y = (x_k + y_k)$$

for each $\alpha \in C$ where C denotes the set of all complex numbers. If $\lambda = (\lambda_k)$ is a scalar sequence and $x \in S(X)$ then we shall write $\lambda x = (\lambda_k x_k)$. Let U be the set of all sequences $u = (u_k)$ such that $u_k \neq 0$ and complex for all $k = 1, 2, \ldots$. Let $p = (p_k)$ be a sequence of positive real numbers and M be an Orlicz function. Given $u \in U$. Let $s \geq 0$. Then we define the sequence space

$$\ell_M(u, p, q, s) = \{x \in S(X) : \sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty, \ for \ some \ \rho > 0\}.$$

The following inequality and $p = (p_k)$ sequence will be used frequently throughout this paper.

$$|a_k + b_k|^{p_k} \leq D\{|a_k|^{p_k} + |b_k|^{p_k}\},$$

where $a_k, b_k \in C$, $0 < p_k \leq \sup_k p_k = G$, $D = \max(1, 2^{G-1})$[4].

A sequence space E is said to be solid (or normal) if $(\alpha_k x_k) \in E$ whenever $(x_k) \in E$ for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$.

2. Main Results

Theorem 2.1. *The sequence space $\ell_M(u, p, q, s)$ is a linear space over the field C complex numbers.*

Proof. Let $x, y \in \ell_M(u, p, q, s)$ and $\alpha, \beta \in C$. Then there exist some positive numbers ρ_1 and ρ_2 such that

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho_1}))]^{p_k} < \infty$$

and

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k y_k}{\rho_2}))]^{p_k} < \infty.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M is non-decreasing and convex, and since q is a seminorm, we have

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k(\alpha x_k + \beta y_k)}{\rho_3}))]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s}[M(q(\frac{\alpha u_k x_k}{\rho_3}) + q(\frac{\beta u_k y_k}{\rho_3})]^{p_k}$$

$$\begin{aligned}
&\leq \sum_{k=1}^{\infty} \frac{1}{2^{p_k}} k^{-s}[M(q(\frac{u_k x_k}{\rho_1})) + M(q(\frac{u_k y_k}{\rho_2}))]^{p_k} \qquad (1')\\
&\leq \sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho_1})) + M(q(\frac{u_k y_k}{\rho_2}))]^{p_k}\\
&\leq D\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho_1}))]^{p_k} + D\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k y_k}{\rho_2}))]^{p_k}\\
&< \infty.
\end{aligned}$$

This proves that $\ell_M(u,p,q,s)$ is a linear space.
Theorem 2.2. *The space $\ell_M(u,p,q,s)$ is paranormed (not necessarily totaly paranormed) with*

$$g_u(x) = \inf\{\rho^{p_n/H} : (\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k})^{1/H} \leq 1, \quad n = 1,2,3,\dots\}$$

where $H = \max(1, \sup_k p_k)$.
Proof. Clearly $g_u(x) = g_u(-x)$. The subadditivity of g_u follows from (1'), on taking $\alpha = 1$ and $\beta = 1$. Since $q(\theta) = 0$ and $M(0) = 0$, we get $\inf\{\rho^{p_n/H}\} = 0$ for $x = \theta$. Finally, we prove that the scalar multiplication is continuous. Let λ be any number. By definition,

$$g_u(\lambda x) = \inf\{\rho^{p_n/H} : (\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{\lambda u_k x_k}{\rho}))]^{p_k})^{1/H} \leq 1, \quad n = 1,2,3,\dots\}.$$

Then

$$g_u(\lambda x) = \inf\{(\lambda r)^{p_n/H} : (\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{r}))]^{p_k})^{1/H} \leq 1, \quad n = 1,2,3,\dots\}$$

where $r = \rho/\lambda$. Since $|\lambda|^{p_k} \leq \max(1, |\lambda|^H)$, then $|\lambda|^{p_k/H} \leq (\max(1,|\lambda|^H))^{1/H}$. Hence

$$g_u(\lambda x) \leq (\max(1,|\lambda|^H))^{1/H} \inf\{(r)^{p_n/H} : (\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{r}))]^{p_k})^{1/H} \leq 1, \quad n = 1,2,3,\dots\}$$

and therefore $g_u(\lambda x)$ converges to zero when $g_u(x)$ converges to zero in $\ell_M(u,p,q,s)$. Now suppose that $\lambda_n \to 0$ and x is in $\ell_M(u,p,q,s)$. For arbitrary $\varepsilon > 0$, let N be a positive integer such that

$$\sum_{k=\mathrm{N}+1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < (\frac{\varepsilon}{2})^H$$

for some $\rho > 0$. This implies that

$$(\sum_{k=\mathrm{N}+1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k})^{1/H} \leq \frac{\varepsilon}{2}.$$

Let $0 < |\lambda| < 1$, then using Remark 1.1 we get

$$\sum_{k=\mathrm{N}+1}^{\infty} k^{-s}[M(q(\frac{\lambda u_k x_k}{\rho}))]^{p_k} < \sum_{k=\mathrm{N}+1}^{\infty} k^{-s}[|\lambda| M(q(\frac{u_k x_k}{\rho}))]^{p_k} < (\frac{\varepsilon}{2})^{H}.$$

Since M is continuous everywhere in $[0, \infty)$, then

$$f(t) = \sum_{k=1}^{\mathrm{N}} k^{-s}\left[M(q(\frac{t u_k x_k}{\rho}))\right]^{p_k}$$

is continuous at 0. So there is $1 > \delta > 0$ such that $|f(t)| < \frac{\varepsilon}{2}$ for $0 < t < \delta$. Let K be such that $|\lambda_n| < \delta$ for $n > K$, then for $n > K$ we have

$$(\sum_{k=1}^{\mathrm{N}} k^{-s}[M(q(\frac{\lambda_n u_k x_k}{\rho}))]^{p_k})^{1/H} < \frac{\varepsilon}{2}.$$

Since $0 < \varepsilon < 1$ we have

$$(\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{\lambda_n u_k x_k}{\rho}))]^{p_k})^{1/H} < 1, \quad \text{for} \quad n > K.$$

If we take limit on $\inf\{\rho^{p_n/H}\}$ we get $g_u(\lambda x) \to 0$.

3. Some Particular Cases

We get the following sequence spaces from $\ell_M(u, p, q, s)$ on giving particular values to p and s. Taking $p_k = 1$ for all $k \in N$, we have

$$\ell_M(u, q, s) = \{x \in S(X) : \sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))] < \infty, s \geq 0, \text{ for some } \rho > 0\}.$$

If we take $s = 0$, then we have

$$\ell_M(u, p, q) = \{x \in S(X) : \sum_{k=1}^{\infty}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

If we take $p_k = 1$ for all $k \in N$ and $s = 0$, then we have

$$\ell_M(u,q) = \{x \in S(X) : \sum_{k=1}^{\infty}[M(q(\frac{u_k x_k}{\rho}))] < \infty, \text{ for some } \rho > 0\}.$$

If we take $s = 0$, $q(x) = |x|$ and $X = C$, then we have

$$\ell_M(u,p) = \{x \in S(X) : \sum_{k=1}^{\infty}[M(\frac{|u_k x_k|}{\rho})]^{p_k} < \infty, \text{ for some } \rho > 0\}.$$

In addition to the above sequence spaces, we write $\ell_M(u,p,q,s) = \ell_M(p)$ due to Parashar and Choudhary [5], on taking $u_k = 1$ for all $k \in N$, $s = 0$, $q(x) = |x|$ and $X = C$. If we take $u_k = 1$ for all $k \in N$, we have $\ell_M(u,p,q,s) = \ell_M(p,q,s)$ [1].

Theorem 3.1. **(i)** *Let $0 < p_k \leq t_k < \infty$ for each $k \in N$. Then $\ell_M(u,p,q) \subseteq \ell_M(u,t,q)$.*

(ii) *$\ell_M(u,q) \subseteq \ell_M(u,q,s)$.*

(iii) *$\ell_M(u,p,q) \subseteq \ell_M(u,p,q,s)$.*

Proof. **(i)** Let $x \in \ell_M(u,p,q)$. Then there exists some $\rho > 0$ such that

$$\sum_{k=1}^{\infty}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.$$

This implies that $M(q(\frac{u_i x_i}{\rho})) \leq 1$ for sufficiently large values of i, say $i \geq k_0$ for some fixed $k_0 \in N$.

Since M is non-decreasing, we get

$$\sum_{k=1}^{\infty}[M(q(\frac{u_k x_k}{\rho}))]^{t_k} < \infty,$$

since

$$\sum_{k\geq k_0}^{\infty}[M(q(\frac{u_k x_k}{\rho}))]^{t_k} \leq \sum_{k\geq k_0}^{\infty}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.$$

Hence $x \in \ell_M(u,t,q)$.

The proof of (ii) and (iii) is trivial.

Theorem 3.2. *Let $0 < p_k \leq t_k < \infty$ for each k. Then $\ell_M(u,p) \subseteq \ell_M(u,t)$.*

Proof. Proof can be proved by the same way as Theorem 3.1(i).

Theorem 3.3. **(i)** *If $0 < p_k \leq 1$ for all $k \in N$, then $\ell_M(u,p,q) \subseteq \ell_M(u,q)$.*

(ii) *If $p_k \geq 1$ for all $k \in N$, then $\ell_M(u,q) \subseteq \ell_M(u,p,q)$.*

Proof. **(i)** If we take $t_k = 1$ for all $k \in N$, in Theorem 3.1(i), then $\ell_M(u,p,q) \subseteq \ell_M(u,q)$.

(ii) If we take $p_k = 1$ for all $k \in N$, in Theorem 3.1(i), then $\ell_M(u,q) \subseteq \ell_M(u,p,q)$.

Proposition 3.4 *For any two sequences $p = (p_k)$ and $t = (t_k)$ of positive real numbers and any two seminorms q_1 and q_2 we have $\ell_M(u,p,q_1,r) \cap \ell_M(u,t,q_2,s) \neq \emptyset$ for $r, s > 0$.*

Proof. Since the zero element belongs to $\ell_M(u,p,q_1,r)$ and $\ell_M(u,t,q_2,s)$, thus the intersection is nonempty.

Theorem 3.5. *The sequence space $\ell_M(u,p,q,s)$ is solid.*

Proof. Let $(x_k) \in \ell_M(u,p,q,s)$, i.e,

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.$$

Let (α_k) be sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then we have

$$\sum_{k=1}^{\infty} k^{-s}[M(q(\frac{\alpha_k u_k x_k}{\rho}))]^{p_k} \leq \sum_{k=1}^{\infty} k^{-s}[M(q(\frac{u_k x_k}{\rho}))]^{p_k} < \infty.$$

Hence $(\alpha_k x_k) \in \ell_M(u,p,q,s)$ for all sequences of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in N$, whenever $(x_k) \in \ell_M(u,p,q,s)$.

Therefore the space $\ell_M(u,p,q,s)$ is a solid sequence space.

Corollary 3.6. (i) *Let $|u_k| \leq 1$ for all $k \in N$. Then $\ell_M(p,q,s) \subseteq \ell_M(u,p,q,s)$.*

(ii) *Let $|u_k| \geq 1$ for all $k \in N$. Then $\ell_M(u,p,q,s) \subseteq \ell_M(p,q,s)$.*

Proof. Proof is trivial.

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