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NEIGHBOURHOODS AND PARTIAL SUMS OF CERTAIN SUBCLASS OF ANALYTIC FUNCTIONS

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Abstract. The main purpose of the present paper is to derive the neighborhoods and partial sums of certain subclass of analytic functions which was introduced and investigated recently by Liu and Liu [Z.-W. Liu and M.-S. Liu, Properties and characteristics of certain subclasses of analytic functions, J. South China Normal Univ. Natur. Sci. Ed. 28 (2010), 11–15].

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1.Introduction

Let $\mathcal{A}_m$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=m+1}^{\infty} a_k z^k \qquad (m \in \mathbb{N} := \{1, 2, 3, \ldots\}), \tag{1}$$

which are *analytic* in the *open* unit disk

$$\mathbb{U} := \{z : \ z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

In 2002, Owa and Nishiwaki [12] introduced the class $\mathcal{M}_m(\beta)$ which was defined by

$$\Re\left(\frac{zf'(z)}{f(z)}\right) < \beta \qquad (f \in \mathcal{A}_m;\ z \in \mathbb{U};\ \beta > 1). \tag{2}$$

Assuming that $\alpha \geqq 0,\ \beta > 1$ and $f \in \mathcal{A}_m$, we say that a function $f \in \mathcal{M}_m(\alpha, \beta)$ if it satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}\right) < \alpha\beta\left(\beta + \frac{m}{2} - 1\right) + \beta - \frac{m\alpha}{2} \qquad (z \in \mathbb{U}). \tag{3}$$

Obviously, $\mathcal{M}_m(0,\beta) = \mathcal{M}_m(\beta)$.

For convenience, throughout this paper, we write

$$\gamma_m := \alpha\beta\left(\beta + \frac{m}{2} - 1\right) + \beta - \frac{m\alpha}{2}. \tag{4}$$

Recently, Liu and Liu [11] proved that $\mathcal{M}_m(\alpha,\beta) \subset \mathcal{M}_m(\beta)$ and derived various properties and characteristics such as inclusion relationships, sufficient conditions and Fekete-Szegö inequality for the class $\mathcal{M}_m(\alpha,\beta)$. In the present paper, we aim at proving the neighborhoods and partial sums of the class $\mathcal{M}_m(\alpha,\beta)$.

2.Main Results

Following the earlier works (based upon the familiar concept of neighborhood of analytic functions) by Goodman [9] and Ruscheweyh [13], and (more recently) by Altintaş *et al.* [1,2,3,4], Cătaş [5], Frasin [7], Keerthi *et al.* [10] and Srivastava *et al.* [15], we begin by introducing here the δ-neighborhood of a function $f \in \mathcal{A}_m$ of the form (1) by means of the definition

$$\mathcal{N}_\delta(f) := \Bigg\{ g \in \mathcal{A}_m :\ g(z) = z + \sum_{k=m+1}^{\infty} b_k z^k \quad \text{and}$$
$$\sum_{k=m+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_m}{\gamma_m - 1}|a_k - b_k| \leqq \delta \qquad (\delta,\ \alpha \geqq 0;\ \beta > 1;\ \gamma_m > 1)\Bigg\}. \tag{5}$$

By making use of the definition (5), we now derive the following result.

Theorem 1. *If $f \in \mathcal{A}_m$ satisfies the condition*

$$\frac{f(z)+\varepsilon z}{1+\varepsilon} \in \mathcal{M}_m(\alpha,\beta) \qquad (\varepsilon \in \mathbb{C};\ |\varepsilon| < \delta;\ \delta > 0), \tag{6}$$

then

$$\mathcal{N}_\delta(f) \subset \mathcal{M}_m(\alpha,\beta). \tag{7}$$

Proof. By noting that the condition (3) can be rewritten as follows:

$$\left| \frac{\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)}}{\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} - 2\gamma_m} \right| < 1 \qquad (z \in \mathbb{U}), \tag{8}$$

we easily find from (8) that a function $g \in \mathcal{M}_m(\alpha,\beta)$ if and only if

$$\frac{zg'(z) + \alpha z^2 g''(z)}{zg'(z) + \alpha z^2 g''(z) - 2\gamma_m g(z)} \neq \sigma \qquad (z \in \mathbb{U};\ \sigma \in \mathbb{C};\ |\sigma| = 1),$$

which is equivalent to

$$\frac{(g*h)(z)}{z} \neq 0 \qquad (z \in \mathbb{U}), \tag{9}$$

where

$$h(z) = z + \sum_{k=m+1}^{\infty} c_k z^k \qquad \left(c_k := \frac{k + \alpha k(k-1) - [k + \alpha k(k-1) - 2\gamma_m]\sigma}{(2\gamma_m - 1)\sigma} \right). \tag{10}$$

It follows from (10) that

$$\begin{aligned} |c_k| &= \left| \frac{k + \alpha k(k-1) - [k + \alpha k(k-1) - 2\gamma_m]\sigma}{(2\gamma_m - 1)\sigma} \right| \\ &\leqq \frac{k + \alpha k(k-1) + [k + \alpha k(k-1) - 2\gamma_m] |\sigma|}{2(\gamma_m - 1) |\sigma|} \\ &= \frac{k(1 + k\alpha - \alpha) - \gamma_m}{\gamma_m - 1} \qquad (|\sigma| = 1). \end{aligned}$$

If $f \in \mathcal{A}_m$ satisfies the condition (6), we deduce from (9) that

$$\frac{(f*h)(z)}{z} \neq -\varepsilon \qquad (|\varepsilon| < \delta;\ \delta > 0),$$

or equivalently,

$$\left| \frac{(f*h)(z)}{z} \right| \geqq \delta \qquad (z \in \mathbb{U};\ \delta > 0). \tag{11}$$

We now suppose that

$$q(z) = z + \sum_{k=m+1}^{\infty} d_k z^k \in \mathcal{N}_\delta(f).$$

It follows from (5) that

$$\left| \frac{((q-f)*h)(z)}{z} \right| = \left| \sum_{k=m+1}^{\infty} (d_k - a_k) c_k z^{k-1} \right| \leqq \sum_{k=m+1}^{\infty} \frac{k(1 + k\alpha - \alpha) - \gamma_m}{\gamma_m - 1} |d_k - a_k| < \delta. \tag{12}$$

Combining (11) and (12), we easily find that

$$\left| \frac{(q*h)(z)}{z} \right| = \left| \frac{([f + (q-f)]*h)(z)}{z} \right| \geqq \left| \frac{(f*h)(z)}{z} \right| - \left| \frac{((q-f)*h)(z)}{z} \right| > 0,$$

which implies that

$$\frac{(q*h)(z)}{z} \neq 0 \qquad (z \in \mathbb{U}).$$

Therefore, we conclude that

$$q(z) \in \mathcal{N}_\delta(f) \subset \mathcal{M}_m(\alpha, \beta).$$

We thus complete the proof of Theorem 1. □

Next, we derive the partial sums of the class $\mathcal{M}_m(\alpha, \beta)$. For some recent investigations involving the partial sums in analytic function theory, one can find in [6, 8, 14] and the references cited therein.

Theorem 2. *Let $f \in \mathcal{A}_m$ be given by* (1) *and define the partial sums $f_n(z)$ of f by*

$$f_n(z) = z + \sum_{k=m+1}^{n} a_k z^k \qquad (n \in \mathbb{N};\ n \geqq m+1). \tag{13}$$

If

$$\sum_{k=m+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_m}{\gamma_m - 1} |a_k| \leqq 1 \qquad (\alpha \geqq 0;\ \beta > 1;\ \gamma_m > 1)\,, \tag{14}$$

then

1. $f \in \mathcal{M}_m(\alpha, \beta)$;
2. $$\Re\left(\frac{f(z)}{f_n(z)}\right) \geqq \frac{(n+1)(1+n\alpha)+1-2\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}), \tag{15}$$

   and

   $$\Re\left(\frac{f_n(z)}{f(z)}\right) \geqq \frac{(n+1)(1+n\alpha)-\gamma_m}{n(1+\alpha+n\alpha)} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}). \tag{16}$$

   The bounds in (15) *and* (16) *are sharp with the extremal function given by* (20).

Proof. (1) Suppose that $f_1(z) = z$. We know that $z \in \mathcal{M}_m(\alpha, \beta)$, which implies that

$$\frac{f_1(z) + \varepsilon z}{1+\varepsilon} = z \in \mathcal{M}_m(\alpha, \beta).$$

From (14), we easily find that

$$\sum_{k=m+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_m}{\gamma_m - 1} |a_k - 0| \leqq 1,$$

which implies that $f \in \mathcal{N}_1(z)$. In view of Theorem 1, we deduce that

$$f \in \mathcal{N}_1(z) \subset \mathcal{M}_m(\alpha, \beta).$$

(2) It is easy to verify that

$$\frac{(n+1)[1+(n+1)\alpha-\alpha]-\gamma_m}{\gamma_m-1} = \frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} > \frac{n(1+n\alpha-\alpha)-\gamma_m}{\gamma_m-1} > 1$$

where $(n \in \mathbb{N})$. Therefore, we have

$$\sum_{k=m+1}^{n} |a_k| + \frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} \sum_{k=n+1}^{\infty} |a_k| \leqq \sum_{k=m+1}^{\infty} \frac{k(1+k\alpha-\alpha)-\gamma_m}{\gamma_m-1} |a_k| \leqq 1. \tag{17}$$

We now suppose that

$$\begin{aligned} \psi(z) &= \frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} \left(\frac{f(z)}{f_n(z)} - \frac{(n+1)(1+n\alpha)+1-2\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} \right) \\ &= 1 + \frac{\frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} \sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1 + \sum\limits_{k=m+1}^{n} a_k z^{k-1}}. \end{aligned} \tag{18}$$

It follows from (17) and (18) that

$$\left| \frac{\psi(z)-1}{\psi(z)+1} \right| \leqq \frac{\frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} \sum\limits_{k=n+1}^{\infty} |a_k|}{2 - 2\sum\limits_{k=m+1}^{n} |a_k| - \frac{(n+1)(1+n\alpha)-\gamma_m}{\gamma_m-1} \sum\limits_{k=n+1}^{\infty} |a_k|} \leqq 1 \qquad (z \in \mathbb{U}),$$

which shows that

$$\Re\left(\psi(z)\right) \geqq 0 \qquad (z \in \mathbb{U}). \tag{19}$$

Combining (18) and (19), we deduce that the assertion (15) holds true.

Moreover, if we put

$$f(z) = z + \frac{\gamma_m - 1}{(n+1)(1+n\alpha)-\gamma_m} z^{n+1} \qquad (n \in \mathbb{N} \setminus \{1, 2, \ldots, m-1\};\ m \in \mathbb{N}), \tag{20}$$

then for $z = re^{i\pi/n}$, we have

$$\frac{f(z)}{f_n(z)} = 1 + \frac{\gamma_m - 1}{(n+1)(1+n\alpha)-\gamma_m} z^n \to \frac{(n+1)(1+n\alpha)+1-2\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} \qquad (r \to 1^-),$$

which implies that the bound in (15) is the best possible for each $n \in \mathbb{N}\backslash\{1, 2, \ldots, m-1\}$.

Similarly, we suppose that

$$\begin{aligned}\varphi(z) &= \frac{n(1+\alpha+n\alpha)}{\gamma_m - 1}\left(\frac{f_n(z)}{f(z)} - \frac{(n+1)(1+n\alpha)-\gamma_m}{n(1+\alpha+n\alpha)}\right) \\ &= 1 - \frac{\frac{n(1+\alpha+n\alpha)}{\gamma_m-1}\sum\limits_{k=n+1}^{\infty} a_k z^{k-1}}{1+\sum\limits_{k=m+1}^{\infty} a_k z^{k-1}}.\end{aligned} \tag{21}$$

In view of (17) and (21), we conclude that

$$\left|\frac{\varphi(z)-1}{\varphi(z)+1}\right| \leqq \frac{\frac{n(1+\alpha+n\alpha)}{\gamma_m-1}\sum\limits_{k=n+1}^{\infty}|a_k|}{2-2\sum\limits_{k=m+1}^{n}|a_k| - \frac{(n+1)(1+n\alpha)+1-2\gamma_m}{\gamma_m-1}\sum\limits_{k=n+1}^{\infty}|a_k|} \leqq 1 \qquad (z \in \mathbb{U}),$$

which implies that

$$\Re\left(\varphi(z)\right) \geqq 0 \qquad (z \in \mathbb{U}). \tag{22}$$

Combining (21) and (22), we readily get the assertion (16) of Theorem 2. The bound in (16) is sharp with the extremal function f given by (20).

The proof of Theorem 2 is thus completed. $\square$

Taking $\alpha = 0$ in Theorem 2, we obtain the following corollary.

Corollary 1. *Let $f \in \mathcal{A}_m$ be given by* (1) *and define the partial sums $f_n(z)$ of f by* (13)*. If f satisfies*

$$\sum_{k=m+1}^{\infty}(k-\beta)|a_k| \leqq \beta - 1 \qquad (\beta > 1), \tag{23}$$

then $f \in \mathcal{M}_m(\beta)$,

$$\Re\left(\frac{f(z)}{f_n(z)}\right) \geqq \frac{n+2-2\beta}{n+1-\beta} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}), \tag{24}$$

and

$$\Re\left(\frac{f_n(z)}{f(z)}\right) \geqq \frac{n+1-\beta}{n} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}). \tag{25}$$

The bounds in (24) *and* (25) *are sharp with the extremal function given by*

$$f(z) = z + \frac{\beta-1}{n+1-\beta}z^{n+1} \qquad (n \in \mathbb{N}\setminus\{1,2,\ldots,m-1\};\ m \in \mathbb{N}). \tag{26}$$

Finally, we turn to ratios involving derivatives. The proof of Theorem 3 below is much akin to that of Theorem 2, we here choose to omit the analogous details.

Theorem 3. *Let* $f \in \mathcal{A}_m$ *be given by* (1) *and define the partial sums* $f_n(z)$ *of* f *by* (13). *If the condition* (14) *holds, then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geqq \frac{(n+1)(2+n\alpha-\gamma_m)-\gamma_m}{(n+1)(1+n\alpha)-\gamma_m} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}), \quad (27)$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geqq \frac{(n+1)(1+n\alpha)-\gamma_m}{(n+1)(n\alpha+\gamma_m)-\gamma_m} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}). \quad (28)$$

The bounds in (27) *and* (28) *are sharp with the extremal function given by* (20).

Taking $\alpha = 0$ in Theorem 3, we obtain the following corollary.

Corollary 2. *Let* $f \in \mathcal{A}_m$ *be given by* (1) *and define the partial sums* $f_n(z)$ *of* f *by* (13). *If the condition* (23) *holds, then*

$$\Re\left(\frac{f'(z)}{f_n'(z)}\right) \geqq \frac{(n+1)(2-\beta)-\beta}{n+1-\beta} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}), \quad (29)$$

and

$$\Re\left(\frac{f_n'(z)}{f'(z)}\right) \geqq \frac{n+1-\beta}{n\beta} \qquad (n \in \mathbb{N};\ n \geqq m+1\,;\ z \in \mathbb{U}). \quad (30)$$

The bounds in (29) *and* (30) *are sharp with the extremal function given by* (26).

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References

[1] O. Altintaş, *Neighborhoods of certain p-valently analytic functions with negative coefficients*, Appl. Math. Comput. 187 (2007), 47–53.

[2] O. Altintaş and S. Owa, *Neighborhoods of certain analytic functions with negative coefficients*, Int. J. Math. Math. Sci. 19 (1996), 797–800.

[3] O. Altintaş, Ö. Özkan and H. M. Srivastava, *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Lett. 13 (2000), 63–67.

[4] O. Altintaş, Ö. Özkan and H. M. Srivastava, *Neighborhoods of a certain family of multivalent functions with negative coefficients*, Comput. Math. Appl. 47 (2004), 1667–1672.

[5] A. Cătaş, *Neighborhoods of a certain class of analytic functions with negative coefficients*, Banach J. Math. Anal. 3 (2009), 111–121.

[6] B. A. Frasin, *Partial sums of certain analytic and univalent functions*, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 21 (2005), 135–145. (electronic)

[7] B. A. Frasin, *Neighborhoods of certain multivalent functions with negative coefficients*, Appl. Math. Comput. 193 (2007), 1–6.

[8] B. A. Frasin, *Generalization of partial sums of certain analytic and univalent functions*, Appl. Math. Lett. 21 (2008), 735–741.

[9] A. W. Goodman, *Univalent functions and nonanalytic curves*, Proc. Amer. Math. Soc. 8 (1957), 598–601.

[10] B. S. Keerthi, A. Gangadharan and H. M. Srivastava, *Neighborhoods of certain subclasses of analytic functions of complex order with negative coefficients*, Math. Comput. Modelling 47 (2008), 271–277.

[11] Z.-W. Liu and M.-S. Liu, *Properties and characteristics of certain subclasses of an. functions*, J. South China Normal Univ. Natur. Sci. Ed. 28 (2010), 11–15.

[12] S. Owa, J. Nichiwaki, *Coefficient for certain classes of analytic functions*, J. Inequal. Pure Appl. Math. 3 (2002), Article 72, pp. 1–5. (electronic)

[13] S. Ruscheweyh, *Neighborhoods of univalent functions*, Proc. Amer. Math. Soc. 81 (1981), 521–527.

[14] H. Silverman, *Partial sums of starlike and convex functions*, J. Math. Anal. Appl. 209 (1997), 221–227.

[15] H. M. Srivastava, S. S. Eker, and B. Seker, *Inclusion and neighborhood properties for certain classes of multivalently analytic functions of complex order associated with the convolution structure*, Appl. Math. Comput. 212 (2009), 66–71.

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