Acta Universitatis Apulensis No. 26/2011
ISSN: 1582-5329 pp. 237-244

YOUNG'S INEQUALITY REVISITED

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Abstract. Some companions and a reverse of Young's inequality are proved in the context of absolutely continuous measures. We avoid the continuity condition that appears in the classical statement of Young's inequality, extending some known results to the class of all nondecreasing functions.

2000 *Mathematics Subject Classification*: 26A51, 26D15.

1. Introduction

It is well-known that:

Theorem 1. [Young's inequality] *Every strictly increasing continuous function* $f : [0, \infty) \longrightarrow [0, \infty)$ *with* $f(0) = 0$ *and* $\lim_{x\to\infty} f(x) = \infty$ *verifies*

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(y)\,dy,$$

whenever a *and* b *are nonnegative real numbers. The equality occurs if and only if* $f(a) = b$.

The inequality referred to the title appears for the first time in a paper of W. H. Young [5]. At the core of this inequality is a geometric identity telling us that the curve

$$y = f(x), \quad x \in [a, b]$$

decomposes the rectangle $[0, a] \times [0, f(a)]$ into two parts of areas $\int_0^a f(x)\,dx$ and $\int_0^{f(a)} f^{-1}(x)\,dx$, so that

$$af(a) = \int_0^a f(x)\,dx + \int_0^{f(a)} f^{-1}(x)\,dx.$$

See also [2,Theorem 1.2.1].

In the sequel $f:[0,\infty)\longrightarrow[0,\infty)$ will denote a nondecreasing function such that $f(0)=0$ and $\lim\limits_{x\to\infty}f(x)=\infty$. We attach to f a *pseudo-inverse,* with the convention $f(0-)=0$,

$$f_{\sup}^{-1}(y)=\sup\{x:y\in[f(x-),f(x+)]\},$$

where $f(x-)$ and $f(x+)$ represent the lateral limits at x. Notice that we have

$$f_{\sup}^{-1}(y)=\max\{x\geq 0:y=f(x)\}$$

when f is continuous.

Theorem 2.*[Young's inequality for nondecreasing functions] Under the above assumptions, for $K:[0,\infty)\times[0,\infty)\longrightarrow[0,\infty)$ is Lebesgue locally integrable and for every pair of nonnegative numbers $a<b$ and every number $c\geq f(a)$, we have*

$$\begin{aligned}&\int_a^b\int_{f(a)}^c K(x,y)\,dydx\\&\leq\int_a^b\left(\int_{f(a)}^{f(x)}K(x,y)\,dy\right)dx+\int_{f(a)}^c\left(\int_a^{f_{\sup}^{-1}(y)}K(x,y)\,dx\right)dy.\end{aligned}$$

If K is strictly positive almost everywhere, then the equality occurs if and only if $c\in[f(b-),f(b+)]$.

This result can be find in a recent paper of F. C. Mitroi and C. P. Niculescu [1].

In what follows we consider the particular case of it $K(x,y)=p(x)\,q(y)$, where $p,q:[0,\infty)\longrightarrow[0,\infty)$ are continuous functions. We immediately infer the following inequality:

$$\begin{aligned}&\int_a^b p(x)\,dx\cdot\int_{f(a)}^c q(y)\,dy\\&\leq\int_a^b\left(\int_{f(a)}^{f(x)}q(y)\,dy\right)p(x)\,dx+\int_{f(a)}^c\left(\int_a^{f_{\sup}^{-1}(y)}p(x)\,dx\right)q(y)\,dy.\end{aligned}$$

2. Companions of Young's inequality

Before stating the main result of this section, we require the following well-known lemma:

Lemma 1. *For $m,\ n\geq 0$ and $p,\ q>1$such that $\frac{1}{p}+\frac{1}{q}=1$ one has $\frac{m^p}{p}+\frac{n^q}{q}\geq mn$.*

We may now give an extension of Young's Inequality:

Theorem 3. *Let $p \geq 1$ be an arbitrary real number. We have:*

$$p \int_a^b p(x)\,dx \cdot \int_{f(a)}^c q(y)\,dy + (p-1)\left(\int_a^b p(x)\,dx + \int_{f(a)}^c q(y)\,dy\right)$$
$$\leq \int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy\right)^p p(x)\,dx + \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right)^p q(y)\,dy.$$

Proof. Applying Lemma 1 we take firstly $m = \int_{f(a)}^{f(x)} q(y)\,dy$ and $n = 1$. Then $\frac{1}{p}\left(\int_{f(a)}^{f(x)} q(y)\,dy\right)^p + \frac{1}{q} \geq \int_{f(a)}^{f(x)} q(y)\,dy$. We integrate this with respect to $p(x)\,dx$.

$$\frac{1}{p}\int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy\right)^p p(x)\,dx + \frac{1}{q}\int_a^b p(x)\,dx$$
$$\geq \int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy\right) p(x)\,dx.$$

After that we consider $m = \int_a^{f_{\sup}^{-1}(y)} p(x)\,dx$ and $n = 1$. Then

$$\frac{1}{p}\left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right)^p + \frac{1}{q} \geq \int_a^{f_{\sup}^{-1}(y)} p(x)\,dx$$

and we integrate it with respect to $q(y)\,dy$:

$$\frac{1}{p}\int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right)^p q(y)\,dy + \frac{1}{q}\left(\int_{f(a)}^c q(y)\,dy\right)$$
$$\geq \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right) q(y)\,dy.$$

The sum of these two inequalities gives us:

$$\frac{1}{p}\int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy\right)^p p(x)\,dx + \frac{1}{p}\int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right)^p q(y)\,dy$$
$$\geq \int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy\right) p(x)\,dx + \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx\right) q(y)\,dy$$
$$- \frac{1}{q}\left(\int_a^b p(x)\,dx + \int_{f(a)}^c q(y)\,dy\right).$$

Multiplying this inequality with p and applying Theorem 2 we finally get the claimed result:

$$\int_a^b \left(\int_{f(a)}^{f(x)} q(y)\,dy \right)^p p(x)\,dx + \int_{f(a)}^c \left(\int_a^{f_{\sup}^{-1}(y)} p(x)\,dx \right)^p q(y)\,dy$$
$$\geq p \int_a^b p(x)\,dx \cdot \int_{f(a)}^c q(y)\,dy - (p-1) \left(\int_a^b p(x)\,dx + \int_{f(a)}^c q(y)\,dy \right).$$

The case $p = 1$ coincides to the case $K(x, y) = p(x)\,q(y)$ in Theorem 2.

The following corollary appears in a paper of W.T. Sulaiman [3]:

Corollary 1. *The above inequality, for $p(x) = q(x) = 1$, with $a = f(a) = 0$, f continuous and strictly increasing, reduces to:*

$$pbc - (p-1)(b+c) \leq \int_0^b f^p(x)\,dx + \int_0^c \left(f^{-1}(y)\right)^p dy, \text{ for every } p \geq 1.$$

The case $p = 1$ is exactly Young's inequality.

Theorem 4. *Let $p, q \geq 1$ be arbitrary real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\int_0^a \left(\int_0^{f(x)} q(y)\,dy \right) p(x)\,dx \cdot \int_0^b \left(\int_0^{f_{\sup}^{-1}(y)} p(x)\,dx \right) q(y)\,dy$$
$$\leq \frac{\int_0^b q(y)\,dy}{p} \int_0^a \left(\int_0^{f(x)} q(y)\,dy \right)^p p(x)\,dx$$
$$+ \frac{\int_0^a p(x)\,dx}{q} \int_0^b \left(\int_0^{f_{\sup}^{-1}(y)} p(x)\,dx \right)^q q(y)\,dy.$$

Proof. Via Lemma 1, if we integrate $\int_0^a \left(\int_0^b (.)\, q(y)\,dy \right) p(x)\,dx$ both sides of the following inequality:

$$\int_0^{f(x)} q(y)\,dy \cdot \int_0^{f_{\sup}^{-1}(y)} p(x)\,dx \leq \frac{\left(\int_0^{f(x)} q(y)\,dy \right)^p}{p} + \frac{\left(\int_0^{f_{\sup}^{-1}(y)} p(x)\,dx \right)^q}{q},$$

we obtain the clamed result.

3. Reverse of Young's inequality

Noticing that the functions

$$F:(a,\infty)\longrightarrow\mathbb{R}_+,\quad F(x)=\frac{\int_a^x\left(\int_{f(a)}^{f(t)}q(y)\,dy\right)p(t)\,dt}{\int_a^x p(t)\,dt}$$

and

$$G:(f(a),\infty)\longrightarrow\mathbb{R}_+,\quad G(x)=\frac{\int_{f(a)}^x\left(\int_a^{f_{\sup}^{-1}(y)}p(t)\,dt\right)q(y)\,dy}{\int_{f(a)}^x q(y)\,dy}$$

are nondecreasing, we can state the following result:

Theorem 5. *Let a, b be two positive real numbers with $a<b$. The following inequality is true:*

$$\begin{aligned}&\min\left\{1,\frac{\int_{f(a)}^c q(y)\,dy}{\int_{f(a)}^{f(b)}q(y)\,dy}\right\}\int_a^b\left(\int_{f(a)}^{f(t)}q(y)\,dy\right)p(t)\,dt \\ &+\min\left\{1,\frac{\int_a^b p(t)\,dt}{\int_a^{f_{\sup}^{-1}(c)}p(t)\,dt}\right\}\int_{f(a)}^c\left(\int_a^{f_{\sup}^{-1}(y)}p(t)\,dt\right)q(y)\,dy \\ &\leq\int_a^b p(t)\,dt\cdot\int_{f(a)}^c q(y)\,dy.\end{aligned}\tag{1}$$

Equality holds iff we have $\int_b^{f_{\sup}^{-1}(c)}p(t)\,dt=0$ for $b\leq f_{\sup}^{-1}(c)$ or $\int_c^{f(b)}q(y)\,dy=0$ for $b\geq f_{\sup}^{-1}(c)$.

Proof. Consider first that $b\leq f_{\sup}^{-1}(c)$. Then $F(f_{\sup}^{-1}(c))\geq F(b)$ and more explicitely this reads as

$$\frac{\int_a^{f_{\sup}^{-1}(c)}\left(\int_{f(a)}^{f(t)}q(y)\,dy\right)p(t)\,dt}{\int_a^{f_{\sup}^{-1}(c)}p(t)\,dt}\geq\frac{\int_a^b\left(\int_{f(a)}^{f(t)}q(y)\,dy\right)p(t)\,dt}{\int_a^b p(t)\,dt}.$$

We already know that

$$\begin{aligned}&\int_a^{f_{\sup}^{-1}(c)}\left(\int_{f(a)}^{f(t)}q(y)\,dy\right)p(t)\,dt+\int_{f(a)}^c\left(\int_a^{f_{\sup}^{-1}(y)}p(t)\,dt\right)q(y)\,dy \\ &=\int_a^{f_{\sup}^{-1}(c)}p(t)\,dt\cdot\int_{f(a)}^c q(y)\,dy.\end{aligned}$$

Consequently

$$\frac{\int_a^{f_{\sup}^{-1}(c)} p(t)\,dt}{\int_a^b p(t)\,dt} \int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy$$

$$\leq \int_a^{f_{\sup}^{-1}(c)} p(t)\,dt \cdot \int_{f(a)}^{c} q(y)\,dy.$$

We multiply this by $\frac{\int_a^b p(t)dt}{\int_a^{f_{\sup}^{-1}(c)} p(t)dt}$ and we immediately obtain the following partial result:

$$\int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \frac{\int_a^b p(t)\,dt}{\int_a^{f_{\sup}^{-1}(c)} p(t)\,dt} \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy \quad (2)$$

$$\leq \int_a^b p(t)\,dt \cdot \int_{f(a)}^{c} q(y)\,dy.$$

The equality holds if

$$\int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt$$

$$+ \left(1 - \frac{\int_b^{f_{\sup}^{-1}(c)} p(t)\,dt}{\int_a^{f_{\sup}^{-1}(c)} p(t)\,dt} \right) \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy$$

$$= \int_a^b p(t)\,dt \cdot \int_{f(a)}^{c} q(y)\,dy,$$

and that happens if and only if $\int_b^{f_{\sup}^{-1}(c)} p(t)\,dt = 0$.

Considering now the reverse $b \geq f_{\sup}^{-1}(c)$, we use the fact $G(f(b)) \geq G(c)$, written more explicitely as

$$\frac{\int_{f(a)}^{f(b)} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy}{\int_{f(a)}^{f(b)} q(y)\,dy} \geq \frac{\int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy}{\int_{f(a)}^{c} q(y)\,dy}$$

Since

$$\int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \int_{f(a)}^{f(b)} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy$$

$$= \int_a^b p(t)\,dt \cdot \int_{f(a)}^{f(b)} q(y)\,dy$$

we obtain

$$\int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \frac{\int_{f(a)}^{f(b)} q(y)\,dy}{\int_{f(a)}^{c} q(y)\,dy} \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy$$
$$\leq \int_a^b p(t)\,dt \cdot \int_{f(a)}^{f(b)} q(y)\,dy.$$

We multiply by $\frac{\int_{f(a)}^{c} q(y)dy}{\int_{f(a)}^{f(b)} q(y)dy}$. Finally:

$$\frac{\int_{f(a)}^{c} q(y)\,dy}{\int_{f(a)}^{f(b)} q(y)\,dy} \int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy \qquad (3)$$
$$\leq \int_a^b p(t)\,dt \cdot \int_{f(a)}^{c} q(y)\,dy.$$

The equality holds when

$$\left(1 - \frac{\int_{c}^{f(b)} q(y)\,dy}{\int_{f(a)}^{f(b)} q(y)\,dy} \right) \int_a^b \left(\int_{f(a)}^{f(t)} q(y)\,dy \right) p(t)\,dt + \int_{f(a)}^{c} \left(\int_a^{f_{\sup}^{-1}(y)} p(t)\,dt \right) q(y)\,dy$$
$$= \int_a^b p(t)\,dt \cdot \int_{f(a)}^{c} q(y)\,dy,$$

and that happens if and only if $\int_c^{f(b)} q(y)\,dy = 0$.

The inequality (1) appears now as a shorter version of the inequalities (2) and (3).

As a particular case of it we obtain a result due to A. Witkowski [4] .

Corollary 2. *For $p(x) = q(x) = 1$ on $[0, \infty)$, $a = f(a) = 0$, f continuous and strictly increasing, we obtain an inequality with upper bound :*

$$\min\left\{1, \frac{c}{f(b)}\right\} \int_0^b f(x)\,dx + \min\left\{1, \frac{b}{f^{-1}(c)}\right\} \int_0^c f^{-1}(y)\,dy \leq bc,$$

Equality holds iff $c = f(b)$.

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