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RUSCHEWEYH-TYPE UNIVALENT HARMONIC FUNCTIONS STARLIKE OF THE COMPLEX ORDER

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Abstract. In this paper, we have defined the class $\mathcal{R}T^*_{\mathcal{H}}(\gamma, \lambda, \beta)$ by making use of the Ruscheweyh derivatives and we give necessary and sufficient conditions for the functions to be in $\mathcal{R}T^*_{\mathcal{H}}(\gamma, \lambda, \beta)$.

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1. Introduction

A continuous function $f = u + iv$ is a complex-valued harmonic function in a domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D. In any simply connected domain, we can write $f = h + \overline{g}$, where h and g are analytic in D. We call h the analytic part and g the co-analytic part of f. Clunie and Sheil-Small [2] proved a necessary and sufficient condition for f to be locally univalent and sense-preserving in D is that $|h'(z)| > |g'(z)|$ in D.

Let $\mathcal{H}$ denote the class of functions $f = h + \overline{g}$ that are harmonic univalent and sense-preserving in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ with $f(0) = f_z(0) - 1 = 0$. Therefore we can express analytic and co-analytic parts of the function $f = h + \overline{g}$ as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1. \tag{1}$$

We can note that $\mathcal{H}$ reduces to S, the class of normalized univalent analytic functions whenever the co-analytic part $g \equiv 0$.

Let $\mathcal{R}T_{\mathcal{H}}$ denote the family of functions $f = h + \overline{g}$ that are harmonic in $\mathcal{U}$ with the normalization

$$h(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad a_n \geq 0, \quad b_n \geq 0, \quad b_1 < 1. \tag{2}$$

For $\lambda > -1, \gamma \in \mathbb{C} \setminus \{0\}$ and $0 \leq \beta \leq 1$, we let $\mathcal{RT}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$ denote the class of all functions in $\mathcal{RT}_{\mathcal{H}}$ for which

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{z(D^\lambda f(z))'}{\beta z(D^\lambda f(z))' + (1-\beta)D^\lambda f(z)} - 1\right)\right\} > 0. \tag{3}$$

Here, the operator $D^\lambda f(z)$ is the Ruscheweyh derivative of $\phi(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ given by

$$D^\lambda \varphi(z) = \frac{z}{(1-z)^{1+\lambda}} * \varphi(z) = \sum_{n=1}^{\infty} B_n(\lambda) c_n z^n,$$

where $*$ stands for the convolution or Hadamard product of two power series and

$$B_n(\lambda) = \frac{(\lambda+1)(\lambda+2)\cdots(\lambda+n-1)}{(n-1)!}, \quad \text{see [5]}.$$

Also if $f(z) = h(z) + \overline{g}(z)$ then

$$D^\lambda f(z) = D^\lambda h(z) + \overline{D^\lambda g(z)}, \quad \text{see [4]}. \tag{4}$$

We note that $\mathcal{RT}^*_{\mathcal{H}}(\gamma, 0, 0)$ is the class of harmonic function in the unit disk studied by Sibel et al. [6].

Furthermore, let $\mathcal{ST}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$ denote the subclass at $\mathcal{RT}_{\mathcal{H}}$ consisting of functions $f = h + \overline{g} \in \mathcal{RT}_{\mathcal{H}}$ that satisfy the following

$$\sum_{n=1}^{\infty}[2((n-1)(1-\beta) + (\beta(n-1)+1)|\gamma|)B_n(\lambda)a_n$$
$$+((n+1)(1-\beta) + |(n+1)(1-\beta) - 2\gamma(1-\beta(n+1))|)B_n(\lambda)b_n] \leq 4|\gamma| \tag{5}$$

We also consider $\mathcal{LR}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$ the subclass of $\mathcal{RT}_{\mathcal{H}}$ consisting of functions $f = h + \overline{g} \in \mathcal{RT}_{\mathcal{H}}$ that satisfy the following

$$\sum_{n=1}^{\infty}\left[(n-1)(1-\beta)\frac{Re(\gamma)}{|\gamma|} + (1+\beta(n-1)|\gamma|\right]B_n(\lambda)a_n$$
$$+\left[(n+1)(1-\beta)\frac{Re(\gamma)}{|\gamma|} - (1-\beta(n+1))|\gamma|\right]B_n(\lambda)b_n \leq (2+\beta)|\gamma| \tag{6}$$

The harmonic starlike functions studied by Avci and Zlotkiewicz [1], Jahangiri [3], Silverman [7], and Silverman and Silvia [8].

The coefficient condition $\sum\limits_{n=2}^{\infty} n(|a_n| + |b_n|) \leq 1$, with $b_1 = 0$ is sufficient for $f = h + \overline{g}$ to be harmonic starlike proved by Avci and Zlotkiewicz [1] while Silverman

[7] proved that this coefficient condition is also necessary if $b_1 = 0$ and if a_n and b_n in (1) are negative. Jahangiri [3] proved that if $f = h + \overline{g}$ is given by (1) and if

$$\sum_{n=1}^{\infty}\left(\frac{n-\alpha}{1-\alpha}|a_n| + \frac{n+\alpha}{1-\alpha}|b_n|\right) \leq 2, \quad 0 \leq \alpha < 1, \quad a_1 = 1, \tag{8}$$

then f is harmonic, univalent, and starlike of order α in $\mathcal{U}$. This condition is proved to be also necessary if h and g are of the form (2). the case when $\alpha = 0$ is given in [8], and for $\alpha = b_1 = 0$, see [7].

2.Main Results

Theorem 1. $\mathcal{ST}^*_{\mathcal{H}}(\gamma, \lambda, \beta) \subset \mathcal{RT}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$.

Proof. Let $f \in \mathcal{ST}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$. We need to show that the condition (3) holds, therefore

$$\begin{aligned}&Re\{[(\gamma-1)[\beta z(D^{\lambda}h(z))' - \overline{\beta z(D^{\lambda}g(z))'} + (1-\beta)D^{\lambda}h(z) + (1-\beta)\overline{D^{\lambda}g(z)}]\\ &+z(D^{\lambda}h(z))' - \overline{z(D^{\lambda}g(z))'}]/[\gamma[\beta z(D^{\lambda}h(z))' - \overline{\beta z(D^{\lambda}g(z))'} + (1-\beta)D^{\lambda}h(z)\\ &+(1-\beta)\overline{D^{\lambda}g(z)}]\} > 0, \quad \text{where} \quad 0 \leq \beta < 1, \gamma \in \mathbb{C}\setminus\{0\}, \quad \lambda > -1.\end{aligned}$$

Using the fact that $Re\ w > 0$ if and only if $|w+1| > |1-w|$, then we have and by (2)

$$\begin{aligned}&|(2\gamma-1)[\beta z(D^{\lambda}h(z))' - \overline{\beta z(D^{\lambda}g(z))'} + (1-\beta)D^{\lambda}h(z) + (1-\beta)\overline{D^{\lambda}g(z)}]\\ &+ z(D^{\lambda}h(z))' - \overline{z(D^{\lambda}g(z))'}| - |\beta z(D^{\lambda}h(z))' - \beta \overline{z(D^{\lambda}g(z))'}\\ &+ (1-\beta)D^{\lambda}h(z) + (1-\beta)\overline{D^{\lambda}g(z)} - z(D^{\lambda}h(z))' + \overline{z(D^{\lambda}g(z))'}|\\ &= |(2\gamma-1)[\beta z - \textstyle\sum_{n=2}^{\infty}\beta n B_n(\lambda)a_n z^n - \sum_{n=1}^{\infty}\beta n B_n(\lambda)b_n\overline{z}^n + (1-\beta)z\\ &- \textstyle\sum_{n=2}^{\infty}(1-\beta)B_n(\lambda)a_n z^n + \sum_{n=1}^{\infty}(1-\beta)B_n(\lambda)b_n\overline{z}^n] + z - \sum_{n=2}^{\infty} nB_n(\lambda)a_n z^n\\ &- \textstyle\sum_{n=1}^{\infty} nB_n(\lambda)b_n\overline{z}^n|\\ &-|\beta z - \textstyle\sum_{n=2}^{\infty}\beta n B_n(\lambda)a_n z^n - \sum_{n=1}^{\infty}\beta n B_n(\lambda)b_n\overline{z}^n + (1-\beta)z - \sum_{n=2}^{\infty}(1-\beta)B_n(\lambda)a_n z^n\\ &+ \textstyle\sum_{n=1}^{\infty}(1-\beta)B_n(\lambda)b_n\overline{z}^n - z + \sum_{n=2}^{\infty} nB_n(\lambda)a_n z^n + \sum_{n=1}^{\infty} nB_n(\lambda)b_n\overline{z}^n|\\ &= 2\gamma z - \textstyle\sum_{n=2}^{\infty}(2\gamma\beta n - \beta n + 2\gamma - 2\gamma\beta - 1 + \beta + n)B_n(\lambda)a_n z^n\\ &- \textstyle\sum_{n=1}^{\infty}(2\gamma\beta n - \beta n - 2\gamma + 2\gamma\beta + 1 - \beta + n)B_n(\lambda)b_n\overline{z}^n|\\ &- |\textstyle\sum_{n=2}^{\infty}(n - \beta n - 1 + \beta)B_n(\lambda)a_n z^n + \sum_{n=1}^{\infty}(n + 1 - \beta n - \beta)B_n(\lambda)b_n\overline{z}^n|\\ &> 2|\gamma| - (\textstyle\sum_{n=2}^{\infty} 2((n-1)(1-\beta) + (\beta(n-1)+1)|\gamma|)B_n(\lambda)a_n + \sum_{n=1}^{\infty}((n+1)(1-\beta)\\ &+ |(n+1)(1-\beta) - 2\gamma(1-\beta(n+1))|)B_n(\lambda)b_n) \geq 0.\end{aligned}$$

For sharpness consider the function

$$\begin{aligned}f(z) \quad &= \quad z - \sum_{n=2}^{\infty}\frac{|\gamma|}{(n-1)(1-\beta) + (\beta(n-1)+1)|\gamma|}s_n z^n\\ &+\sum_{n=1}^{\infty}\frac{2|\gamma|}{(n+1)(1-\beta) + |(n+1)(1-\beta) - 2\gamma(1-\beta(n+1))}t_n\overline{z}^n \end{aligned} \tag{8}$$

where s_n, t_n are non-negative and $\sum\limits_{n=2}^{\infty} s_n + \sum\limits_{n=1}^{\infty} t_n = 1$ and all the functions of the form (8) are in $\mathcal{R}T_H^*(\gamma,\lambda,\beta)$, since

$$\sum_{n=2}^{\infty}(2((n-1)(1-\beta)+(\beta(n-1)+1)|\gamma|)B_n(\lambda)a_n+\sum_{n=1}^{\infty}((n+1)(1-\beta)$$
$$+|(n+1)(1-\beta)-2\gamma(1-\beta(n+1))|)B_n(\lambda)b_n=2|\gamma|(1+\sum_{n=2}^{\infty}s_n+\sum_{n=1}^{\infty}t_n)=4|\gamma|.$$

Theorem 2. $\mathcal{R}T_{\mathcal{H}}^*(\gamma,\lambda,\beta)\subset\mathcal{LR}_{\mathcal{H}}^*(\gamma,\lambda,\beta)$.

Proof. Let $f\in\mathcal{R}T_{\mathcal{H}}^*(\gamma,\lambda,\beta)$, then from the condition (3) we have

$$Re\left\{\frac{1}{\gamma}\left(\frac{z(D^\lambda h(z))'-\overline{z(D^\lambda g(z))'}}{\beta z(D^\lambda h(z))'-\overline{\beta z(D^\lambda g(z))'}+(1-\beta)D^\lambda h(z)+(1-\beta)\overline{D^\lambda g(z))}}-1\right)\right\}>-1.$$

By using (2), we obtain $Re\{\frac{1}{\gamma}([z-\sum_{n=2}^{\infty}nB_n(\lambda)a_nz^n-\sum_{n=1}^{\infty}nB_n(\lambda)b_n\overline{z}^n]/[\beta z-\sum_{n=2}^{\infty}\beta nB_n(\lambda)a_nz^n-\sum_{n=1}^{\infty}\beta nB_n(\lambda)b_n\overline{z}^n+(1-\beta)z$ $-\sum_{n=2}^{\infty}(1-\beta)B_n(\lambda)a_nz^n+\sum_{n=1}^{\infty}(1-\beta)B_n(\lambda)b_n\overline{z}^n]-1\}>-1$, then, we have

$$Re\left\{\frac{1}{\gamma}\frac{-\sum\limits_{n=2}^{\infty}(n-\beta n-1+\beta)B_n(\lambda)a_nz^n-\sum\limits_{n=1}^{\infty}(n-\beta n+1-\beta)B_n(\lambda)b_n\overline{z}^n}{z-\sum\limits_{n=2}^{\infty}(\beta n+1-\beta)B_n(\lambda)a_nz^n+\sum\limits_{n=1}^{\infty}(1-\beta-\beta n)B_n(\lambda)b_n\overline{z}^n}\right\}>-1.$$

Choosing $z\to 1^-$ on the real axis, we obtain

$$\frac{\sum\limits_{n=2}^{\infty}(n-1)(1-\beta)B_n(\lambda)a_n+\sum\limits_{n=1}^{\infty}(n+1)(1-\beta)B_n(\lambda)b_n}{1-\sum\limits_{n=2}^{\infty}(1+\beta(n-1))B_n(\lambda)a_n+\sum\limits_{n=1}^{\infty}(1-\beta(n+1))B_n(\lambda)b_n}Re\left(\frac{1}{\gamma}\right)\le 1,$$

thus,

$$\sum_{n=2}^{\infty}(n-1)(1-\beta)B_n(\lambda)a_n+\sum_{n=1}^{\infty}(n+1)(1-\beta)B_n(\lambda)b_n$$
$$\le\frac{|\gamma|^2}{Re(\gamma)}\left(1-\sum_{n=2}^{\infty}(1+\beta(n-1))B_n(\lambda)a_n+\sum_{n=1}^{\infty}(1-\beta(n+1))B_n(\lambda)b_n\right),$$

then we get $\sum_{n=1}^{\infty}\left[(n-1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}+(1+\beta(n-1)|\gamma|\right]B_n(\lambda)a_n$
$+\left[(n+1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}-(1-\beta(n+1))|\gamma|\right]B_n(\lambda)b_n \le 2|\gamma|$. Then by (6) we have $f\in\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$.

Theorem 3. *$\mathcal{ST}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)=\mathcal{RT}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)=\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$, where $0<\gamma\le 1, 0\le \beta<1$ and $\lambda>-1$.*

Proof. If $\gamma\in(0,1]$, then the condition (5) and (6) are equivalent and here $\mathcal{ST}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)=\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$. By making use the previous two theorems, we get the result and this complete the proof.

Theorem 4. *$\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)\not\subseteq\mathcal{RT}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$, if $Re(\gamma)\le 0$ and $Re(\gamma)\ne -\frac{1}{2}$ or $\gamma\in(\frac{3}{2},\infty)$.*

Proof. Consider the function $f(z)=z-\frac{1}{\lambda+1}z^2, \lambda>-1, f\in\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$, since $\sum_{n=1}^{\infty}\left[(n-1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}+(1+\beta(n-1))|\gamma|\right]B_n(\lambda)a_n$
$+\left[(n+1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}-(1-\beta(n+1)|\gamma|\right]B_n(\lambda)b_n=|\gamma|+(1-\beta)\frac{Re(\gamma)}{|\gamma|}+(1+\beta)|\gamma|=(2+\beta)|\gamma|+(1-\beta)\frac{Re(\gamma)}{|\gamma|}\le(2+\beta)|\gamma|$ when $\gamma\in\mathbb{C}\setminus\{0\}$ and $Re(\gamma)<0$.

Also, let $r=Re(\gamma)<0$ and t be negative real number such that $(1-\beta)+2r(1+\beta)(1-t)>0$. If we choose $z=\frac{\gamma(1-t)}{1-\beta+\gamma(1+\beta)(1-t)}$, then $z\in\mathcal{U}$ and by $D^{\lambda}f(z)=z-z^2$, we have $1+\frac{1}{\gamma}\left(\frac{z(D^{\lambda}f(z))'}{\beta z(D^{\lambda}f(z))'+(1-\beta)D^{\lambda}f(z)}-1\right)=t<0$, then $f(z)\notin\mathcal{RT}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$.

By the same way, let $f(z)=z+\frac{1}{\lambda+1}\overline{z}^2$, then if $\gamma\in\left(\frac{3(1-\beta)}{2},\infty\right)$, we obtain $f\in\mathcal{LR}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$, since

$$\sum_{n=1}^{\infty}\left[(n-1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}+(1+\beta(n-1))|\gamma|)\right]B_n(\lambda)a_n$$
$$+\left[(n+1)(1-\beta)\frac{Re(\gamma)}{|\gamma|}-(1-\beta(n+1))|\gamma|\right]B_n(\lambda)b_n$$
$$=3(1-\beta)\frac{Re(\gamma)}{|\gamma|}+3\beta|\gamma|\le(2+\beta)|\gamma|.$$

Now let t be a negative real number such that $3(1-\beta)+\gamma(t-1)<0$, choose $z=-\frac{\gamma(t-1)}{3(1-\beta)+\gamma(t-1)}$, then $z\in\mathcal{U}$ and by definition of f we have

$$1+\frac{1}{\gamma}\left(\frac{z(D^{\lambda}f(z))'}{\beta z(D^{\lambda}f(z))'+(1-\beta)D^{\lambda}f(z)}-1\right)=t<0,$$

therefore $f\notin\mathcal{RT}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$.

Theorem 5. *$\mathcal{RT}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)\not\subseteq\mathcal{ST}_{\mathcal{H}}^{*}(\gamma,\lambda,\beta)$, whenever $\gamma<-1,\lambda>-1$ and $\beta\in[0,1)$.*

Proof. Consider the function $f_\sigma(z) = z - \frac{\sigma}{1+\lambda} z^2, \lambda > -1$ and $\sigma > \frac{\gamma}{(1-\beta)+\gamma(1+\beta)}$, then $f \in \mathcal{RT}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$, since

$$Re\left\{1 + \frac{1}{\gamma}\left(\frac{z(D^\lambda f_\sigma(z))'}{\beta z(D^\lambda f_\sigma(z))' + (1-\beta)D^\lambda f_\sigma(z)} - 1\right)\right\}$$
$$= Re\left\{1 + \frac{\sigma z(1-\beta)}{\gamma(\sigma z(1+\beta) - 1)}\right\} > 0.$$

We have also

$$\sum_{n=1}^{\infty}[2((n-1)(1-\beta) + (\beta(n-1)+1)|\gamma|)B_n(\lambda)a_n + ((n+1)(1-\beta)$$
$$+|(n+1)(1-\beta) - 2\gamma(1-\beta(n+1))|)B_n(\lambda)b_n]$$
$$= 2|\gamma| + [2(1-\beta) + 2(\beta+1)|\gamma|]\sigma > 4|\gamma|,$$

because $\sigma > \frac{\gamma}{(1-\beta)+\gamma(1+\beta)} > 1$, then $f \notin \mathcal{ST}^*_{\mathcal{H}}(\gamma, \lambda, \beta)$.

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