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HIGH ORDER ACCELERATIONS AND POLES UNDER THE ONE-PARAMETER PLANAR HOMOTHETIC MOTIONS

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Abstract. In this paper, after presenting a summary of the one-parameter planar homothetic motion on the complex plane given by Kuruoğlu [1], high ordered velocities, accelerations and poles are analyzed under the one-parameter homothetic motions on the complex plane.

Also, high ordered velocities and accelerations are presented by taking the angle of the rotation instead of the parameter of the motion.

In the case of the homothetic rate $h \equiv 1$, some results, which were given by Müller [2] are obtained.

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1.Introduction

Let E and E' be moving and fixed complex planes and $\{O; e_1, e_2\}$, $\{O'; e'_1, e'_2\}$ be their coordinate systems, respectively. If the vector OO' is represented by the complex number u', the motion defined by the transformation

$$x' = u' + hxe^{i\varphi} \tag{1}$$

is called a one-parameter planar homothetic motion and denoted by $H_1 = E/E'$, where h is the homothetic scale and φ is the rotation angle of the motion H_1, that is the angle between the vectors e_1 and e'_1, and the complex numbers $x = x_1 + ix_2$, $x' = x'_1 + ix'_2$ represent the point $X \in E$ with respect to the moving and the fixed rectangular coordinate systems, respectively. The homothetic scale $h = h(t)$, the rotation angle $\varphi = \varphi(t)$, x, x' and u' are continuously differentiable functions of a real parameter t.Furthermore, at the initial time $t = 0$ the coordinate systems are coincident. To avoid the cases of the pure translation and the pure rotation we assume that $\dot{\varphi} = \dot{\varphi}(t)$, $\qquad h = h(t) \neq$ constant (see [1], [3], [4], for homothetic motion.)

Let the complex number $u = u_1 + iu_2$ represents the origin of the fixed point system with respect to the moving system. Then, if we take $X' = O'$ we obtain $x' = 0$ and $hx = u$. Thus, we have from Eq. (1)

$$u' = -ue^{i\varphi}. \tag{2}$$

Let X be a moving point of E. Then, the velocity of X with respect to E is known as relative velocity of the motion H_1 and it is shown by X_r. This vector is given with $X_r = \frac{dx}{dt} = \dot{x}$. This vector is expressed with the complex number

$$X'_r = X_r e^{i\varphi} = \dot{x}e^{i\varphi} \tag{3}$$

in the fixed coordinate system.

If we differentiate Eq. (1) with respect to t , then we obtain the absolute velocity of the motion H_1 as $X'_a = \dot{u}' + \dot{h}xe^{i\varphi} + h\dot{x}e^{i\varphi} + hi\dot{\varphi}xe^{i\varphi}$ or

$$X'_a = \dot{u}' + (\dot{h} + hi\dot{\varphi})xe^{i\varphi} + h\dot{x}e^{i\varphi}, \tag{4}$$

where

$$X'_f = \dot{u}' + (\dot{h} + hi\dot{\varphi})xe^{i\varphi} \tag{5}$$

is the sliding velocity of the motion H_1.

Theorem 1. *Let X be a moving point in E and, X_a, X_f and X_r be the absolute, sliding and relative velocities of X, respectively under the one-parameter planar homothetic motion $H_1 = E/E'$. Then*

$$X_a = X_f + hX_r. \tag{6}$$

The proof is obvious by using the definitions of velocities above, [1].□

Under the motion H_1 with the homothetic scale $h(t) \neq 0$, $\forall t \in I$, using Eqs. (1) and (5), we can rewrite

$$X'_f = \dot{u}' + \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)(x' - u'). \tag{7}$$

For a general planar homothetic motions, there is a point that does not move, which means that its coordinates are the same in both reference coordinate systems $\{O; e_1, e_2\}$ and $\{O'; e'_1, e'_2\}$. This point is called the pole point or the instantaneous rotation pole center. In this case, we obtain $X'_f = 0$. From Eq. (7), for the pole point $p' = p'_1 + ip'_2$, we get

$$p' = u' - \frac{\dot{u}'}{\left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)}. \tag{8}$$

Special case 1. In the case of $h(t) \equiv 1$, from Eq. (8) we get $p' = u' - \frac{\dot{u}'}{i\dot{\varphi}}$ which was given by Müller [2].□

Let p' be the instantaneous rotation pole center of the motion H_1 and X be a moving point of E. Then the pole ray $P'X'$ is given by

$$P'X' = x' - p'. \tag{9}$$

Moreover, from Eqs. (7) and (8), we can rewrite

$$X_f' = \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\left(x' - p'\right). \tag{10}$$

Then, from Eq. (10), we get

$$\|X_f'\| = \sqrt{\left(\frac{\dot{h}^2}{h^2} + i\dot{\varphi}^2\right)\langle\left(x' - p'\right), \left(x' - p'\right)\rangle} = \sqrt{\frac{\dot{h}^2}{h^2} + i\dot{\varphi}^2} \parallel P'X' \parallel .$$

Special case 2. In the case of $h(t) \equiv 1$, we obtain $\parallel X_f' \parallel = \mid \dot{\varphi} \mid \parallel P'X' \parallel$. which was given by Müller [2].□

During the motion H_1, locus of pole points (which are fixed in both planes at all t) are called the moving and fixed pole curves in E and E' and denoted by (P) and (P') , respectively. Due to, the pole point defined with $X_f' = 0$ and from Eqs. (3), (4), (5) for $X = P$, we get

$$X_a' = hX_r' = h\dot{p}e^{i\varphi}. \tag{11}$$

Let ds and ds' be the arc elements of the moving and fixed pole curves, respectively. Then, we can write $ds = \parallel X_r' \parallel dt$ and $ds' = \parallel X_a' \parallel dt$. Hence, using Eq. (11) and the latter equations we obtain$ds' = \mid h \mid ds$. So, during the motion H_1, the velocities of (P) and (P') are different at each time t and the pole curves slide and roll upon each other.

Special case 3. In the case of the homothetic scale $h(t) \equiv 1$, we get $ds = ds'$. The pole curves only roll upon each other without sliding, [2].□

Now, let us find the acceleration vectors of the motion H_1, on the complex plane: If we differentiate the relative velocity X_r with respect to t, we obtain the relative acceleration vector as$b_r = \dot{X}_r = \ddot{x}$. This vector is expressed with

$$b_r' = b_r e^{i\varphi} = \ddot{x}e^{i\varphi} \tag{12}$$

in the fixed coordinate system. During the motion H_1, if we differentiate the absolute velocity vector X_a' with respect to t, then we get the absolute acceleration vector b_a'

as follows:

$$\begin{aligned} b'_a &= \dot{X}'_a = (x-p)\left[\ddot{h} - h\dot{\varphi}^2 + i\left(h\ddot{\varphi} + 2\dot{h}\dot{\varphi}\right)\right] e^{i\varphi} - \dot{p}\left(\dot{h} + ih\dot{\varphi}\right) e^{i\varphi} \\ &\quad + 2\dot{x}\left(\dot{h} + ih\dot{\varphi}\right) e^{i\varphi} + h\ddot{x} e^{i\varphi}, \end{aligned} \tag{13}$$

where

$$b'_f = \dot{X}'_f = (x-p)\left[\ddot{h} - h\dot{\varphi}^2 + i\left(h\ddot{\varphi} + 2\dot{h}\dot{\varphi}\right)\right] e^{i\varphi} - \dot{p}\left(\dot{h} + ih\dot{\varphi}\right) e^{i\varphi} \tag{14}$$

and

$$b'_c = 2\dot{x}\left(\dot{h} + ih\dot{\varphi}\right) e^{i\varphi} \tag{15}$$

are called the sliding acceleration vector and the coriolis acceleration vector, respectively. Thus, for the acceleration vectors of the motion H_1, we can give the following theorem:

Theorem 2. *Let X be a moving point in E. Then,*

$$b'_a = b'_f + b'_c + hb'_r. \tag{16}$$

during the one-parameter planar homothetic motion $H_1 = E/E'$, [1]. □

Under the motion H_1, the acceleration pole is characterized by vanishing the sliding b_f. Therefore, if we take $b_f = 0$, then for the acceleration pole point of the motion H_1, we get

$$x = p + \frac{\dot{p}\left(\dot{h} + ih\dot{\varphi}\right)}{\ddot{h} - h\dot{\varphi}^2 + \left(2\dot{h}\dot{\varphi} + h\ddot{\varphi}\right)}. \tag{17}$$

Special case 4. In the case of the homothetic scale $h(t) \equiv 1$, we get

$$x = p + \frac{\dot{\varphi}\dot{p}\left(\ddot{\varphi} - i\dot{\varphi}^2\right)}{\ddot{\varphi}^2 + \dot{\varphi}^4}$$

which is the acceleration pole point of the motion H_1, given by Müller [2].

2. High order accelerations and poles under the one-parameter homothetic motions on the complex plane

Let $H_1 = E/E'$ be the one-parameter planar motion of E according to E' and $X \in E$ be a fixed point. We assume that the homothetic scale $h(t) \neq 0$ at each time t.

2.1 The high order accelerations

Under the motion H_1, with the homothetic scale $h(t) \neq 0$, $\forall t \in I$, the absolute velocity and the sliding velocity is equal to each other and this velocity is given as

$$\dot{x}' = \dot{u}' + \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\left(x' - u'\right). \tag{18}$$

If we differentiate Eq. (18) with respect to t, then we obtain the velocity (absolute velocity or sliding velocity) of second order such as

$$\ddot{x}' = \ddot{u}' + \left[\left(\frac{\ddot{h}}{h} - \frac{\dot{h}^2}{h^2} + i\ddot{\varphi}\right) + \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\right]\left(x' - u'\right). \tag{19}$$

Let $(x' - u')$'s coefficient from Eq. (18) be $\Phi_1' = \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right) \neq 0$ and $(x' - u')$'s coefficient from Eq. (19) be$\Phi_2' = \left[\left(\frac{\ddot{h}}{h} - \frac{\dot{h}^2}{h^2} + i\ddot{\varphi}\right) + \left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\left(\frac{\dot{h}}{h} + i\dot{\varphi}\right)\right] \neq 0$. Thus, we obtain Φ_2' in terms of Φ_1' as $\Phi_2' = \dot{\Phi}_1' + \Phi_1'\Phi_1'$.Hence, from Eqs. (18) and (19), we can rewrite

$$\dot{x}' = \dot{u}' + \Phi_1'\left(x' - u'\right) \tag{20}$$

and

$$\ddot{x}' = \ddot{u}' + \Phi_2'\left(x' - u'\right). \tag{21}$$

If we differentiate Eq. (21) with respect to t, then we obtain the velocity of the third order (or the acceleration of the second order) such as

$$x' = u' + \dot{\Phi}_2'\left(x' - u'\right) + \Phi_2'\left(\dot{x}' - \dot{u}'\right).$$

From Eq. (20) and the latter equation, we have

$$x' = u' + \left(\dot{\Phi}_2' + \Phi_2'\Phi_1'\right)\left(x' - u'\right)$$

or

$$x' = u' + \Phi_3'\left(x' - u'\right), \tag{22}$$

where $\Phi_3' = \dot{\Phi}_2' + \Phi_2'\Phi_1' \neq 0$. If we differentiate Eq. (22) with respect to t, then we obtain the velocity of the fourth order (or the acceleration of the third order) such as

$$\overset{(4)}{x'} = \overset{(4)}{u'} + \dot{\Phi}_3'\left(x' - u'\right) + \Phi_3'\left(\dot{x}' - \dot{u}'\right).$$

Using Eq. (20) and the latter equation, we get

$$\overset{(4)}{x'} = \overset{(4)}{u'} + \left(\dot{\Phi}_3' + \Phi_3'\Phi_1'\right)\left(x' - u'\right)$$

or

$$\overset{(4)}{x'} = \overset{(4)}{u'} + \Phi'_4 \left(x' - u' \right). \tag{23}$$

where $\Phi'_4 = \dot{\Phi}'_3 + \Phi'_3 \Phi'_1 \neq 0$.

If we continue with consecutive differentiations, we obtain the high order velocities and accelerations. Then the velocity from n th order and the acceleration from $(n-1)$th order is equal to each other. We can give the following theorem for n th order velocities (or $(n-1)$th order accelerations):

Theorem 3. *During the one-parameter planar homothetic motion $H_1 = E/E'$ on the complex plane, we assume that $h(t) \neq 0$ for each t,*

$$\Phi'_1 = \left(\frac{\dot{h}}{h} + i\dot{\varphi} \right), \ \Phi'_0 = 1, \ \overset{(0)}{x'} = x'$$

and

$$\Phi'_n = \dot{\Phi}'_{n-1} + \Phi'_{n-1} \Phi'_1 \neq 0. \tag{24}$$

Then, the n th order velocities (or $(n-1)$th order accelerations) is

$$\overset{(n)}{x'} = \overset{(n)}{u'} + \Phi'_n \left(x' - u' \right). \tag{25}$$

Special case 5. In the case of the homothetic scale $h(t) \equiv 1$, we get the results which was given by Müller [2], [5].

2.2 The High Order Poles

Under the motion H_1 with the homothetic scale $h(t) \neq 0$, $\forall t \in I$, from Eq. (20), we get the first order pole point as

$$x' = p'_1 = u' - \frac{\dot{u}'}{\Phi'_1}.$$

From Eq. (21), we get the second order pole point as

$$x' = p'_2 = u' - \frac{\ddot{u}'}{\Phi'_2}.$$

Similarly, using Eq. (25) we obtain the $(n-1)$th order acceleration pole as

$$x' = p'_n = u' - \frac{\overset{(n)}{u'}}{\Phi'_n}. \tag{26}$$

Now, let us write $\overset{(n)}{x'}$, in terms of p'_n:
From Eq. (26) we have

$$\overset{(n)}{u'} = \Phi'_n \left(u' - p'_n\right).$$

Using Eq. (25) and the latter equation, we can rewrite

$$\overset{(n)}{x'} = \Phi'_n \left(x' - p'_n\right). \tag{27}$$

Special case 6. In the case of the homothetic scale $h(t) \equiv 1$ and $n = 1$, we get

$$p'_1 = u' - \frac{\dot{u}'}{i\dot{\varphi}} = u' + i\frac{\dot{u}'}{\dot{\varphi}}$$

which was given by Müller [2].

Special case 7. In the case of $h(t) \neq$constant and $n = 1$, we get for the first order pole point p'_1 as

$$p'_1 = \frac{\dot{u}\dot{h} + uh\dot{\varphi}^2 + i\left(u\dot{h}\dot{\varphi} - \dot{u}h\dot{\varphi}\right)}{\dot{h}^2 + h^2\dot{\varphi}^2}$$

which was given by Kuruoğlu [1].

2.2 Rotation Angle As a Parameter

During the one-parameter planar homothetic motion $H_1 = E/E'$, if we choose the rotation angle φ of the motion H_1 as parameter t, then we get as

$$\Phi'_n = \sum_{k=0}^{n-1} \binom{n}{k} \frac{\overset{(n-k)}{h}}{h} (i)^k + i^n \tag{28}$$

and

$$\overset{(n)}{x'} = \left[\sum_{k=0}^{n-1} \binom{n}{k} \frac{\overset{(n-k)}{h}}{h} (i)^k + i^n\right] \left(x' - u'\right). \tag{29}$$

Also, from Eq. (29), we obtain the n th p'_n as

$$p'_n = x' + \left[-\sum_{k=0}^{n-1} \binom{n}{k} \frac{\overset{(n-k)}{h}}{h} (i)^k + i^n\right]^{-1} \overset{(n)}{x'}. \tag{30}$$

The expression

$$\left[-\sum_{k=0}^{n-1}\left(\begin{array}{c} n \\ k \end{array}\right)\frac{\overset{(n-k)}{h}}{h}(i)^k+i^n\right]^{-1}$$

in Eq. (30) is a complex number such as $z=a+ib$. This complex number rotates $\overset{(n)}{x'}$ as $\arg(z)$ and translates it as module z. So we may give the following theorem:

Theorem 4. *Let $H_1=E/E'$ be a one-parameter planar motion of E according to E', and $X\in E$ be a fixed point. If we choose the rotation angle φ of the motion H_1 as a parameter, then the sum of the vector which is the multiple $\overset{(n)}{x'}$ by a complex number z and the vector x' is the $(n-1)$th order acceleration pole p'_n .*

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