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ON P-VALENT STRONGLY STARLIKE AND STRONGLY CONVEX FUNCTIONS

Ali muhammad

Abstract. In this paper, we define some new classes $S^*(a, c, \lambda, p, \alpha, \beta)$ and $C(a, c, \lambda, p, \alpha, \beta)$ of strongly starlike and strongly convex functions or order $\alpha$and type β by using Cho- Kown -Srivastava integral operators. We also derive some interesting properties, such as inclusion relationships of these classes.

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1. Introduction

Let $A(p)$ denote the class of functions $f(z)$ normalized by

$$z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k},\ p \in \mathbb{N} = \{1, 2, 3, ...\}. \tag{1.1}$$

which are analytic and p-valent in the unit disk $E = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$.

A function $f(z) \in A(p)$ is said to be in the class $S^*(p, \beta)$ of p-valently starlike function of order β in E if it satisfies the following inequality:

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \beta,\ z \in E,\ 0 \le \beta < p. \tag{1.2}$$

The class $S^*(p, \beta)$ was introduced and studied by Goodman [4], see also [11]. For recent investigation and more detail the interested reader are refers to the work by [12, 13]. Also, we note that $S^*(p, 0) = S_p^*$, where S_p^* is the class of p-valently starlike functions in E.

A function $f(z) \in A(p)$ is said to be in the class $C(p, \beta)$ of p-valently convex function of order β in E if it satisfies the following inequality:

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \beta, \ z \in E, \ 0 \leq \beta < p. \tag{1.3}$$

The class $C(p, \beta)$ was introduced and studied by Goodman [4], see also [10]. Also we note that $C(p, 0) = C_p$, where C_p is the class of p-valently convex functions in E.

A function $f(z) \in A(p)$ is said to be strongly starlike of order α and type β in E if it satisfies the following inequality:

$$\left|\arg\left(\frac{zf'(z)}{f(z)} - \beta\right)\right| < \frac{\alpha\pi}{2}, \ z \in E, \ 0 \leq \alpha < p \text{ and } 0 \leq \beta < p. \tag{1.4}$$

We denote this class by $S^*(p, \alpha, \beta)$.A function $f(z) \in A(p)$ is said to be strongly convex of order α and type β in E if it satisfies the following inequality:

$$\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - \beta\right)\right| < \frac{\alpha\pi}{2}, \ z \in E, \ 0 \leq \alpha < p \text{ and } 0 \leq \beta < p. \tag{1.5}$$

We denote this class by $C(p, \alpha, \beta)$.

It is obvious that $f(z) \in A(p)$ belongs to $C(p, \alpha, \beta)$ if and only if $\frac{zf'(z)}{p} \in S^*(p, \alpha, \beta)$. Also, we note that $S^*(p, \alpha, \beta) = S^*(p, \beta)$ and $C(p, \alpha, \beta) = C(p, \beta)$.

In our present investigation we shall make use of the Gauss hypergeometric functions defined by

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k \ (b)_k \ z^k}{(c)_k \ (1)_k}, \tag{1.6}$$

where $a, b, c \in \mathbb{C}$, $c \notin \mathbb{Z}_0^- = \{0, -1, -2, ...\}$ and $(k)_n$ denote the Pochhammer symbol (or the shifted factorial) given, in terms of the Gamma function Γ, by

$$(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)} = \begin{cases} k(k+1)...(k+n-1) & n \in \mathbb{N}, \\ 1 & n = 0. \end{cases}$$

We note that the series defined by (1.6) converges absolutely for $z \in E$ and hence represents an analytic function in the open unit disk E, see [14].

We define a function $\phi_p(a, c; z)$ by

$$\phi_p(a, c; z) = z^p + \sum_{k=1}^{\infty} \frac{(a)_k \ z^{p+k}}{(c)_k}, \ a \in \mathbb{R}; \ c \in \mathbb{R} \backslash \mathbb{Z}_0^- = \{0, -1, -2, ...\} \ (z \in E).$$

Using the function $\phi_p(a,c;z)$, we consider a function $\phi_p^{\dagger}(a,c;z)$ defined by

$$\phi_p(a,c;z) * \phi_p^{\dagger}(a,c;z) = \frac{z^p}{(1-z)^{\lambda+p}},\ z \in E,$$

where $\lambda > -p$. This function yields the following family of linear operators

$$I_p^{\lambda}(a,c)f(z) = \phi_p^{\dagger}(a,c;z) * f(z),\ z \in E, \tag{1.7}$$

where $a,\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-$. For a function $f(z) \in A(p)$, given by (1.1), it follows from (1.7) that for $\lambda > -p$ and $a,\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-$

$$\begin{aligned} I_p^{\lambda}(a,c)f(z) &= z^p + \sum_{k=1}^{\infty} \frac{(a)_k\ (\lambda+p)}{(c)_k\ (1)_k} a_{p+k}\ z^{p+k} \\ &= z^P{}_2F_1(c,\lambda+p;a;z) * f(z),\ z \in E. \end{aligned} \tag{1.8}$$

From equation (1.8) we deduce that

$$z(I_p^{\lambda}(a,c)f(z))' = (\lambda+p)I_p^{\lambda+1}(a,c)f(z) - \lambda I_p^{\lambda}(a,c)f(z), \tag{1.9}$$

$$z(I_p^{\lambda}(a+1,c)f(z))' = aI_p^{\lambda}(a,c)f(z) - (a-p)I_p^{\lambda}(a+1,c)f(z). \tag{1.10}$$

We also note that

$I_p^0(p+1,1)f(z) = p\int\limits_0^z \frac{f(t)}{t}dt,\ \ I_p^0(p,1)f(z) = I_p^1(p+1,1)f(z) = f(z),$

$I_p^1(p,1)f(z) = \frac{zf'(z)}{p},\ \ I_p^2(p,1)f(z) = \frac{2zf'(z)+z^2f''(z)}{p(p+1)},$

$I_p^2(p+1,1)f(z) = \frac{f(z)+zf'(z)}{p+1},\ \ I_p^n(a,a)f(z) = D^{n+p-1}f(z),\ n \in \mathbb{N},\ n > -p,$

where $D^{n+p-1}$is the Ruscheweyh derivative of $(n+p-1)th$ order, see [5].

The operator $I_p^{\lambda}(a,c)$ $(\lambda > -p, a,\ c \in \mathbb{R}\backslash\mathbb{Z}_0^-)$ was recently introduced by Cho et.al [1], who investigated (among other things) some inclusion relationships and argument properties of various subclasses of multivalent functions in $A(p)$, which were defined by means of the operator $I_p^{\lambda}(a,c)$.

For $\lambda = c = 1$ and $a = n+p$, Cho-kown-Srivastava operator $I_p^{\lambda}(a,c)$ yields

$$I_p^1(n+p,1) = I_{n,p}\ (n > -p),$$

where $I_{n,p}$ denotes an integral operator of the (n+p-1)th order, which was studied by Liu and Noor [6], see also [7, 8]. The linear operator $I_1^{\lambda}(\mu+2,1)$ $(\lambda > -1, \mu > -2)$ was also recently introduced and studied by Choi et.al [2]. For relevant details about further special cases of the Choi-Saigo Srivastava operator the interested reader may refer to the works by Choi et. al [2] and Cho et. al [1], see also [3].

Using the operator $I_p^\lambda(a,c)$ we now define new subclasses of $A(p)$ as follows:

Definition 1.1.

$$S^*(a,c,\lambda,p,\alpha,\beta)=\left\{f(z)\in A(p): I_p^\lambda(a,c)f(z)\in S^*(p,\alpha,\beta),\ \frac{z(I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)}\neq\beta\right\}.$$

where $z\in E$.

It is easy to see that $S^*(p,1,0,p,\alpha,\beta)=S^*(p,\alpha,\beta)$ and $S^*(p+1,1,1,p,\alpha,\beta)=S^*(p,\alpha,\beta)$, is the class of strongly starlike functions of order α and type β as given by (1.4).

Definition 1.2.

$$C(a,c,\lambda,p,\alpha,\beta)=\left\{f(z)\in A(p): I_p^\lambda(a,c)f(z)\in C(p,\alpha,\beta), 1+\frac{z(I_p^\lambda(a,c)f(z))''}{(I_p^\lambda(a,c)f(z))'}\neq\beta\right\}.$$

where $z\in E$.

Obviously, $f(z)\in C(a,c,\lambda,p,\alpha,\beta)$ if and only if $\frac{zf'(z)}{p}\in S^*(a,c,\lambda,p,\alpha,\beta)$, $C(p,1,0,p,\alpha,\beta)=C(p,\alpha,\beta)$ and $C(p+1,1,1,p,\alpha,\beta)=C(p,\alpha,\beta)$ is the class of strongly convex functions of order α and type β as given by (1.5).

2. Preliminaries and Main results

In order to prove our main results we shall need the following result.

Lemma 2.1.[9] *Let a function* $h(z)=1+c_1z+c_2z^2+...$ *be analytic in* E *and* $h(z)\neq 0$, $z\in E$. *If there exists a point* $z_0\in E$ *such that*

$$|\arg h(z)|<\frac{\pi}{2}\alpha\ \ (|z|<|z_0|)\ \text{ and }\ |\arg h(z_0)|=\frac{\pi}{2}\alpha\ \ (0\leq\alpha<1),\tag{2.1}$$

then we have,

$$\frac{z_0h'(z_0)}{h(z_0)}=ik\alpha,\tag{2.2}$$

where

$$k\geq\frac{1}{2}(a+\frac{1}{a})\ \text{ when }\ \arg h(z_0)=\frac{\pi}{2}\alpha,\tag{2.3}$$

$$k\leq-\frac{1}{2}((a+\frac{1}{a})\ \text{ when }\ \arg h(z_0)=\frac{\pi}{2}\alpha,\tag{2.4}$$

and

$$(h(z_0))^{\frac{1}{\alpha}}=\pm i\alpha,\ \ (\alpha>0).$$

Theorem 2.2. $S^*(a,c,\lambda+1,p,\alpha,\beta) \subset S^*(a,c,\lambda,p,\alpha,\beta) \subset S^*(a+1,c,\lambda,p,\alpha,\beta)$.
Proof. Set

$$\frac{z(I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)} = \beta + (p-\beta)h(z). \tag{2.5}$$

Then $h(z)$ is analytic in E with $h(0)=1$ and $h\ (z) \neq 0$, $z \in E$. Applying the identity (1.9) in (2.1) and differentiating the resulting equation with respect to z we have

$$\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)} - \beta = (p-\beta)h(z) + \frac{(p-\beta)zh'(z)}{(\lambda+\beta)+(p-\beta)h(z)}.$$

Suppose there exists a point $z_0 \in E$ such that the conditions (2.1) to (2.4) of Lemma 2.1 are satisfied. Thus, if

$$\arg h(z_0) = \frac{-\pi}{2}\alpha,\ \ z_0 \in E,$$

then

$$\begin{aligned}\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)} - \beta &= (p-\beta)h(z_0)\left[1 + \frac{\frac{z_0h'(z_0)}{h(z_0)}}{(\lambda+\beta)+(p-\beta)h(z_0)}\right] \\ &= (p-\beta)a^\alpha e^{-\frac{\pi\alpha}{2}}\left[1 + \frac{ik\alpha}{(\lambda+\beta)+(p-\beta)a^\alpha e^{-\frac{\pi\alpha}{2}}}\right].\end{aligned}$$

This implies that

$$\arg\left(\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)} - \beta\right) = \frac{-\pi\alpha}{2} + \arg\left[1 + \frac{ik\alpha}{(\lambda+\beta)+(p-\beta)a^\alpha e^{-\frac{\pi\alpha}{2}}}\right] = \frac{-\pi\alpha}{2}$$

$$+\tan^{-1}\left[\frac{k\alpha((\lambda+\beta)+(p-\beta)a^\alpha\cos\left(\frac{\pi\alpha}{2}\right)}{(\lambda+\beta)^2+2(\lambda+\beta)(p-\beta)a^\alpha\cos\left(\frac{\pi\alpha}{2}\right)+(p-\beta)^2a^{2\alpha}-k\alpha(p-\beta)a^\alpha\sin\left(\frac{\pi\alpha}{2}\right)}\right].$$

This gives that

$$\arg\left(\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)} - \beta\right) \leq \frac{-\pi\alpha}{2},$$

since

$$k \leq \frac{-1}{2}(a+\frac{1}{a}) \leq -1 \text{ and } z_0 \in E,$$

this contradicts the condition $f(z) \in S^*(a,c,\lambda,p,\alpha,\beta)$.

On the other hand if we set

$$\arg h(z_0) = \frac{\pi\alpha}{2},$$

then it can similarly be shown that

$$\arg\left(\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)}-\beta\right)\geq\frac{\pi\alpha}{2},$$

since

$$k\geq\frac{1}{2}(a+\frac{1}{a})\text{ and }z_0\in E,$$

which again contradicts the hypothesis that $f(z)\in S^*(a,c,\lambda,p,\alpha,\beta)$.

Thus the function defined by (2.5) has to satisfy the following inequality:

$$\arg h(z)\leq\frac{\pi\alpha}{2},\ z\in E.$$

which implies that

$$\left|\arg\left(\frac{z(I^{\lambda+1}(a,c)f(z))'}{I^{\lambda+1}(a,c)f(z)}-\beta\right)\right|<\frac{\pi\alpha}{2},\ z\in E.$$

The proof of part (ii) lies on the similar lines. This completes the proof of Theorem 2.2.

Theorem 2.3.$C(a,c,\lambda+1,p,\alpha,\beta)\subset C(a,c,\lambda,p,\alpha,\beta)\subset C(a+1,c,\lambda,p,\alpha,\beta)$
Proof. To prove this inclusion relationship, we observe from Theorem 2.2 that

$$\begin{aligned}f(z) &\in C(a,c,\lambda+1,p,\alpha,\beta)\Leftrightarrow\frac{zf'(z)}{p}\in S^*(a,c,\lambda+1,p,\alpha,\beta)\\ &\Rightarrow \frac{zf'(z)}{p}\in S^*(a,c,\lambda,p,\alpha,\beta)\Leftrightarrow f(z)\in C(a,c,p,\alpha,\beta).\end{aligned}$$

The proof of second part lies on similar lines. This completes the proof of Theorem 2.3.

For a function $f(z)\in A(p)$ the integral operator, $\mathcal{F}_{\delta,p}:A(p)\longrightarrow A(p)$ is defined by

$$\begin{aligned}\mathcal{F}_{\delta,p}(f)(z) &= \frac{\delta+p}{z^p}\int_0^z t^{\delta-1}f(t)dt=\left(z^p+\sum_{k=1}^{\infty}\frac{\delta+p}{\delta+p+k}z^{p+k}\right)*f(z) \quad (2.6)\\ &= z^p\ {}_2F_1(1,\delta+p,\delta+p+1;z)*f(z),\quad z\in E,\ \text{see [2]}.\end{aligned}$$

It follows from (2.6) that

$$z\left((I_p^{\lambda}(a,c)\mathcal{F}_{\delta,p}(f)(z)\right)'=(\delta+p)I_p^{\lambda}(a,c)f(z)-\delta I_p^{\lambda}(a,c)\mathcal{F}_{\delta,p}(f)(z). \qquad (2.7)$$

We have the following result.

Theorem 2.4. *Let* $\delta > -\beta$ *and* $0 \leq \beta < p$.*If* $f(z) \in S^*(a, c, \lambda, p, \alpha, \beta)$ *with*

$$\frac{z(I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z))'}{I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z)} \neq \beta, \text{ for all } z \in E,$$

then we have

$$\mathcal{F}_{\delta,p}(f)(z) \in S^*(a, c, \lambda, p, \alpha, \beta).$$

Proof. We begin by setting

$$\frac{z\ (\ I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z))'}{I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z)} = \beta + (p-\beta)h(z), \quad z \in E. \tag{2.8}$$

Then $h(z)$ is analytic in E with $h(0) = 1$. Using (2.7) and (2.8), we find that

$$(\delta + p)\frac{(I_p^\lambda(a,c)f(z)}{I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z)} = (\delta + p) + (p-\beta)h(z). \tag{2.9}$$

Differentiation both sides of (2.9) logarithmically, we obtain

$$\frac{z(\ I_p^\lambda(a,c)f(z))'}{I_p^\lambda(a,c)f(z)} - \beta = (p-\beta)h(z) + \frac{(p-\beta)zh'(z)}{(\delta+\beta)+(p-\beta)h(z)}.$$

Suppose now there exists a point $z_0 \in E$ such that

$$|\arg h(z)| < \frac{\pi}{2}\alpha \ (|z| < |z_0|) \text{ and } |\arg h(z)| = \frac{\pi}{2}\alpha.$$

Then by Lemma 2.1, we can write that

$$\frac{z_0 h'(z_0)}{h(z_0)} = ik\alpha \text{ and } (h(z_0))^{\frac{1}{\alpha}} = \pm i\alpha \ (\alpha > 0).$$

If

$$\arg h(z_0) = \frac{\pi}{2}\alpha, \ z_0 \in E,$$

then

$$\begin{aligned}\frac{z(I_p^\lambda(a,c)f(z_0))'}{I_p^\lambda(a,c)f(z_0)} - \beta &= (p-\beta)h(z_0)\left[1 + \frac{\frac{zh'(z_0)}{h(z_0)}}{(\delta+\beta)+(p-\beta)h(z_0)}\right] \\ &= (p-\beta)a^\alpha e^{i\frac{\pi\alpha}{2}}\left[1 + \frac{ik\alpha}{(\delta+\beta)+(p-\beta)a^\alpha e^{i\frac{\pi\alpha}{2}}}\right].\end{aligned}$$

This shows that

$$\arg\left(\frac{z(I_p^\lambda(a,c)f(z_0))'}{I_p^\lambda(a,c)f(z_0)}-\beta\right)=\frac{\pi\alpha}{2}+\arg\left\{1+\frac{ik\alpha}{(\delta+\beta)+(p-\beta)a^\alpha e^{i\frac{\pi\alpha}{2}}}\right\}=\frac{\pi\alpha}{2}$$

$$+\quad\tan^{-1}\left\{\frac{k\alpha\left(\delta+\beta+(p-\beta)\right)a^\alpha\cos\left(\frac{\pi\alpha}{2}\right)}{(\delta+\beta)^2+2(\delta+\beta)(p-\beta)a^\alpha\cos\left(\frac{\pi\alpha}{2}\right)+(p-\beta)^2a^{2\alpha}+k\alpha(p-\beta)a^\alpha\sin\left(\frac{\pi\alpha}{2}\right)}\right\}$$

$$\geq\quad\frac{\pi\alpha}{2},$$

since

$$k\geq\frac{1}{2}(a+\frac{1}{a})\geq 1 \text{ and } z_0\in E,$$

which contradicts the assumption that $f(z)\in S^*(a,c,\lambda,p,\alpha,\beta)$.

Similarly in the case when

$$\arg h(z_0)=-\frac{\pi}{2}\alpha,\ z_0\in E,$$

we can prove that

$$\arg\left(\frac{z(I_p^\lambda(a,c)f(z_0))'}{I_p^\lambda(a,c)f(z_0)}-\beta\right)\leq\frac{-\pi\alpha}{2},$$

since

$$k\leq-\frac{1}{2}(a+\frac{1}{a})\leq -1 \text{ and } z_0\in E,$$

contradicting once again the condition that $f(z)\in S^*(a,c,\lambda,p,\alpha,\beta)$.

We thus conclude that $h(z)$ must satisfy the following inequality

$$|\arg h(z)|<\frac{\pi}{2}\alpha,\ z\in E.$$

This shows that

$$\left|\arg\frac{z(I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z))'}{I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z)}-\beta\right|<\frac{\pi\alpha}{2},\ z\in E,$$

and hence the proof is complete.

Theorem 2.5. *Let* $\delta>-\beta$ *and* $0\leq\beta<p$. *If* $f(z)\in C(a,c,\lambda,p,\alpha,\beta)$ *and*

$$1+\frac{z(I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z))''}{(I_p^\lambda(a,c)\mathcal{F}_{\delta,p}(f)(z))'}\neq\beta \text{ for all } z\in E,$$

then we have

$$\mathcal{F}_{\delta,p}(f)(z) \in C(a,c,\lambda,p,\alpha,\beta).$$

Proof. Since

$$\begin{aligned} f(z) &\in C(a,c,\lambda,p,\alpha,\beta) \Leftrightarrow \frac{zf'(z)}{p} \in S^*(a,c,\lambda,p,\alpha,\beta) \\ &\Rightarrow \frac{\mathcal{F}_{\delta,p}(f)(z)}{p} \in S^*(a,c,\lambda,p,\alpha,\beta) \Leftrightarrow \frac{z(\mathcal{F}_{\delta,p}(f)(z))'}{p} \in S^*(a,c,\lambda,p,\alpha,\beta) \\ &\Leftrightarrow \mathcal{F}_{\delta,p}(f)(z) \in C(a,c,\lambda,p,\alpha,\beta). \end{aligned}$$

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References

[1] N. E. Cho, O. S. Kown, and H. M. Srivastava, *Inclusion relationships and argument properties for certain subclasses of multivalent functions associated with a family of linear operators*, J. Math. Anal. App. 292 (2004), 470 -483.

[2] J. H. Choi, M. Siago, and H .M. Srivastava, *Some inclusion properties of certain family of integral operators*, J. Math. Anal. Appl. 276 (2002), 432 - 445.

[3] N. E. Cho, O. H. Kown, and H. M. Srivastava, *Inclusion and argument properties for certain subclasses of meromorphic functions associated with a family of multiplier transformations*, J. Math. Anal. Appl. 300 (2004), 505- 520.

[4] A. W. Goodman, On the Schwarz Christoffel transformation and P-valent functions, Trans. Amer. Math. Soc. 68 (1950), 204 - 233.

[5] R. M. Goel and N. S. Sohi, *A new criterion for p-valent functions*, Proc. Amer. Math. Soc. 78 (1980), 353 - 357.

[6] J. L. Liu and K. I. Noor, *Some properties of Noor integral operator*, J. Natur. Geom. 21 (2002),81-90.

[7] K. I. Noor, *Some classes of p-valent analytic functions defined by certain integral operator*, Appl. Math. Compute. 157 (2004), 835-840.

[8] K. I. Noor and M. A. Noor, *On certain classes of analytic functions defined by Noor integral operator*, J. Math. Appl. 281 (2003), 244-252.

[9] M. Nunokawa, *On the order of strongly starlikness of strongly convex functions*, Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), 234-237.

[10] S. Owa, *On certain classes of P-valent functions with negative coefficients*, Simon Stevin 59 (1985), 385-402.

[11] D. A. Patel and N. K. Thakare, *On convex hulls and extreme points of P-valent starlike and convex classes with applications*, Bull. Math. Soc. Sci. Math. Roumanie (N.S) 27 (1983), 145 -160.

[12] J. Patel, N. E. Cho and H. M. Srivastava, *Certain subclasses of multivalent functions associated with a family of linear operators*, Math. Comput. Modelling 43 (2006), 320-338.

[13] H. M. Srivastava, J. Patel, and G. P. Mohapatra, *A certain class of p-valently analytic functions*, Math. Comput. Modelling 41 (2005), 321-334.

[14] E. T. Whittaker and G. N. Watson, *A course on Modern Analysis: An introduction to the general Theory of Infinite processes and of Analytic Functions; With an account of the principle of Transcendental Functions*, 4th Edition, Cambridge University Press, Cambridge, 1927.

Ali Muhammad
Depatment of Basic Sciences
University of Engineering and Technology
Technology Peshawar Pakistan
E-mail: *ali7887@gmail.com*